Oblique incidence off a non-rigid wall

As discussed in class, the rigid wall obeys the 4th order inhomogeneous partial differential equation

$$m\frac{\partial^2 \xi}{\partial t^2} + B\frac{\partial^4 \xi}{\partial y^4} = p_i + p_r - p_t \tag{1}$$

We guess that the solution is of the form

$$\xi(y,t) = \xi_0 e^{j\omega\left(t - y/c_{\mathrm{tr}}\right)} \tag{2}$$

Substituting equation (2) into equation (1) and noting that $j\omega\xi = u_w = u_t\cos\theta = p_t\cos\theta/\rho_0 c_0$,

$$\left(jm\omega + \frac{B}{j\omega(c_{\mathrm{tr}/\omega})^4}\right)u_w = p_i + p_r - p_t$$
$$\left(jm\omega + \frac{B}{j\omega(c_{\mathrm{tr}/\omega})^4}\right)\frac{p_t\cos\theta}{\rho_0 c_0} = p_i + p_r - p_t$$

Dividing by p_i on both sides,

$$\left(jm\omega + \frac{B}{j\omega(c_{\mathrm{tr}/\omega})^4}\right)\frac{\cos\theta}{\rho_0 c_0}T = 1 + R - T \tag{3}$$

(4)

The other boundary condition gave the relation

$$T = 1 - R \tag{5}$$

Adding equations (3) and (5),

$$\left(\left(jm\omega + \frac{B}{j\omega(c_{\mathrm{tr}}/\omega)^4}\right)\frac{\cos\theta}{\rho_0 c_0} + 2\right)T = 2 \tag{6}$$

Dividing equation (6) by 2,

$$\left(\left(jm\omega + \frac{B}{j\omega(c_{\mathrm{tr}}/\omega)^4}\right)\frac{\cos\theta}{2\rho_0 c_0} + 1\right)T = 1 \tag{7}$$

Solving equation (7) for T,

$$T = \frac{1}{1 + \frac{jm\omega\cos\theta}{2\rho_0 c_0} + \frac{B\cos\theta}{2j\omega\left(c_{\mathrm{tr}}/\omega\right)^4 \rho_0 c_0}} \tag{8}$$

Let us focus on the last term in denominator of equation (8):

$$\frac{B\cos\theta}{2j\omega\left(c_{\mathrm{tr}}/\omega\right)^4 \rho_0 c_0}$$

This can be written more suggestively as

$$-\frac{jm\omega^2\cos\theta}{2\omega\rho_0 c_0}\left(\frac{B\omega^2}{m}\right)\frac{1}{c_{\mathrm{tr}}^4}$$

Noting that $c_b^4 = \frac{B\omega^2}{m}$ (as defined at the end of class on Wednesday), the above becomes

$$-\frac{jm\omega\cos\theta}{2\rho_0 c_0}\left(\frac{c_b}{c_{\mathrm{tr}}}\right)^4$$

Substituting the above into equation (8),

$$\begin{aligned} T &= \frac{1}{1 + \frac{jm\omega\cos\theta}{2\rho_0 c_0} - \frac{jm\omega\cos\theta}{2\rho_0 c_0}\left(\frac{c_b}{c_{\mathrm{tr}}}\right)^4} \\ &= \frac{1}{1 + \frac{jm\omega\cos\theta}{2\rho_0 c_0}\left(1 - \left(\frac{c_b}{c_{\mathrm{tr}}}\right)^4\right)} \end{aligned} \tag{C-23 ✓}$$

In class I had incorrectly written

$$T = \frac{1}{\left(1 + \frac{jm\omega\cos\theta}{2\rho_0 c_0}\right)\left(1 - \left(\frac{c_b}{c_{\mathrm{tr}}}\right)\right)^4} \tag{✗}$$

Similarly, when writing in terms of the critical frequency, I had written

$$T = \frac{1}{\left(1 + \frac{jm\omega\cos\theta}{2\rho_0 c_0}\right)\left(1 - (f/f_0)^2\sin^4\theta\right)^4} \tag{✗}$$