Coriolis & Centrifugal Forces

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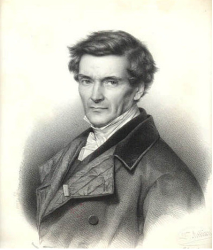


MÉMOIRE

Sur le principe des forces vives dans les mouvemens relatifs des Machines;

PAR M. G. CORIOLIS.

LU À L'ACADÉMIE DES SCIENCES, LE 6 JUIN 1831.

LA détermination du mouvement d'un système de corps liés d'une manière quelconque à des points qui sont entraînés dans l'espace, est une des questions qui intéressent le plus la théorie des machines, particulièrement celle des roues hydrauliques. Jean Bernouilli a traité le mouvement d'un point matériel pesant dans un tube droit tournant horizontalement d'un mouvement uniforme autour d'un de ses points.

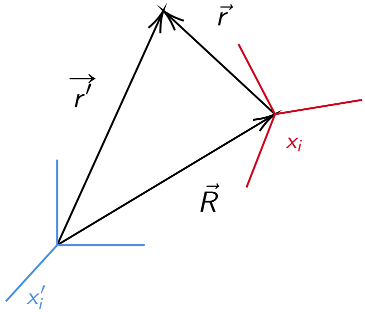
Defining a Rotating Frame

Primed coordinates x_i' refer to the inertial axes.

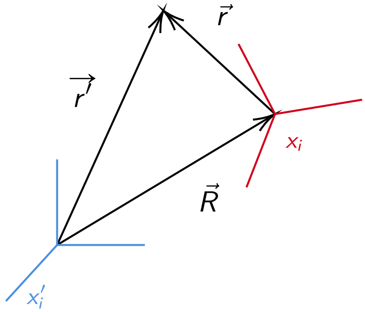
Un-primed coordinates x_i refer to the rotating axes.

The frames are related by

$$\vec{r'} = \vec{R} + \vec{r}$$



$$\vec{r'} = \vec{R} + \vec{r}$$



Each of the quantities in $\vec{r'} = \vec{R} + \vec{r}$ can be measured in either the inertial frame (denoted by subscript $_{\text{Inertial}}$) or the non-inertial frame (denoted by subscript $_{\text{Non-Inertial}}$).

For example, $\vec{R}_{\text{Inertial}} = -\vec{R}_{\text{Non-Inertial}}$

Introducing "Rotation"

If the non-inertial system experiences some infinitesimal rotation $\delta\theta$, then

$$(\mathrm{d}\vec{r})_{\text{Inertial}} = \mathrm{d}\vec{\theta} \times \vec{r}$$

Dividing by $\mathrm{d}t$,

$$\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_{\text{Inertial}} = \frac{\mathrm{d}\vec{\theta}}{\mathrm{d}t} \times \vec{r}$$

Defining $\vec{\omega} \equiv \frac{\mathrm{d}\vec{\theta}}{\mathrm{d}t}$,

$$\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_{\text{Inertial}} = \vec{\omega} \times \vec{r} \tag{1}$$

Equation (1) is in terms of $\vec{r}$ (*not* $\vec{r'}$), so this is *not* the velocity of a rigid body as measured in the inertial frame. Does anyone have a physical interpretation for equation (1)?

Non-Rigid Bodies

In the more general case that the object moves with respect to the non-inertial frame, equation (1) becomes

$$\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_{\text{Inertial}} = \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_{\text{Non-Inertial}} + \vec{\omega} \times \vec{r} \tag{2}$$

Equation (2) is generically true for an arbitrary vector linear in $\vec{r}$. If equation (2) is physically valid, a torque on the object must be calculated to be the same in both frames. Since $\vec{\tau} = I\vec{\alpha} = I\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}$, it suffices to check if $\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}$ is the same in both frames:

$$\begin{aligned}\left(\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}\right)_{\text{Inertial}} &= \left(\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}\right)_{\text{Non-Inertial}} + \vec{\omega} \times \vec{\omega} \\ &= \left(\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}\right)_{\text{Non-Inertial}}\end{aligned}$$

Yay!

