Three-medium problem

Suppose there are three media, labelled I, II, and III, as described in David Blackstock's *Fundamentals of Physical Acoustics*, page 163.

In medium I, the spatial pressure field is

$$P_{\mathrm{I}} = A_1 e^{-jk_1 x} + B_1 e^{jk_1 x}$$

In medium II, the spatial wave pressure field is

$$P_{\mathrm{II}} = A_2 e^{-jk_2 x} + B_2 e^{jk_2 x}$$

And in medium III, we take for convenience $x = l$ to be the origin for the transmitted wave, so the spatial pressure field is

$$P_{\mathrm{III}} = A_3 e^{-jk_3 (x-l)}$$

Boundary Conditions at the Interface of I & II

Continuity of pressure at the interface between media I and II demands that $P_{\mathrm{I}} = P_{\mathrm{II}}$ at $x = 0$, which simplifies to

$$A_1 + B_1 = A_2 + B_2 \tag{E-4}$$

Meanwhile, to match the particle velocity U at $x = 0$, we appeal to the conservation of momentum, with the time-dependence $e^{j\omega t}$ factored out:

$$U = \frac{j}{\rho_0 \omega} \frac{\partial P}{\partial x} \tag{Momentum equation}$$

Applying the (Momentum equation) to P_{I}, we find the particle velocity in medium I at $x = 0$ to be

$$\begin{aligned} U_{\mathrm{I}} &= \frac{j}{\rho_1 \omega} (-jA_1 k_1 + jB_1 k_1) \\ &= \frac{A_1}{Z_1} - \frac{B_1}{Z_1} \end{aligned}$$

Similarly applying the (Momentum equation) to P_{II}, we find the particle velocity in medium II at $x = 0$ to be

$$\begin{aligned} U_{\mathrm{II}} &= \frac{j}{\rho_2 \omega} (-jA_2 k_2 + jB_2 k_2) \\ &= \frac{A_2}{Z_2} - \frac{B_2}{Z_2} \end{aligned}$$

Matching the velocities (i.e., equating U_{I} and U_{II}),

$$A_1 - B_1 = \frac{Z_1}{Z_2}(A_2 - B_2) \tag{E-5}$$

Boundary Conditions at the Interface of II & III

Continuity of pressure at the interface between media II and III demands that $P_{\text{II}} = P_{\text{III}}$ at $x = l$, which simplifies to

$$A_2 e^{-jk_2 l} + B_2 e^{jk_2 l} = A_3 \tag{E-6}$$

Meanwhile, to match the particle velocity U at $x = l$, we again apply the (Momentum equation) to P_{II} and P_{III}. We find the particle velocity in medium II at $x = l$ to be

$$\begin{aligned} U_{\text{II}} &= \frac{j}{\rho_1 \omega}(-jk_2 A_2 e^{-jk_2 l} + jk_2 B_2 e^{jk_2 l}) \\ &= \frac{A_2}{Z_2} e^{-jk_2 l} - \frac{B_2}{Z_2} e^{jk_2 l} \end{aligned}$$

Similarly, we find the particle velocity in medium III at $x = l$ to be

$$\begin{aligned} U_{\text{III}} &= \frac{j}{\rho_3 \omega}(-jA_2 k_3) \\ &= \frac{A_3}{Z_3} \end{aligned}$$

Matching the velocities (i.e., equating U_{I} and U_{II}),

$$A_2 e^{-jk_2 l} - B_2 e^{jk_2 l} = \frac{Z_2}{Z_3} A_3 \tag{E-7}$$

Solving the system

We want to find the pressure transmission coefficient, which is the ratio of the pressure amplitude in to the pressure amplitude out, A_3/A_1. To do this, we should eliminate the other pressure coefficients, B_1, A_2, and B_2.

Adding equations (E-4) and (E-5) eliminates B_1:

$$2A_1 = \left(1 + \frac{Z_1}{Z_2}\right) A_2 + \left(1 - \frac{Z_1}{Z_2}\right) B_2 \tag{E-8}$$

If we can write equation (E-8) in terms of A_1 and A_3, we can then solve for $A_1/A_3 = T$. To do this, we can write A_2 and B_2 in terms of A_3. Adding (E-6) and (E-7) gives us A_2, and subtracting (E-6) and (E-7) gives us B_2,

$$\text{adding...}\quad 2A_2e^{-jk_2l} = A_3\left(1+\frac{Z_2}{Z_3}\right)$$
$$\text{subtracting...}\ 2B_2e^{jk_2l} = A_3\left(1-\frac{Z_2}{Z_3}\right)$$

Solving the above equations for A_2 and B_2,

$$\text{adding...}\quad A_2 = \frac{1}{2}A_3e^{jk_2l}\left(1+\frac{Z_2}{Z_3}\right) \qquad (A_2 \text{ in terms of } A_3)$$
$$\text{subtracting...}\ B_2 = \frac{1}{2}A_3e^{-jk_2l}\left(1-\frac{Z_2}{Z_3}\right) \qquad (B_2 \text{ in terms of } A_3)$$

