Practice problem

Jackson learned calculus in the 9th grade. Because his brain is suffused with advanced mathematics, his head of radius a can be modeled as a pressure-release sphere. Meanwhile, his iconic clear-frame glasses, which sit at $\theta = 45°$, impose band of rigidity around his head. The boundary conditions for Jackson's head are therefore

$$p(a,\theta) = \begin{cases} 0 & \text{for } \theta \neq \pi/4 \\ p_0 & \text{for } \theta = \pi/4 \end{cases}$$

On the other hand, Chirag did not study calculus until the 12th grade. As such, the top half of his head can be modeled be as a hemisphere with a rigid boundary. The bottom half of his head is pressure-release. The boundary conditions for Chirag's head are therefore

$$p(a,\theta) = \begin{cases} p_0 & \text{for } 0 \leq \theta \leq \pi/2 \\ 0 & \text{for } \pi/2 < \theta < \pi \end{cases}$$

You are given the orthogonality integral

$$\int_1^{-1} P_n(z) P_m(z) \, \mathrm{d}z = \frac{2}{2n+1} \delta_{nm}$$

as well as the integral

$$\int_0^{\theta_0} P_m(\cos\theta) \sin\theta \, \mathrm{d}\theta = \frac{1}{2m+1} \left(P_{m-1}(\cos\theta_0) - P_{m+1}(\cos\theta_0) \right)$$

1. Solve the pressure wave equation for Jackson's head.

   The Neumann functions are discarded since they diverge at the origin. Also, since there is no dependence on ψ, $m = 0$. The general solution is therefore

   $$p(r,\theta,t) = \sum_{n=0}^{\infty} A_n j_n(kr) P_n(\cos\theta) e^{j\omega t} \tag{1}$$

   Meanwhile, the boundary condition on Jackson's head can be written using the Dirac delta function as

   $$p(a,\theta) = p_0 \delta(\theta - \pi/4) \tag{2}$$

   Equating the time-independent part of equation (1) at $r = a$ and equation (2),

$$\sum_{n=0}^{\infty} A_n j_n(ka) P_n(\cos\theta) e^{j\omega t} = p_0 \delta(\theta - \pi/4)$$

Multiplying both sides by $P_m(\cos\theta)\sin\theta$, and integrating from $\theta = 0$ to $\theta = \pi$,

$$\sum_{n=0}^{\infty} A_n j_n(ka) \int_0^{\pi} P_m(\cos\theta) P_n(\cos\theta) \sin\theta \,\mathrm{d}\theta = p_0 \int_0^{\pi} P_m(\cos\theta) \sin\theta \delta(\theta - \pi/4)\,\mathrm{d}\theta \tag{$\star$}$$

Making the substitution $\cos\theta \mapsto z$, $\sin\theta\,\mathrm{d}\theta \mapsto -\mathrm{d}z$, the orthogonality integral is employed on the left-hand-side (LHS) of ($\star$):

$$\begin{aligned}
\sum_{n=0}^{\infty} A_n j_n(ka) \int_0^{\pi} P_m(\cos\theta) P_n(\cos\theta) \sin\theta \,\mathrm{d}\theta &= -\sum_{n=0}^{\infty} A_n j_n(ka) \int_{-1}^{1} P_m(z) P_n(z)\,\mathrm{d}z \\
&= \sum_{n=0}^{\infty} \frac{2 A_n j_n(kr)}{2n+1} \delta_{nm} \\
&= \frac{2 A_m j_m(ka)}{2m+1} \qquad \text{(Simplified LHS)}
\end{aligned}$$

Meanwhile, the right-hand-side (RHS) of ($\star$) is

$$\begin{aligned}
p_0 \int_0^{\pi} P_m(\cos\theta) \sin\theta \delta(\theta - \pi/4)\,\mathrm{d}\theta &= p_0 P_m(\cos(\pi/4)) \sin(\pi/4) \\
&= p_0 \frac{\sqrt{2}}{2} P_m(\sqrt{2}/2) \quad \text{(Simplified RHS)}
\end{aligned}$$