Velocity as measured in the inertial frame

We are interested in the velocity as measured in the inertial frame, so we must write $\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_{\text{Inertial}} = \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_{\text{Non-Inertial}} + \vec{\omega} \times \vec{r}$ (equation 2) in terms of $\vec{r'}$. To do this, we differentiate $\vec{r'} = \vec{R} + \vec{r}$ with respect to time:

$$\left(\frac{\mathrm{d}\vec{r'}}{\mathrm{d}t}\right)_{\text{Inertial}} = \left(\frac{\mathrm{d}\vec{R}}{\mathrm{d}t}\right)_{\text{Inertial}} + \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_{\text{Inertial}}$$

Substituting equation (2) into the above gives

$$\left(\frac{\mathrm{d}\vec{r'}}{\mathrm{d}t}\right)_{\text{Inertial}} = \left(\frac{\mathrm{d}\vec{R}}{\mathrm{d}t}\right)_{\text{Non-Inertial}} + \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_{\text{Non-Inertial}} + \vec{\omega} \times \vec{r} \tag{3}$$

More conveniently in terms of velocities,

$$\vec{v}_{\text{Inertial}} = \vec{V} + \vec{v}_{\text{Non-Inertial}} + \vec{\omega} \times \vec{r} \tag{4}$$

Applying Newton's Second Law...

$$\begin{aligned}\vec{F} &= m\frac{\mathrm{d}\,\vec{v}_{\text{Inertial}}}{\mathrm{d}t}\\ &= m\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{V} + \vec{v}_{\text{Non-Inertial}} + \vec{\omega}\times\vec{r}\right)\\ &= m\left(\ddot{\vec{R}} + \frac{\mathrm{d}\,\vec{v}_{\text{Non-Inertial}}}{\mathrm{d}t} + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times\left(\frac{\mathrm{d}\,\vec{r}}{\mathrm{d}t}\right)_{\text{Inertial}}\right)\end{aligned}$$

But $\vec{\omega}\times\left(\frac{\mathrm{d}\,\vec{r}}{\mathrm{d}t}\right)_{\text{Inertial}} = \vec{\omega}\times\left(\frac{\mathrm{d}\,\vec{r}}{\mathrm{d}t}\right)_{\text{Non-Inertial}} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$, so

$$\vec{F} = m\left(\ddot{\vec{R}} + \vec{a}_{\text{Non-Inertial}} + \dot{\vec{\omega}}\times\vec{r} + 2\vec{\omega}\times\vec{v}_{\text{Non-Inertial}} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right)$$

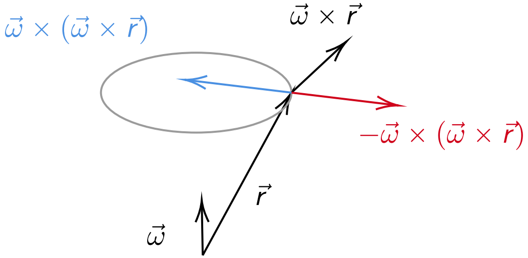
Solving for $m\vec{a}_{\text{Non-Inertial}} \equiv \vec{F}_{\text{effective}}$,

$$\vec{F}_{\text{effective}} = \vec{F} - m\ddot{\vec{R}} - m\dot{\vec{\omega}}\times\vec{r} - 2m\vec{\omega}\times\vec{v}_{\text{Non-Inertial}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

Physical Interpretation

$$\vec{F}_{\text{effective}} = \vec{F} - m\ddot{\vec{R}} - m\dot{\vec{\omega}} \times \vec{r} - 2m\vec{\omega} \times \vec{v}_{\text{Non-Inertial}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

1. $\vec{F}$ is the force on the object measured in the inertial frame.
2. $-m\ddot{R}$ is the translational acceleration of the entire non-inertial frame.
3. $-m\dot{\vec{\omega}} \times \vec{r}$ is the rotational acceleration of the non-inertial frame.
4. $-2m\vec{\omega} \times \vec{v}_{\text{Non-Inertial}} \propto \vec{v}_{\text{Non-Inertial}}$ is the Coriolis force.
5. $-m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is the centrifugal force. It is directed outward.



