Virial Theorem for Transverse Modes on String

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Derivation of the Virial Theorem[1]

In this discussion, f is a function of canonical variables (q_k, p_k), $\mathcal{H} = \sum_j p_j \dot{q}_j - \mathcal{L}$ is the Hamiltonian, and[2] $[f, \mathcal{H}] \equiv \frac{\partial f}{\partial q_k}\frac{\partial \mathcal{H}}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial \mathcal{H}}{\partial q_k} = \frac{\partial f}{\partial q_k}\dot{q}_k - \frac{\partial f}{\partial p_k}\dot{p}_k$ (the last equality by Hamilton's equations).

Suppose we have an n-dimensional system of s degrees of freedom that we choose to represented by N points in a $2n$-dimensional phase space. The density of points in the space is $\rho(q_k, p_k, t)$. Then, a differential volume element of the phase space is

$$dv = (dq_1 dq_2 \ldots dq_s)(dp_1 dp_2 \ldots dp_s)$$

and

$$N = \rho dv$$

For some area $dq_k dp_k$ in the q_k-p_k plane, the number of points per unit time moving into the area across a region of constant q_k

$$\rho \dot{q}_k dp_k \tag{1}$$

while the number of points per unit time moving into the area across a region of constant p_k

$$\rho \dot{p}_k dq_k \tag{2}$$

The total number of points per unit time moving **into** the area $dq_k dp_k$ is then the sum of of equations (1) and (2):

$$\rho(\dot{q}_k dp_k + \dot{p}_k dq_k) \tag{3}$$

To find the total number of points per unit time move **out** of the area $dq_k dp_k$, we can Taylor expand equation (3) to first order, giving

$$\left(\rho\dot{q}_k + \frac{\partial}{\partial q_k}(\rho\dot{q}_k)dq_k\right)dp_k + \left(\rho\dot{p}_k + \frac{\partial}{\partial p_k}(\rho\dot{p}_k)dp_k\right)dq_k \tag{4}$$

Subtracting equation (4) from equation (3) is the total increase in density in the area, $\frac{\partial \rho}{\partial t} dq_k dp_k$ per unit time:

[1] I have adapted the derivation found in *Classical Dynamics of Particles and Systems*, 5th ed. by Stephen Thornton, Jerry Marion, 276-278. Their derivation leaves out several steps and is a bit convoluted.

[2] Definition of the Poisson bracket

$$\frac{\partial \rho}{\partial t} dq_k dp_k = -\left(\tfrac{\partial}{\partial q_k}(\rho \dot{q}_k) + \tfrac{\partial}{\partial p_k}(\rho \dot{p}_k) \right) dq_k dp_k \tag{5}$$

Dividing equation (5) by the area $dq_k dp_k$,

$$\frac{\partial \rho}{\partial t} = -\left(\tfrac{\partial}{\partial q_k}(\rho \dot{q}_k) + \tfrac{\partial}{\partial p_k}(\rho \dot{p}_k) \right)$$

and moving everything to the left-hand-side gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q_k}(\rho \dot{q}_k) + \frac{\partial}{\partial p_k}(\rho \dot{p}_k) = 0 \tag{6}$$

Let's look at the second two terms of on the left-hand-side of equation (6) individually. Applying the product rule to each,

$$\frac{\partial}{\partial q_k}(\rho \dot{q}_k) = \frac{\partial \rho}{\partial q_k}\dot{q}_k + \rho \frac{\partial \dot{q}_k}{\partial q_k} \tag{7}$$

and

$$\frac{\partial}{\partial p_k}(\rho \dot{p}_k) = \frac{\partial \rho}{\partial p_k}\dot{p}_k + \rho \frac{\partial \dot{p}_k}{\partial p_k} \tag{8}$$

Now recall Hamilton's equations:

$$\begin{cases} \dot{q}_k = \frac{\partial \mathcal{H}}{\partial p_k} \\ \dot{p}_k = -\frac{\partial \mathcal{H}}{\partial q_k} \end{cases}$$

Taking the derivative with respect to q_k of the top equation and the derivative with respect to p_k of the bottom equation gives

$$\begin{cases} \frac{\partial \dot{q}_k}{\partial q_k} = \frac{\mathcal{H}}{\partial q_k \partial p_k} \\ \frac{\partial \dot{p}_k}{\partial p_k} = -\frac{\mathcal{H}}{\partial p_k \partial q_k} \end{cases}$$

Adding the two equations above and assuming that the Hamiltonian has equal mixed partials, we get

$$\frac{\partial \dot{q}_k}{\partial q_k} + \frac{\partial \dot{p}_k}{\partial p_k} = 0 \tag{9}$$

