Spherical symmetry in acoustics and quantum mechanics

Introduction

The following spherical coordinates are used, in accordance with [1] and [2].[1][2]

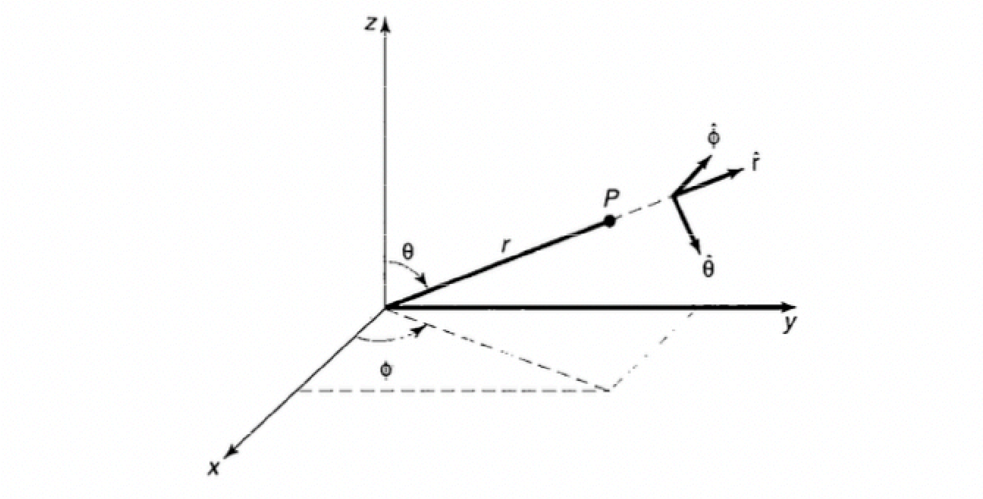

FIGURE 4.1: Spherical coordinates: radius r, polar angle θ, and azimuthal angle ϕ.

Eigenfunctions in acoustics

The solution to the Helmholtz equation

$$\nabla^2 p - \partial^2 p / \partial t^2 = 0$$

in spherical coordinates is given by linear combinations of the eigenfunctions [2]

$$\boxed{p_{nlm}(r,\theta,\phi) = \begin{Bmatrix} j_n(k_l r) \\ n_n(k_l r) \end{Bmatrix} \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix} P_n^m(\cos\theta)} \tag{1}$$

[1] Figure 4.1 from [1]

[2] [2] uses ψ in place of ϕ, so as to not confuse the velocity potential with the angular coordinate. Here the more conventional ϕ is used to denote the azimuthal angle, which is consistent with [1].

where

$$
\begin{aligned}
n &= 0, 1, 2, \ldots \infty, \\
l &= 1, 2, 3, \ldots \infty, \\
\text{and } |m| &\leqslant n
\end{aligned}
$$

Three cases arise:

1. $n = 0$, $l = 0, 1, 2, \ldots \infty$, and $m = 0$ describes sound with spherical symmetry. This means the sound has polar and azimuthal symmetry, i.e., $p = p(r)$.
2. $n = 0, 1, 2, \ldots \infty$, $l = 0, 1, 2, \ldots \infty$, and $m = 0$ describes sound with azimuthal symmetry, but not polar symmetry, i.e., $p = p(r, \theta)$.
3. $n = 0, 1, 2, \ldots \infty$, $l = 0, 1, 2, \ldots \infty$, and $|m| \leqslant n$ describes sound without symmetry, i.e., $p = p(r, \phi, \theta)$.

Naïve students like Chirag may wonder why there is no fourth case, sound with azimuthal symmetry and without polar symmetry, i.e., $p = p(r, \phi)$. Such inept students simply point to the inequality $|m| \leqslant n$ and say, "That's just the way it is. If $n = 0$, then $m = 0$. That is, azimuthal symmetry implies polar symmetry."