Mass on a frictionless, flat plane rotating at constant ω

$$\vec{F}_{\text{effective}} = \vec{F} - m\ddot{\vec{R}} - m\dot{\vec{\omega}} \times \vec{r} - 2m\vec{\omega} \times \vec{v}_{\text{Non-Inertial}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad (5)$$

But...

- Since there are no external forces in the inertial frame, $\vec{F} = 0$.
- Since the non-inertial frame has no translational acceleration, $-m\ddot{\vec{R}} = 0$
- Since ω is constant, $-m\dot{\vec{\omega}} \times \vec{r} = 0$

...equation (5) becomes

$$\vec{F}_{\text{effective}} = -2m\vec{\omega} \times \vec{v}_{\text{Non-Inertial}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Dividing by the mass,

$$\vec{a}_{\text{effective}} = -2\vec{\omega} \times \vec{v}_{\text{Non-Inertial}} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad (6)$$

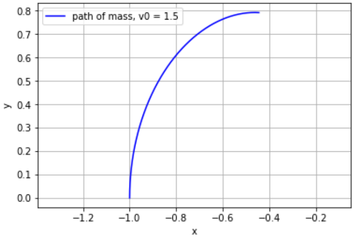
Since $\vec{\omega} = \omega \hat{z}$,

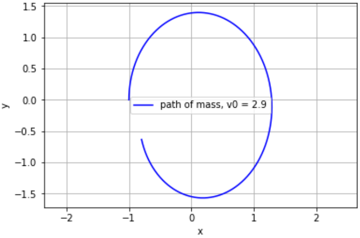
$$\vec{a}_{\text{effective}} = -2\vec{\omega} \times \vec{v}_{\text{Non-Inertial}} - \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

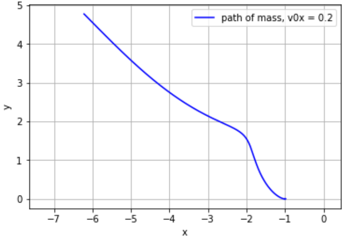
is reduced to two coupled second-order ODEs:

$$\begin{cases} \ddot{r}_x = \omega^2 r_x + 2\omega \dot{r}_y \\ \ddot{r}_y = -\omega^2 r_y - 2\omega \dot{r}_x \end{cases}$$

These equations can be numerically integrated to find $\vec{r}(t) = r_x(t)\hat{x} + r_y(t)\hat{y}$ under given initial conditions.







For those interested, these numerical solutions are publicly available at *https://cag170030.github.io/chirag/physics.html*.

Foucault Pendulum

Pendule de Foucault, Panthéon, Paris, France

$$\vec{F}_{\text{effective}} = \vec{F} - m\ddot{\vec{R}} - m\dot{\vec{\omega}} \times \vec{r} - 2m\vec{\omega} \times \vec{v}_{\text{Non-Inertial}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

- The motion is confined to the x-y plane, so any vertical motion can be neglected ($\dot{z} \sim 0$, for example)
- The only external force in the inertial frame is tension $\vec{T}$
- The Earth is not appreciably spinning up or down ($\dot{\vec{\omega}} \sim 0$).
- The centrifugal term $m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ does not change the angular direction of the motion, but this is primarily what the Foucault pendulum measures, so it will be neglected.

The only external force in the inertial frame is tension $\vec{T}$, and Equation (5) becomes

$$\vec{F}_{\text{effective}} = m\vec{g} + \vec{T} - 2m\vec{\omega} \times \vec{v}_{\text{Non-Inertial}}$$

or dividing by m,

$$\vec{a}_{\text{effective}} = \vec{g} + \frac{\vec{T}}{m} - 2\vec{\omega} \times \vec{v}_{\text{Non-Inertial}}$$

We want to solve

$$\vec{a}_{\text{effective}} = \vec{g} + \frac{\vec{T}}{m} - 2\vec{\omega} \times \vec{v}_{\text{Non-Inertial}} \tag{7}$$

for $x(t)$ and $y(t)$.