Substituting (A_2 in terms of A_3) and (B_2 in terms of A_3) into equation (E-8),

$$2A_1 = \left(1+\frac{Z_1}{Z_2}\right)\frac{1}{2}A_3e^{jk_2l}\left(1+\frac{Z_2}{Z_3}\right) + \left(1-\frac{Z_1}{Z_2}\right)\frac{1}{2}A_3e^{-jk_2l}\left(1-\frac{Z_2}{Z_3}\right)$$
$$T = \frac{A_3}{A_1} = \frac{2*2}{\left(1+\frac{Z_1}{Z_2}\right)\left(1+\frac{Z_2}{Z_3}\right)e^{jk_2l} + \left(1-\frac{Z_1}{Z_2}\right)\left(1-\frac{Z_2}{Z_3}\right)e^{-jk_2l}}$$
$$= \frac{4}{\left(1+\frac{Z_2}{Z_3}+\frac{Z_1}{Z_2}+\frac{Z_1}{Z_3}\right)e^{jk_2l} + \left(1-\frac{Z_2}{Z_3}-\frac{Z_1}{Z_2}+\frac{Z_1}{Z_3}\right)e^{-jk_2l}}$$
$$= \frac{4}{\left(1+\frac{Z_1}{Z_3}\right)\left(e^{jk_2l}+e^{-jk_2l}\right) + \left(\frac{Z_2}{Z_3}+\frac{Z_1}{Z_2}\right)\left(e^{jk_2l}-e^{-jk_2l}\right)}$$
$$= \frac{4}{2\left(1+\frac{Z_1}{Z_3}\right)\left(\frac{e^{jk_2l}+e^{-jk_2l}}{2}\right) + 2j\left(\frac{Z_2}{Z_3}+\frac{Z_1}{Z_2}\right)\left(\frac{e^{jk_2l}-e^{-jk_2l}}{2j}\right)}$$
$$= \frac{2}{\left(1+\frac{Z_1}{Z_3}\right)\left(\frac{e^{jk_2l}+e^{-jk_2l}}{2}\right) + j\left(\frac{Z_2}{Z_3}+\frac{Z_1}{Z_2}\right)\left(\frac{e^{jk_2l}-e^{-jk_2l}}{2j}\right)}$$
$$= \frac{2}{\left(1+\frac{Z_1}{Z_3}\right)\cos k_2l + j\left(\frac{Z_2}{Z_3}+\frac{Z_1}{Z_2}\right)\sin k_2l} \qquad \text{(E-9)}$$

Similarly, the pressure reflection coefficient is the ratio of the reflected pressure out to the pressure in, $B_1/A_1 = R$.

We can solve for this ratio by dividing (E-4) by A_1:

$$1+R = \frac{A_2}{A_1} + \frac{B_2}{A_1}$$

Substituting in (A_2 in terms of A_3) and (B_2 in terms of A_3) into the above,

$$
\begin{aligned}
1+R &= \frac{1}{A_1}\left(\frac{1}{2}A_3 e^{jk_2l}\left(1+\frac{Z_2}{Z_3}\right)\right)+\frac{1}{A_1}\left(\frac{1}{2}A_3 e^{-jk_2l}\left(1-\frac{Z_2}{Z_3}\right)\right)\\
&= \frac{A_3}{A_1}\left(\frac{1}{2}e^{jk_2l}\left(1+\frac{Z_2}{Z_3}\right)\right)+\frac{A_3}{A_1}\left(\frac{1}{2}e^{-jk_2l}\left(1-\frac{Z_2}{Z_3}\right)\right)\\
&= T\left(\frac{1}{2}e^{jk_2l}\left(1+\frac{Z_2}{Z_3}\right)+\frac{1}{2}e^{-jk_2l}\left(1-\frac{Z_2}{Z_3}\right)\right)\\
&= T\left(\frac{e^{jk_2l}+e^{-jk_2l}}{2}+j\frac{Z_2}{Z_3}\left(\frac{e^{jk_2l}-e^{-jk_2l}}{2j}\right)\right)\\
&= T\left(\cos k_2l+j\frac{Z_2}{Z_3}\sin k_2l\right)\\
\implies R &= T\left(\cos k_2l+j\frac{Z_2}{Z_3}\sin k_2l\right)-1 \qquad (R \text{ in terms of } T)
\end{aligned}
$$

Substituting equation (E-9) in (R in terms of T),

$$
\begin{aligned}
R &= \left(\frac{2}{\left(1+\frac{Z_1}{Z_3}\right)\cos k_2l+j\left(\frac{Z_2}{Z_3}+\frac{Z_1}{Z_2}\right)\sin k_2l}\right)\left(\cos k_2l+j\frac{Z_2}{Z_3}\sin k_2l\right)-1\\
&= \frac{2\cos k_2l+2j\frac{Z_2}{Z_3}\sin k_2l-\left(1+\frac{Z_1}{Z_3}\right)\cos k_2l-j\left(\frac{Z_2}{Z_3}+\frac{Z_1}{Z_2}\right)\sin k_2l}{\left(1+\frac{Z_1}{Z_3}\right)\cos k_2l+j\left(\frac{Z_2}{Z_3}+\frac{Z_1}{Z_2}\right)\sin k_2l}\\
&= \frac{\left(1-\frac{Z_1}{Z_3}\right)\cos k_2l+j\left(\frac{Z_2}{Z_3}-\frac{Z_1}{Z_2}\right)\sin k_2l}{\left(1+\frac{Z_1}{Z_3}\right)\cos k_2l+j\left(\frac{Z_2}{Z_3}+\frac{Z_1}{Z_2}\right)\sin k_2l} \qquad \text{(E-10)}
\end{aligned}
$$

Phew!