While this reasoning is correct mathematically, brighter students like Jackson provide a more physical answer as to why $p \neq p(r, \phi)$. Jackson displays his intellectual prowess by offering a proof by contradiction:

> Assume there is such a function, $p = p(r, \phi)$. This functional form means that at a given radius r_0, p takes on different values for different azimuthal angles ϕ_A and ϕ_B:
>
> $$p_A(r_0, \phi_1) \neq p_B(r_0, \phi_2)\,. \tag{2}$$
>
> Since equation (2) holds for any given polar angle θ_0, it can equivalently be written as
>
> $$p_A(r_0, \theta_0, \phi_1) \neq p_B(r_0, \theta_0, \phi_2)\,. \tag{3}$$
>
> Choosing $\theta_0 = 0$, equation (3) becomes
>
> $$p_A(r_0, 0, \phi_1) \neq p_B(r_0, 0, \phi_2)\,. \tag{4}$$
>
> Note that the coordinate on the left-hand and right-hand sides refers to the same point, the north pole: $(r_0, 0, \phi_1) = (r_0, 0, \phi_2)$. Equation (4) says that the pressure is multivalued at this point, which contradicts the assumption that p is a function. Therefore, the assumption that $p = p(r, \phi)$ is false.

Dr. Hamilton's response:

Very interesting. I agree with Jackson's math, and we discuss something similar in Acoustics II in connection with cylindrical resonators, when we talk about why there cannot be pure spinning modes $(m, n, N) = (m, 0, 0)$ (see comment at bottom of p. 405 in Blackstock). But I have yet to understand the physics underlying the fact that natural frequencies in a spherical resonator do not depend on the azimuthal mode number m. This is referred to as degeneracy,[3] and while such degeneracy makes sense physically for square membranes and cubic enclosures, the physical significance is elusive for spheres.

Eigenfunctions in quantum mechanics

No matter how hard he tries, Chirag is never able to retain much knowledge. A fool in his own right, he bumbles on from one fiasco to the next, constantly re-learning lessons from years past. Now he reminds himself of the indicial notation used in quantum mechanics, which he often confuses with that of acoustics.

The solution to the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

for a infinite spherical potential

$$V(r) = \begin{cases} 0 & \text{if } r \leqslant a \\ \infty & \text{if } r > a \end{cases} \tag{5}$$

consists of eigenfunctions very similar to equation (1) [1]:[4]

$$\boxed{\psi_{nlm}(r, \theta, \phi) = j_l(\beta_{nl} r/a) e^{im\phi} P_l^m(\cos\theta)}$$

where now[5][6]

$$\begin{aligned} l &= 0, 1, 2, \ldots \infty = \text{orbital quantum number}, \\ n &= 1, 2, 3, \ldots \infty \\ \text{and } |m| &\leqslant l = \text{magnetic quantum number}, \end{aligned}$$

[3]Each eigenfrequency is $2n+1$-fold degenerate, since there are $2n+1$ different values of m for each value of n.

[4]β_{nl} appears in the argument of the cylindrical Bessel function to match the boundary conditions given by (5). The Neumann functions are tossed because they diverge at $r = 0$, and the complex exponential form of the ϕ-dependence is chosen because the eigenfunctions are complex.

[5]Confusingly, the orbital quantum number is sometimes called that "azimuthal quantum number" for historical reasons [1].

[6]n takes on the name, "principal quantum number" when the potential is set to $1/r$, which corresponds to the hydrogen atom.

Three cases arise, similar to the solution to the Helmholtz equation of acoustics in spherical coordinates, only with $n \leftrightarrow l$.

1. $l = 0$, $n = 0, 1, 2, \ldots \infty$, and $m = 0$ describes sound with spherical symmetry. This means the wave function has polar and azimuthal symmetry, i.e., $\psi = \psi(r)$.

2. $l = 0, 1, 2, \ldots \infty$, $n = 0, 1, 2, \ldots \infty$, and $m = 0$ describes sound with azimuthal symmetry, but not polar symmetry, i.e., $\psi = \psi(r, \theta)$.

3. $l = 0, 1, 2, \ldots \infty$, $n = 0, 1, 2, \ldots \infty$, and $|m| \leqslant l$ describes sound without symmetry, i.e., $\psi = \psi(r, \phi, \theta)$.

Again, one cannot have $\psi(r, \phi)$, because this will result in a multivalued wave function.

References

[1] D. J. Griffiths, "Introduction to Quantum Mechanics." Pearson, (2017).

[2] D. T. Blackstock, "Fundamentals of Physical Acoustics." Wiley, (2000).