Use Green's functions to solve the inhomogenous Helmholtz equation

$$\nabla^2 p_\omega + k^2 p_\omega = -f_\omega(\boldsymbol{r}). \tag{1}$$

Since Green's functions $g_\omega(\boldsymbol{r}|\boldsymbol{r}_0)$ are solutions to

$$\nabla^2 g_\omega(\boldsymbol{r}|\boldsymbol{r}_0) + k^2 g_\omega(\boldsymbol{r}|\boldsymbol{r}_0) = -\delta(\boldsymbol{r}-\boldsymbol{r}_0), \tag{2}$$

so are $G_\omega(\boldsymbol{r}_0|\boldsymbol{r}) = g_\omega(\boldsymbol{r}_0|\boldsymbol{r}) + \chi(\boldsymbol{r})$, where $\chi(\boldsymbol{r})$ satisfies the homogeneous Helmholtz equation $\nabla^2\chi + k^2\chi = 0$. Equation (2) for $G_\omega(\boldsymbol{r}_0|\boldsymbol{r})$ reads

$$\nabla^2 G_\omega(\boldsymbol{r}|\boldsymbol{r}_0) + k^2 G_\omega(\boldsymbol{r}|\boldsymbol{r}_0) = -\delta(\boldsymbol{r}-\boldsymbol{r}_0), \tag{3}$$

Equation (3) is multiplied by $p_\omega$ and subtracted from the product of $G_\omega$ and equation (1):

$$G_\omega(\boldsymbol{r}|\boldsymbol{r}_0)\nabla^2 p_\omega - p_\omega\nabla^2 G_\omega(\boldsymbol{r}|\boldsymbol{r}_0) = -f_\omega(\boldsymbol{r})G_\omega(\boldsymbol{r}|\boldsymbol{r}_0) + p_\omega\delta(\boldsymbol{r}-\boldsymbol{r}_0) \tag{4}$$

Now switching the location of the source from $\boldsymbol{r}_0$ to $\boldsymbol{r}$, $f_\omega(\boldsymbol{r}_0) \mapsto f_\omega(\boldsymbol{r})$, so equation (4) becomes

$$G_\omega(\boldsymbol{r}|\boldsymbol{r}_0)\nabla^2 p_\omega - p_\omega\nabla^2 G_\omega(\boldsymbol{r}|\boldsymbol{r}_0) = -f_\omega(\boldsymbol{r}_0)G_\omega(\boldsymbol{r}|\boldsymbol{r}_0) + p_\omega\delta(\boldsymbol{r}-\boldsymbol{r}_0) \tag{5}$$

Further, since $G_\omega$ satisfies reciprocity, $G_\omega(\boldsymbol{r}|\boldsymbol{r}_0) = G_\omega(\boldsymbol{r}_0|\boldsymbol{r})$. Recall also that $\delta(\boldsymbol{r}-\boldsymbol{r}_0) = \delta(\boldsymbol{r}_0-\boldsymbol{r})$. Making these transformations to equation (5) yields

$$G_\omega(\boldsymbol{r}_0|\boldsymbol{r})\nabla^2 p_\omega - p_\omega\nabla^2 G_\omega(\boldsymbol{r}_0|\boldsymbol{r}) = -f_\omega(\boldsymbol{r}_0)G_\omega(\boldsymbol{r}_0|\boldsymbol{r}) + p_\omega\delta(\boldsymbol{r}_0-\boldsymbol{r}) \tag{6}$$

Integrating (6) in the '$_0$' coordinates,

$$\iiint \big\{G_\omega(\boldsymbol{r}_0|\boldsymbol{r})\nabla^2 p_\omega - p_\omega\nabla^2 G_\omega(\boldsymbol{r}_0|\boldsymbol{r})\big\}dv_0 = \iiint \big\{-f_\omega(\boldsymbol{r}_0)G_\omega(\boldsymbol{r}_0|\boldsymbol{r}) + p_\omega\delta(\boldsymbol{r}_0-\boldsymbol{r})\big\}dv_0$$

Applying the sifting property of the delta function on the right-hand-side, writing $G_\omega(\boldsymbol{r}_0|\boldsymbol{r})\nabla^2 p_\omega - p_\omega\nabla^2 G_\omega(\boldsymbol{r}_0|\boldsymbol{r}) = \boldsymbol{\nabla}_0\cdot(G_\omega(\boldsymbol{r}_0|\boldsymbol{r})\boldsymbol{\nabla} p_\omega - p_\omega\boldsymbol{\nabla} G_\omega(\boldsymbol{r}_0|\boldsymbol{r}))$, and solving for $p_\omega(\boldsymbol{r})$,

$$p_\omega(\boldsymbol{r}) = \iiint f_\omega(\boldsymbol{r}_0)G_\omega(\boldsymbol{r}_0|\boldsymbol{r})dv_0 + \iiint \boldsymbol{\nabla}_0\cdot\big\{G_\omega(\boldsymbol{r}_0|\boldsymbol{r})\boldsymbol{\nabla} p_\omega - p\boldsymbol{\nabla} G_\omega(\boldsymbol{r}_0|\boldsymbol{r})\big\}dv_0$$

Utilizing the divergence theorem on the left-hand-side, and writing the gradients as $\frac{\partial}{\partial n_0}$,

$$p_\omega(\boldsymbol{r}) = \iiint f_\omega(\boldsymbol{r}_0)G_\omega(\boldsymbol{r}_0|\boldsymbol{r})dv_0 + \oiint \big\{G_\omega(\boldsymbol{r}_0|\boldsymbol{r})\tfrac{\partial}{\partial n_0}p_\omega - p_\omega\tfrac{\partial}{\partial n_0}G_\omega(\boldsymbol{r}_0|\boldsymbol{r})\big\}dS_0$$

This is the Helmholtz-Kirchoff integral, matching Morse and Ingard's equation (7.1.17).