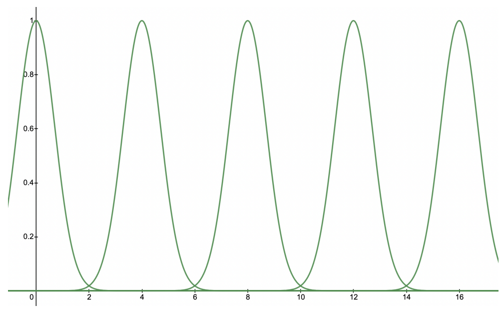
Practice problem

Consider an underwater[1] spherical wave that oscillates in time as shown below:



This "Gaussian comb" profile is defined over one period 4τ, where $\tau = 0.01$ is a dimensionless timescale, as

$$p = p_0 e^{-(t-4\tau)^2}$$

The pressure amplitude p_0 of the wave is 50 Pa when measured 1 m from the source.

1. Is this waveform *physical*? Defend your answer. Proceed with the problem even if your answer is "no."

   Acoustic pressure is a scalar-valued function of space and time. However, the waveform defined above is not actually physical, because the pressure is multivalued, i.e., the pressure has two different values in for one instant in time. Despite this, the Gaussian comb is often used in analytical work because Mother Nature likes Gaussian functions. For example, the Gaussian comb is frequently used to model a a periodic potential (which is also supposed to be scalar-valued) in quantum mechanics.

2. Calculate the sound pressure level 200 m from the source. You may assume that

$$\int_{-2\tau}^{2\tau} e^{-Cx^2}\,\mathrm{d}x \simeq \int_{-\infty}^{\infty} e^{-Cx^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{C}}$$

   The sound pressure level at $r = 200$ m is

$$\mathrm{SPL}(r = 200) = 20\log\left(\frac{p_{\mathrm{rms}}(r = 200)}{10^{-6}}\right)$$

[1] $\rho_0 = 1026$ kg/m^3, 1500 m/s, $\rho_0 c_0 = 1.54$ Mrayls

To find $p_{\text{rms}}(r = 200)$, we must first find p_0, the amplitude of the pressure, at 200 m:

$$\begin{aligned}
\frac{p_0(r = 200)}{p_0(r = 1)} &= \frac{1}{200} \\
\frac{p_0(r = 200)}{50} &= \frac{1}{200} \\
\implies p_0(r = 200) &= 0.25 \text{ Pa}
\end{aligned}$$

So the waveform 200 m away from the source is $p = 0.25e^{-(t-4\tau)^2}$, from which we can calculate p_{rms}:

$$\begin{aligned}
p_{\text{rms}} &= \left(\frac{1}{t_{\text{av}}} \int_{t_{\text{av}}} |p|^2 \, \mathrm{d}t \right)^{\frac{1}{2}} \\
&= \left(\frac{1}{4\tau} \int_{-2\tau}^{2\tau} \left| 0.25e^{-(t-4\tau)^2} \right|^2 \mathrm{d}t \right)^{\frac{1}{2}} \\
&\simeq \left(\frac{1}{4\tau} \int_{-\infty}^{\infty} \left| 0.25e^{-(t-4\tau)^2} \right|^2 \mathrm{d}t \right)^{\frac{1}{2}} \\
&\simeq 0.25 \left(\frac{1}{4\tau} \int_{-\infty}^{\infty} e^{-2(t-4\tau)^2} \, \mathrm{d}t \right)^{\frac{1}{2}} \\
&\simeq \frac{0.135}{\sqrt{\tau}} \left(\sqrt{\frac{\pi}{2}} \right)^{\frac{1}{2}} \\
&\simeq \frac{0.135}{\sqrt{0.01}} \left(\frac{\pi}{2} \right)^{\frac{1}{4}} \simeq 1.51
\end{aligned}$$

So the sound pressure level is

$$\text{SPL} = 20 \log \left(\frac{1.51}{10^{-6}} \right) = 123.58 \text{ dB}$$

3. Calculate the particle velocity amplitude u_0 200 m from the source.

   Since $|Z| = \frac{p_0}{u_0}$,

$$u_0 = \frac{p_0}{|Z|}$$

   Recall that the impedance is $Z = \frac{\rho_0 c_0}{1 + 1/jkr}$, and the magnitude of the impedance is $\frac{\rho_0 c_0 kr}{\sqrt{1+k^2r^2}}$.

$$\begin{aligned} u_0 &= \frac{p_0}{\frac{\rho_0 c_0 kr}{\sqrt{1+k^2r^2}}} \\ &= \frac{0.25}{\frac{1026*1500*(2\pi*0.04/1500)*200}{\sqrt{1+(2\pi*0.04/1500)^2*200^2}}} \\ &= 4.85\ \mu\text{m/s} \end{aligned}$$

4. Suppose the source is Dr. Hamilton's pet hermit crab, Hammy, maniacally laughing at the world as it sits on the ocean floor at the edge of a continental shelf.[2] Calculate the sound power level.

The sound power level is given by

$$\text{PWL} = 10\log(W/10^{-12})$$

where

$$W = \frac{1}{4}(4\pi r^2 I) = \pi r^2 I$$

since the sound is radiated in only one quadrant of a sphere. I found using the exact result of the intensity of a spherical wave:

$$I = \frac{p_{\text{rms}}^2}{\rho_0 c_0} = \frac{1.51^2}{1026 * 1500} = 1.48 * 10^{-6}$$

So $W = 0.186$ W, and

$$\text{PWL} = 10\log(0.186/10^{-12}) = 112\ \text{dB}$$

[2]That is, the sound propagates in only one quadrant of a sphere