With this result in hand, the sum of equations (7) and (8) is

$$\frac{\partial}{\partial q_k}(\rho \dot{q}_k) + \frac{\partial}{\partial p_k}(\rho \dot{p}_k) = \frac{\partial \rho}{\partial q_k}\dot{q}_k + \frac{\partial \rho}{\partial p_k}\dot{p}_k \tag{10}$$

Substituting equation (10) into equation (6) yields

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q_k}\dot{q}_k + \frac{\partial \rho}{\partial p_k}\dot{p}_k = 0 \tag{11}$$

The latter two terms can be consolidated[3] as the Poisson bracket with the Hamiltonian:

$$\frac{\partial \rho}{\partial t} + [\rho, \mathcal{H}] = 0 \tag{12}$$

Equation (12) is the total time derivative of ρ, and for cases when the so-called "distribution function" $\rho(q_k, p_k, t)$ commutes with the Hamiltonian (i.e., $[\rho, \mathcal{H}] = 0$), then[4]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 \tag{13}$$

We can come up with a distribution function for N particles[5] S such that $[S, \mathcal{H}] = 0$:

$$S = \left\langle \sum_k^N \vec{r}_k \cdot \vec{p}_k \right\rangle$$

By equation (13),

$$\begin{aligned}
0 &= \tfrac{\mathrm{d}S}{\mathrm{d}t} \\
&= \left\langle \tfrac{\mathrm{d}}{\mathrm{d}t} \sum_k^N \vec{r}_k \cdot \vec{p}_k \right\rangle \\
&= \left\langle \sum_k^N \tfrac{\mathrm{d}\vec{r}_k}{\mathrm{d}t} \cdot \vec{p}_k + \vec{r}_k \tfrac{\mathrm{d}\vec{p}_k}{\mathrm{d}t} \right\rangle \\
&= \left\langle \sum_k^N \tfrac{\mathrm{d}\vec{r}_k}{\mathrm{d}t} \cdot \vec{p}_k \right\rangle + \left\langle \sum_k^N \vec{r}_k \cdot \tfrac{\mathrm{d}\vec{p}_k}{\mathrm{d}t} \right\rangle \\
-\left\langle \sum_k^N \tfrac{\mathrm{d}\vec{r}_k}{\mathrm{d}t} \cdot \vec{p}_k \right\rangle &= \left\langle \sum_k^N \vec{r}_k \cdot \tfrac{\mathrm{d}\vec{p}_k}{\mathrm{d}t} \right\rangle \\
\langle 2T \rangle &= -\left\langle \sum_k^N \vec{r}_k \cdot \vec{F}_k \right\rangle \\
\langle T \rangle &= -\frac{1}{2} \sum_{k=1}^N \langle \vec{F}_k \cdot \vec{r}_k \rangle \qquad \text{(Virial Theorem)}
\end{aligned}$$

The right-hand-side is what Clausius called the "virial," so the virial theorem says that the average kinetic energy of a system of particles is equal to its virial.

Let's apply this to the string.

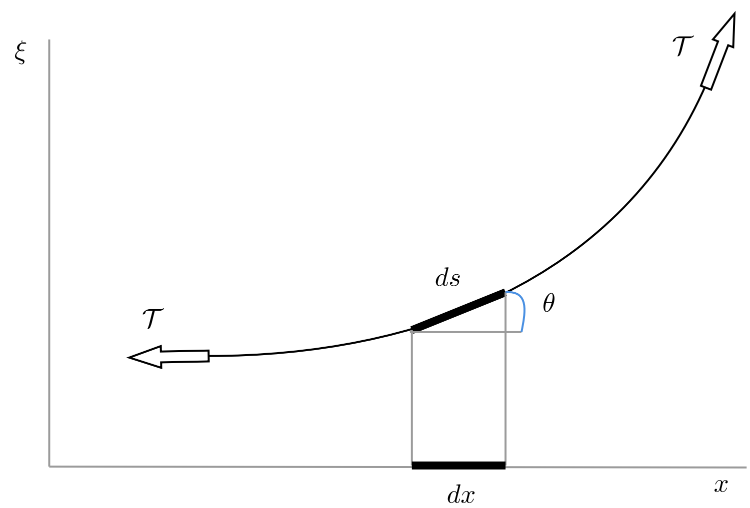
[3] $[f, \mathcal{H}] \equiv \frac{\partial f}{\partial q_k}\frac{\partial \mathcal{H}}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial \mathcal{H}}{\partial q_k} = \frac{\partial f}{\partial q_k}\dot{q}_k - \frac{\partial f}{\partial p_k}\dot{p}_k$

[4] Liouville's theorem

[5] I have been using Einstein's summation convention until now, but I will start writing sums explicitly to match the conventional notation of the virial theorem.

