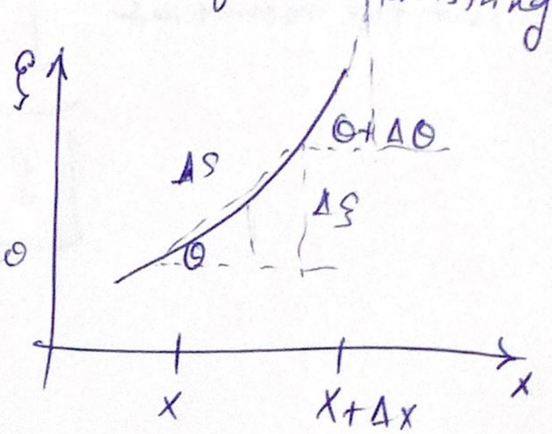
Demystification of Blackstock's nonlinear wave equation for string.

Start with Newton's second law



$$\Delta x \rho \xi_{tt} = T\sin(\theta+\Delta\theta) - T\sin\theta$$

$$\rho\xi_{tt} = \lim_{\Delta x \to 0} \frac{T\sin(\theta+\Delta\theta) - T\sin\theta}{\Delta x}$$

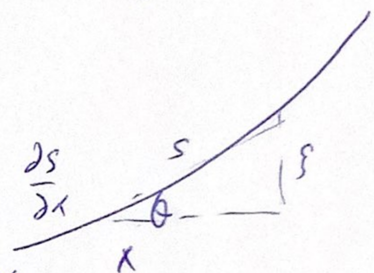
$$= T\frac{\partial}{\partial x}(\sin\theta)$$

$$= T\cos\theta \frac{\partial \theta}{\partial x}.$$

$$\boxed{\theta = \theta(x)}$$

Now we need to find this $\cos\theta$ and $\frac{\partial\theta}{\partial x}$ in terms of ξ:

$$\frac{\partial x}{\partial s} = \cos\theta \qquad \frac{\partial \xi}{\partial s} = \sin\theta$$



Dividing the two gives $\tan\theta = \frac{\partial\xi}{\partial s}\frac{\partial s}{\partial x}$

$$= \partial\xi/\partial x.$$

Since $1 = \frac{1}{\cos^2\theta} - \tan^2\theta \implies \cos\theta = (1+\tan^2\theta)^{-1/2}$

$$= \left[1 + \left(\frac{\partial\xi}{\partial x}\right)^2\right]^{-1/2}$$

$$\sin\theta = \cos\theta\tan\theta = \frac{\partial\xi}{\partial x}\left[1 + \left(\frac{\partial\xi}{\partial x}\right)^2\right]^{-1/2}$$

Take $\frac{\partial}{\partial x}\cos\theta(x) = -\sin\theta(x)\frac{\partial\theta}{\partial x}$

$$= -\frac{1}{2}\left[1 + \left(\frac{\partial\xi}{\partial x}\right)^2\right]^{-3/2} 2\,\frac{\partial^2\xi}{\partial x^2}\frac{\partial\xi}{\partial x}$$

$$\cancel{\frac{\partial\xi}{\partial x}}\left[1 + \left(\frac{\partial\xi}{\partial x}\right)^2\right]^{-1/2}\frac{\partial\theta}{\partial x} = \left[1 + (\partial\xi/\partial x)^2\right]^{-3/2}\frac{\partial^2\xi}{\partial x^2}\cancel{\frac{\partial\xi}{\partial x}}$$

$$\frac{\partial\theta}{\partial x} = \left[1 + (\partial\xi/\partial x)^2\right]^{-1}\left(\frac{\partial^2\xi}{\partial x^2}\right).$$

$$\frac{\partial\theta}{\partial x} = \cos^2\theta\left(\frac{\partial^2\xi}{\partial x^2}\right).$$

Put the result into $\rho\, \xi_{tt} = T\cos\theta\, \frac{\partial\theta}{\partial x}$

$$\boxed{\rho\, \xi_{tt} = T\cos^3\theta\, \xi_{xx}}$$