Since the pendulum is oscillating through a small angle,

$$\begin{cases} T_x \simeq -T\frac{x}{l} \simeq -\frac{mgx}{l} \\ T_y \simeq -T\frac{y}{l} \simeq -\frac{mgy}{l} \\ T_z \simeq T \simeq mg \end{cases} \tag{8}$$

The x, y, and z components of angular velocity $\vec{\omega}$ depends on the latitude λ:

$$\begin{cases} \omega_x = -\omega \cos \lambda \\ \omega_y = 0 \\ \omega_z = \omega \sin \lambda \end{cases} \tag{9}$$

Making substitutions (8) and (9) and defining $\alpha^2 \equiv \frac{g}{l}$, equation (7) becomes

$$\begin{cases} \ddot{x} + \alpha^2 x \simeq 2\omega_z \dot{y} \\ \ddot{y} + \alpha^2 y \simeq -2\omega_z \dot{x} \end{cases} \tag{10}$$

To solve (10), add the first to i times the second:

$$(\ddot{x} + i\ddot{y}) + \alpha^2(x + iy) \simeq -2\omega_z(i\dot{x} - \dot{y}) = -2i\omega_z(\dot{x} + i\dot{y})$$

Defining $q \simeq x + iy$,

$$\ddot{q} + 2i\omega_z \dot{q} + \alpha^2 q \simeq 0 \tag{11}$$

We immediately read off the solution:

$$q(t) = e^{i\omega_z t}\left(Ae^{\sqrt{-\omega_z^2 - \alpha^2}\,t} + Be^{-\sqrt{-\omega_z^2 - \alpha^2}\,t}\right) \tag{12}$$

Note that if the Earth were not rotating, equation (11) would become

$$\ddot{q}' + \alpha^2 q' \simeq 0$$

whose solution is

$$q'(t) = x'(t) + iy'(t) = Ae^{i\alpha t} + Be^{-i\alpha t} \tag{13}$$

where we recognize that α is the angular frequency of the pendulum. Since $\alpha \gg \omega_z$, solution (12) becomes

$$q(t) \simeq e^{i\omega_z t}(Ae^{i\alpha t} + Be^{-i\alpha t}) \tag{14}$$

Comparing equations (13) and (14) we see that

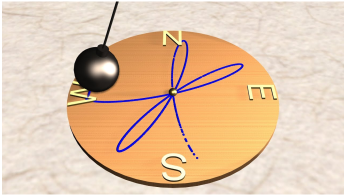
$$q(t) = q'(t)e^{-i\omega_z t}$$

Substituting back $q \simeq x + iy$ and $q' \simeq x' + iy'$,

$$
\begin{aligned}
q(t) &= q'(t)e^{-i\omega_z t} \\
x(t) + iy(t) &= (x'(t) + iy'(t))e^{-i\omega_z t} \\
&= (x'(t) + iy'(t))(\cos\omega_z t - i\sin\omega_z t) \\
&= (x'\cos\omega_z t + y'\sin\omega_z t) + i(-x'\sin\omega_z t + y'\cos\omega_z t)
\end{aligned}
$$

Corresponding the real and imaginary parts,

$$
\begin{cases}
x(t) = x'\cos\omega_z t + y'\sin\omega_z t \\
y(t) = -x'\sin\omega_z t + y'\cos\omega_z t
\end{cases}
\tag{15}
$$



Conclusion & Questions

References

1. Stephen T. Thornton & Jerry B. Marion. *Classical Dynamics of Particles and Systems*, 5th ed. Cengage Learning. 2008.
2. Foucault pendulum animation from http://physics-animations.com/
3. All other images are Public Domain.