Radiation from general axisymmetric ($m = 0$) spherical velocity source[1]

Let the radius of the spherical source be a. At the boundary of the sphere, $r = a$, there is a radially pulsating velocity source given by

$$u_r(a, \theta, t) = u_0 f(\theta) e^{j\omega t} \tag{1}$$

The source causes outgoing (i.e., throw out $h_n^{(1)(kr)}$) pressure waves, which are expressed as an expansion of Legendre polynomials:[2]

$$p(r, \theta, t) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta) h_n(kr) e^{j\omega t} \tag{2}$$

To write equation (2) in terms of particle velocity, the momentum equation $\rho_0 \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} p = 0$, which for harmonic waves becomes $u_r(r, \theta) = -\frac{1}{j\omega\rho_0} \frac{\partial p}{\partial r}$, is applied, giving

$$u_r(r, \theta, t) = -\frac{1}{j c_0 \rho_0} \sum_{n=0}^{\infty} A_n P_n(\cos\theta) h_n'(kr) e^{j\omega t} \tag{3}$$

where $c_0 = \frac{\omega}{k}$.

Expanding the angular distribution function $f(\theta)$ in equation (1) in terms of Legendre polynomials gives

$$u_r(a, \theta) = u_0 e^{j\omega t} \sum_{n=0}^{\infty} U_n P_n(\cos\theta) \tag{4}$$

where the coefficients U_n are found using the orthogonality of Legendre polynomials.

$$U_n = \frac{2n+1}{2} \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta \, \mathrm{d}\theta \tag{5}$$

Equating equations (3) at $r = a$ and (4), the Legendre polynomials and the time dependence cancel:

$$u_0 \sum_{n=0}^{\infty} U_n = -\frac{1}{j\omega c_0} \sum_{n=0}^{\infty} A_n h_n'(ka)$$

Since this must hold for each term in the summations on both sides,

$$u_0 U_n = -\frac{1}{j\omega c_0} A_n h_n'(ka)$$

[1] From Acoustics II class notes, Dr. Mark F. Hamilton

[2] The Hankel functions of the second kind, $h_n^{(2)}$, will be denoted h_n for convenience.

Solving for A_n,

$$A_n = -j\rho_0 c_0 \frac{U_n}{h'_n(ka)}$$

Then, equation (2) becomes

$$p(r,\theta,t) = -j\rho_0 c_0 u_0 e^{j\omega t} \sum_{n=0}^{\infty} \frac{h_n(kr)}{h'_n(ka)} U_n P_n(\cos\theta) \tag{6}$$

Far field limit: $kr \to \infty$

Note that

$$\lim_{kr\to\infty} h_n(kr) = \frac{e^{-jkr}}{kr} e^{j(n+1)\pi/2}$$

Since $e^{j\pi/2} = j$, $e^{j(n+1)\pi/2} = j^{n+1}$, so equation (6) becomes

$$p(r,\theta,t) = \rho_0 c_0 u_0 \frac{e^{j(\omega t - kr)}}{kr} \sum_{n=0}^{\infty} \frac{j_n U_n}{h'_n(ka)} P_n(\cos\theta) \tag{7}$$

Note that the angular dependence has been factored out of the radial dependence.

Small source limit in the far field: $kr \to \infty$, $ka \ll 1$

Continuing in the far field, the small source limit $ka \ll 1$ is now evaluated.

First note that $ka \ll 1 \implies a \ll \lambda$, which means that the source is point-like compared to the wavelength. Therefore, the wavelength is effectively constant along the circumference of the source. So, higher spatial harmonics are excluded, i.e., the $n = 0$ term dominates in this limit for $U_0 \neq 0$. Then, equation (5) becomes

$$U_0 = \frac{1}{2} \int_0^{\pi} f(\theta) \sin\theta \, \mathrm{d}\theta$$

Multiplying the right-hand-side by $1 = \frac{2\pi a^2}{2\pi a^2}$,

$$U_0 = \frac{1}{4\pi a^2} \int_0^{\pi} f(\theta) 2\pi a^2 \sin\theta \, \mathrm{d}\theta$$

Noting that $2\pi a^2 \sin\theta \, \mathrm{d}\theta$ is $\mathrm{d}S$, the differential surface area of a sphere at radius $r = a$, the above becomes

$$U_0 = \frac{1}{S} \int f \, \mathrm{d}S$$

That is, U_0 is just the spatial average of f on the surface of the sphere.

Further, the volume velocity of the source Q_0 is

$$\begin{aligned} Q_0 &= \int u_0 f(\theta) \, \mathrm{d}S \\ &= u_0 S U_0 \\ &= 4\pi a^2 u_0 U_0 \end{aligned}$$

Also note that

$$\begin{aligned} \lim_{ka \to 0} \frac{1}{h_n'(ka)} &= \frac{n! 2^n}{(n+1)(2n)!} (j)(ka)^{n+2} \\ &= (j)(ka)^2, \ n = 0 \\ &= \frac{1}{2} (j)(ka)^3, \ n = 1 \\ &= \dots \end{aligned}$$

Combining these two observations of the $ka \ll 1$ limit, equation (7) becomes

$$\begin{aligned} p(r,t) &= \rho_0 c_0 u_0 \frac{e^{j(\omega t - kr)}}{kr} (j)(ka)^2 U_0 \\ &= j\omega Q_0 \frac{\rho_0 e^{j(\omega t - kr)}}{4\pi r} \end{aligned} \tag{8}$$

This is the so-called "equation for a simple simple source." See page 359, equation D-7. Note that there is no angular dependence in the far-field for $ka \ll 1$ for any source distribution for $U_0 \neq 0$.

Large source limit in the far field: $kr \to \infty$, $ka \gg 1$

Noting that

$$\begin{aligned} \lim_{ka \to \infty} \frac{1}{h_n'(ka)} &= ka e^{j(ka - n\pi/2)} \\ &= j^{-n} ka e^{jka}, \end{aligned}$$

equation (7) becomes

$$p = \rho_0 c_0 u_0 \frac{a}{r} e^{j(\omega t - k(r-a))} \sum_{n=0}^{\infty} U_n P_n(\cos\theta)$$

The sum in the above equation is precisely the expansion of $f(\theta)$ in terms of Legendre polynomials. The above becomes

$$p = \rho_0 c_0 u_0 f(\theta) \frac{a}{r} e^{j(\omega t - k(r-a))} \qquad \text{(Geometric acoustic limit)}$$

The (Geometric acoustic limit) is a radial projection of $f(\theta)$ from a to radius r, i.e., no diffraction.