Pappus's Theorem: An Introduction

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1 Topic of Choice

1.1 Acknowledgment

Thanks to the guidance of Dr. Akbar, we have narrowed our list of topics from six (all of which were quite interesting and worthy of study) to one: Pappus's Theorem.

1.2 Statement of Pappus's theorem

Suppose that points A, B, and C lie on some line l and that points X, Y, and Z lie on line m, where the six points are distinct and the two lines are also distinct. Assume that lines BZ and CY meet at P, lines AZ and CX meet at Q, and lines AY and BX meet at R. Then points P, Q, and R are collinear.[1]

[1] Isaacs, Theorem 4.16

2 What makes this topic interesting

2.1 Comprehensiveness

Pappus's theorem ties together many of the important concepts we have covered in MATH 3321. Its comprehensiveness will serve as a nice way to review the major themes of the course, including collinearity, triangles, Euclid's fifth postulate, and algebraic manipulation of geometric ratios.

The very fabric of this theorem rests on the integrity of Euclid's fifth postulate: we must be careful to not let one of the six points $A, B, C, X, Y,$ and Z described in the hypothesis lie at the intersection point of the two lines l and m as points $P, Q,$ and R would then not be distinct, giving a trivial claim as two points are (by Euclid's first postulate) automatically collinear.

The Cevian product is also central to the proof of Pappus's theorem; the notion of the Cevian, which we have discussed several times in the course, will be introduced in our video (interesting/supplementary results will be referenced as asides). The proof also invokes several other important (and rather advanced) theorems, including the theorem of Menelaus. We will therefore also prove Menelaus's theorem in our introduction.

2.2 Uniqueness

Pappus's theorem "is different in flavor from almost everything else in this book," as Isaacs describes.[2] It is a **nonmetric** result. That is, the notion of length, angular size, etc. (i.e., elements of **metric** geometry—based on measurement) are not relevant to Pappus's theorem, where there is nothing to be measured. Isaacs is quick to note that Pappus's theorem–for this reason–belongs to the field of **nonmetric geometry**.[3]

Isaacs also notes that the theorem's independence from the notion of "points, lines, [and] incidence"[4] casts it in the field of **projective geometry**. He describes projective geometry by providing an interesting physical analogy that we will paraphrase in our presentation. He asks the reader to imagine drawing a diagram fitting the hypothesis of Pappus's theorem with "opaque ink on a sheet of glass," and that a "point source of light causes the figure to cast a shadow onto a planar screen. Since this projection from a point carries points to points and lines to lines, and it preserves incidence, we see that the shadow of a diagram for Pappus' theorem is again a diagram for Pappus' theorem." Isaacs continues by casually defining projective geometry: "In a very rough sense, projective geometry is that part of ordinary (Euclidean) geometry where the shadows of diagrams illustrate the relevant information in the original diagrams." Isaacs notes that other important theorems of geometry (*pons asinorum*, for example) do not belong to projective geometry (since the projection of an

[2]page 149

[3]The distinction between metric and nonmetric geometry is deeply physical; Isaacs notes that "no result involving circles could be called nonmetric because a circle is defined as the locus of points of some fixed distance from a given point."

[4]151

isosceles triangle may not also be isosceles). This appeal to physical projection is satisfying to physics-minded individuals, like Chirag and Tucker (both physics majors).

2.3 Aesthetic and Algebraic Beauty

Pappus's Theorem is actually rather hard to believe, as it is generally true for six generic, distinct points on two randomly oriented lines l and m. We find that theorems of such a general hypothesis yet such a precise conclusion (i.e., collinearity) possess great aesthetic beauty. The use of ratios and their algebraic manipulation only heightens the beauty of the proof.

3 How we will proceed

3.1 Logistics

Below we discuss the various technological challenges we will face, and how we intend to meet these challenges.

3.1.1 Rendering equations and figures

PowerPoint figures are not precise and look unprofessional. The same is true with figures made using Microsoft Word, MS Paint, etc. As students who are ambitious and intend on presenting their work in graduate school and beyond, we choose to turn to a more challenging but more rewarding, professional, and precise means of rendering figures: LaTeX. The extra effort taken will pay off in our future endeavors. We will coordinate our work and associated code in the online LaTeX platform **Overleaf**. Once all code, figures, text, and animation are complete, we will shift to another platform capable of sharing larger video files (likely **UTD Box**).

3.1.2 File management

To distinguish incomplete work with complete work, we will use **this repository** as a space to present finalized documents, code, resources, supplemental topics/proofs, and the eventual video (at this stage, only our initial **Potential Topics for Video Project** document is available at that site). Chirag will be responsible for updating this site on a daily basis. Maintaining this site will also help us keep a track of our progress. This site will be linked in the YouTube description of the video so viewers can access high-resolution versions of all figures, as well as the supplemental topics/proofs, resources, etc. referred to in the presentation.

3.2 Responsibilities

Of course, both Chirag and Tucker will be responsible for the content and presentation of the project, demanding from each Geometer a high level of understanding of Pappus's theorem.

However, Chirag and Tucker each possess unique technological skills which will be employed so as to maximize the quality of the resulting video.

3.2.1 Chirag

Chirag has a background in LaTeX and is intent on finding a package that will render geometric figures and labels well. He has initiated a conversation in the class group chat with hopes to pool resources and knowledge in these syntactical areas. It is our goal to have all text and diagrams created in LaTeX. So far, it looks like `tkz-euclide` is a good option. Chirag will be reading the documentation and attempting his geometry homework/individual assignments using `tkz-euclide` for practice in the coming weeks.

3.2.2 Tucker

Tucker has a strong background in audio/visual areas. He has high-quality microphones and video editing software. It was suggested that we could even delve into animating the project in After Effects (or a similar software) to enhance the quality of the presentation.

Animation would benefit this project because it would increase the visual stimulation of the audience and in turn keep them interested in our topic. Those who have watched popular math channels on YouTube such as 3Blue1Brown or Numberphile can attest to how much better a dynamic presentation works to keep viewers interested as compared to the static nature of your standard PowerPoint presentation. In addition to helping retain the interest of the audience, animation can also help some viewers gain better understanding of topics that require spatial reasoning. The 2D geometric drawings often accompanying geometric proofs can often appear messy or chaotic at first glance; by creating an animated drawing the audience would be able to see the drawing being created in front of them as if they were drawing it themselves. This can help elucidate the spatial relationships of the various lines and points in the drawing which could further aid the viewer in their understanding of what is being demonstrated by the theorem or concept in question.

3.3 Organization of the presentation

We will employ the approach presented by Isaacs in section 4D of the text. This approach draws heavily on previous topics covered in chapter 4, as mentioned. A preliminary outline of the presentation will include:

1. Introductions & interesting/attention-seeking remarks
2. Brief history of Pappus
3. Statement of the theorem
4. Discussion on nonmetric, projective geometry
5. Proof of Menelaus's theorem