Problem

(a) Calculate the energy density of small-signal (i.e., linear) sound.[1]

(b) Show that for a progressive plane wave, the kinetic energy density equals the potential energy density.

Solution

(a) The total energy density is kinetic energy density plus potential energy density:

$$\mathcal{E} = \mathcal{E}_K + \mathcal{E}_P$$

The kinetic energy density is easy:

$$\mathcal{E}_K = \frac{1}{2}(\rho_0 + \rho')u^2 \simeq \frac{1}{2}\rho_0 u^2 \tag{1}$$

where the second step above is the linearization, i.e., throwing out the cubic term $\rho' u^2$.

One has to **work** a little harder to get the potential energy (haha!). First recall the definition of infinitesimal work from thermodynamics: $-p\,dV$. Compressing a gas from volume V_0 (for which the perturbation pressure is 0) to V (for which the pressure is p) corresponds to a finite amount of energy stored, namely $-\int_{V_0}^{V} p\,dV$, or per unit volume,

$$\mathcal{E}_P = -\frac{1}{V_0}\int_{V_0}^{V} p\,dV \tag{2}$$

It is desired to integrate equation (2). To do this, the differential volume element is related to the pressure p. Note that as the gas is compressed, mass is conserved, i.e., $M = \rho V$. Therefore,

$$\begin{aligned} \frac{dV}{d\rho} &= -\frac{M}{\rho^2} = -\frac{V}{\rho} \\ \implies dV &= -\frac{V}{\rho}d\rho = -\frac{V}{(\rho_0 + \rho')}d\rho' \simeq -\frac{V_0}{\rho_0}d\rho' \end{aligned} \tag{3}$$

[1]Note that energies are quadratic in the wave variable, so "linear" in this case means to throw out terms of cubic order and higher, when calculating the energy.

Linearization of ρ and V

There are two quantities to be linearized: ρ and V. Keeping only the linear term in ρ gives

$$dV \simeq -\frac{V}{\rho_0} d\rho'.$$

The change in volume is also assumed to be infinitesimal. In the linear limit, $V \simeq V_0$, so

$$dV \simeq -\frac{V_0}{\rho_0} d\rho'.$$

Next, the linear equation of state $p = c_0^2 \rho'$ is invoked, and equation (3) becomes

$$dV = -\frac{V_0}{\rho_0 c_0^2} dp \tag{4}$$

Equation (4) is substituted into equation (2), giving

$$\mathcal{E}_P = \frac{1}{\rho_0 c_0^2} \int_0^p p' dp' = \frac{p^2}{2\rho_0 c_0^2} \tag{5}$$

Adding equations (1) and (5) gives

$$\boxed{\mathcal{E} = \frac{1}{2}\rho_0 u^2 + \frac{p^2}{2\rho_0 c_0^2}} \tag{6}$$

(b) For a plane wave, $p = \rho_0 c_0 u$, so equation 5 becomes

$$\mathcal{E}_P = \frac{1}{2}\rho_0 u^2 \,,$$

which is equal to equation (1), i.e., the potential and kinetic energies are equal.