Since m is a dummy index, we revert to the original index n. Equating (Simplified LHS) and (Simplified RHS), and solving for A_n,

$$\begin{aligned}
\frac{2 A_n j_n(ka)}{2n+1} &= p_0 \frac{\sqrt{2}}{2} P_n(\sqrt{2}/2) \\
\implies A_n &= p_0 \frac{\sqrt{2}}{2} \frac{2n+1}{2} \frac{P_n(\sqrt{2}/2)}{j_n(ka)}
\end{aligned}$$

The general solution given by equation (1) becomes specific:

$$p(r,\theta,t) = p_0 \sum_{n=0}^{\infty} \frac{\sqrt{2}}{2} \frac{2n+1}{2} \frac{j_n(kr)}{j_n(ka)} P_n(\sqrt{2}/2) P_n(\cos\theta) e^{j\omega t}$$

2. Solve the pressure wave equation for Chirag's head.

The same considerations as in part (1) lead us to start with the general solution

$$p(r,\theta,t) = \sum_{n=0}^{\infty} A_n j_n(kr) P_n(\cos\theta) e^{j\omega t} \tag{3}$$

Equating the time-independent part of equation (3) at $r = a$ to the given boundary condition in the problem statement,

$$\sum_{n=0}^{\infty} A_n j_n(ka) P_n(\cos\theta) e^{j\omega t} = p(a,\theta) = \begin{cases} p_0 & \text{for } 0 \le \theta \le \pi/2 \\ 0 & \text{for } \pi/2 < \theta < \pi \end{cases}$$

Multiplying both sides by $P_m(\cos\theta)\sin\theta$, and integrating from $\theta = 0$ to $\theta = \pi$,

$$\sum_{n=0}^{\infty} A_n j_n(ka) \int_0^{\pi} P_m(\cos\theta) P_n(\cos\theta) \sin\theta \, \mathrm{d}\theta = \int_0^{\pi} p(a,\theta) P_m(\cos\theta) \sin\theta \, \mathrm{d}\theta \tag{$\dagger$}$$

The integral on the left-hand-side of (†) is identical to that of ($\star$) in part (1). The result (Simplified LHS) is therefore used. Meanwhile, the integral on the right-hand-side of (†) is written as the sum $\int_0^{\pi/2} + \int_{\pi/2}^{\pi}$. Equation (†) becomes

$$\begin{aligned}
\frac{2A_m j_m(ka)}{2m+1} &= \int_0^{\pi/2} p(a,\theta) P_m(\cos\theta) \sin\theta \, \mathrm{d}\theta + \int_{\pi/2}^{\pi} p(a,\theta) P_m(\cos\theta) \sin\theta \, \mathrm{d}\theta \\
&= \int_0^{\pi/2} p_0 * P_m(\cos\theta) \sin\theta \, \mathrm{d}\theta + \int_{\pi/2}^{\pi} 0 * P_m(\cos\theta) \sin\theta \, \mathrm{d}\theta \\
&= p_0 \int_0^{\pi/2} P_m(\cos\theta) \sin\theta \, \mathrm{d}\theta \\
&= \frac{p_0}{2m+1} \left(P_{m-1}(\cos\pi/2) - P_{m+1}(\cos\pi/2) \right) \\
&= \frac{p_0}{2m+1} \left(P_{m-1}(0) - P_{m+1}(0) \right)
\end{aligned}$$

Since m is a dummy index, we revert to the original index n. Solving for A_n,

$$\begin{aligned} 2A_n j_n(ka) &= p_0 \left(P_{n-1}(0) - P_{n+1}(0)\right) \\ \implies A_n &= \frac{1}{2j_n(ka)} \left(P_{n-1}(0) - P_{n+1}(0)\right) \end{aligned}$$

The general solution given by equation (3) becomes specific:

$$p(r, \theta, t) = \frac{p_0}{2} \sum_{n=0}^{\infty} \frac{j_n(kr)}{j_n(ka)} \left(P_{n-1}(0) - P_{n+1}(0)\right) P_n(\cos\theta) e^{j\omega t}$$