Energy of transverse modes of string

Given a string under tension $\mathcal{T} = c^2\rho_l$, whose displacement is ξ and mass per unit length is ρ_l, we can compute its kinetic energy T and potential energy U:



By inspection, the kinetic energy of the differential element of string ds is

$$dT = \frac{1}{2}\rho_l dx \left(\frac{\partial \xi}{\partial t}\right)^2 \qquad \text{(Differential kinetic energy)}$$

while the potential energy of the differential element is[6]

$$\begin{aligned} dU &= \mathcal{T}(ds - dx) \\ &\simeq \mathcal{T}\left(\frac{\theta^2}{2}\right) dx \\ &= \frac{1}{2}\mathcal{T}\left(\frac{\partial \xi}{\partial x}\right)^2 dx \\ &= \frac{1}{2}\rho_l c^2 \left(\frac{\partial \xi}{\partial x}\right)^2 dx \qquad \text{(Differential potential energy)} \end{aligned}$$

The differential kinetic and potential energies can be integrated along the length of the string, $x_2 - x_1$, to give the total kinetic and potential energies.

$$T = \int dT = \int_{x_1}^{x_2} \frac{1}{2}\rho_l \left(\frac{\partial \xi}{\partial t}\right)^2 dx \qquad (14)$$

[6]Note that $dx = dx\cos\theta \simeq dx\left(1 - \frac{\theta^2}{2}\right)^{-1} \simeq dx\left(1 + \frac{\theta^2}{2}\right)$. So $ds - dx \simeq \frac{\theta^2}{2}dx$

and

$$U = \int dU = \int_{x_1}^{x_2} \frac{1}{2}\rho_l c^2 \left(\tfrac{\partial \xi}{\partial x}\right)^2 dx \tag{15}$$

We wish to compute the time averages of the above quantities, $\langle T\rangle$ and $\langle U\rangle$. For this, we appeal to the virial theorem.

Since the string represents a continuous set of particles, the sum in the virial theorem becomes an integral:

$$\langle T\rangle = -\frac{1}{2}\int_{x_1}^{x_2} \langle \vec{F}\cdot\vec{r}\rangle\, dx \tag{16}$$

Also, since we are considering the transverse (as opposed to longitudinal) modes, $\vec{F} = F\hat{\xi}$, and $\vec{r}dx = d\xi\hat{\xi} = \frac{\partial \xi}{\partial x}dx\hat{\xi}$ where $\hat{\xi}$ is the unit vector perpendicular to the x-axis. Then equation (16) becomes

$$\langle T\rangle = -\frac{1}{2}\int_{x_1}^{x_2} \left\langle F\frac{\partial \xi}{\partial x}\right\rangle dx \tag{17}$$

Now, recall the definition of a conservative force:

$$F\hat{\xi} = \vec{F} \equiv -\vec{\nabla}U = -\frac{\partial U}{\partial \xi}\hat{\xi} = -\frac{\partial U}{\partial \xi_x dx}\hat{\xi}$$

Differentiating the total potential energy U accordingly,

$$\begin{aligned} F\hat{\xi} &= -\tfrac{\partial}{\partial \xi_x dx}U\hat{\xi} \\ &= -\frac{1}{dx}\tfrac{\partial}{\partial \xi_x}U\hat{\xi} \\ &= -\frac{1}{dx}\tfrac{\partial}{\partial \xi_x}\int_{x_1}^{x_2}\frac{1}{2}\rho_l c^2\left(\tfrac{\partial \xi}{\partial x}\right)^2 dx\hat{\xi} \\ &= -\frac{1}{2}\rho_l c^2 \tfrac{\partial}{\partial \xi_x}\xi_x^2\hat{\xi} \\ &= -\rho_l c^2 \tfrac{\partial \xi}{\partial x}\hat{\xi} \end{aligned} \tag{Force}$$

Putting the force above into equation (17),

$$\begin{aligned} \langle T\rangle &= \frac{1}{2}\int_{x_1}^{x_2}\left\langle \rho_l c^2 \tfrac{\partial \xi}{\partial x}\tfrac{\partial \xi}{\partial x} dx\right\rangle \\ &= \int_{x_1}^{x_2}\left\langle \frac{1}{2}\rho_l c^2\left(\tfrac{\partial \xi}{\partial x}\right)^2\right\rangle dx \\ &\left\langle \int_{x_1}^{x_2}\frac{1}{2}\rho_l c^2\left(\tfrac{\partial \xi}{\partial x}\right)^2 dx\right\rangle \end{aligned}$$

The integral on the last line is exactly equation (15), the total potential energy U of the string. Taking its time average gives us the relationship

$$\langle T \rangle = \langle U \rangle$$

That is, the time-averages of the kinetic and potential energies are equal for transverse waves on a string.

Notice that this result is more general than what was shown in class. We have not made any claims about the boundary conditions. This result therefore applies to any solution to the wave equation for a string (standing, propagating, etc.).