Pappus's Theorem

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> Geometry postulates the solution of these problems from mechanics and teaches the use of the problems thus solved. And geometry can boast that with so few principles obtained from other fields; it can do so much.
>
> –Isaac Newton, 1687
>
> as quoted in *Modern Classical Physics* by Kip Thorne & Roger Blandford

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1 Introduction

1.1 Topic of Choice

In this presentation, we will discuss Pappus's theorem from first principles. It is our intention to provide a sufficient background to the theorem so that a high-school student would be able to follow the proof. Our peers of MATH 3321 can skip to section 5 as they have sufficient background to follow the proof of Menelaus's theorem, which was not covered in MATH 3321 at the time of this writing and is invoked in the proof Pappus's Theorem.

1.2 About the Geometers

Chirag and Tucker are upperclassmen studying physics at UTD. Note that both geometers are responsible for the content and presentation of the project and have a high level of understanding of Pappus's theorem and its proof; readers who have questions—geometric or organizational—should not hesitate to contact either geometer:

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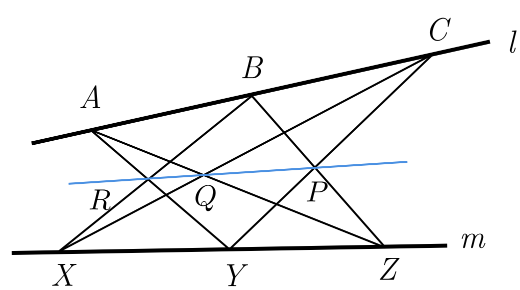
1.3 Statement of the Theorem

Suppose that points A, B, and C lie on some line l and that points X, Y, and Z lie on line m, where the six points are distinct and the two lines are also distinct. Assume that lines BZ and CY meet at P, lines AZ and CX meet at Q, and lines AY and BX meet at R. Then points P, Q, and R are collinear.[1]



2 Appeal of the Theorem

2.1 Comprehensiveness

Pappus's theorem ties together many of the important concepts we have covered in MATH 3321. We hope its comprehensiveness helps other geometers of our MATH 3321 section by reviewing the major themes of the course. An in-depth understanding of our presentation, which includes collinearity, triangles, Euclid's fifth postulate, and algebraic manipulation of geometric ratios, will help us and our peers prepare for the final exam.

[1] Isaacs, Theorem 4.16

The very fabric of this theorem rests on the integrity of Euclid's fifth postulate: we must be careful to not let one of the six points $A, B, C, X, Y,$ and Z described in the hypothesis lie at the intersection point of the two lines l and m as points $P, Q,$ and R would then not be distinct, giving a trivial claim as two points are (by Euclid's first postulate) automatically collinear.

2.2 Uniqueness

Pappus's theorem "is different in flavor from almost everything else in this book," as Isaacs describes.[2] It is a **nonmetric** result. That is, the notion of length, angular size, etc. (i.e., elements of **metric** geometry—based on measurement) are not relevant to Pappus's theorem, where there is nothing to be measured. Isaacs is quick to note that Pappus's theorem–for this reason–belongs to the field of **nonmetric geometry**.[3]

2.3 Discussion on Projective Geometry

Isaacs notes that the theorem's independence from the notion of "points, lines, [and] incidence"[4] casts it in the field of **projective geometry**. This contrasts the field of **affine geometry**, which relies on measurement. He describes projective geometry by providing an interesting physical analogy that we will paraphrase in our presentation. He asks the reader to imagine drawing a diagram fitting the hypothesis of Pappus's theorem with "opaque ink on a sheet of glass," and that a "point source of light causes the figure to cast a shadow onto a planar screen. Since this projection from a point carries points to points and lines to lines, and it preserves incidence, we see that the shadow of a diagram for Pappus' theorem is again a diagram for Pappus' theorem." Isaacs continues by casually defining projective geometry: "In a very rough sense, projective geometry is that part of ordinary (Euclidean) geometry where the shadows of diagrams illustrate the relevant information in the original diagrams." Isaacs notes that other important theorems of geometry (*pons asinorum*, for example) do not belong to projective geometry (since the projection of an isosceles triangle may not also be isosceles).

2.4 Aesthetic and Algebraic Beauty

Pappus's Theorem is actually rather hard to believe, as it is generally true for six generic, distinct points on two randomly oriented lines l and m. We find that theorems of such a general hypothesis yet such a precise conclusion (i.e., collinearity) possess great aesthetic beauty. The use of ratios and their algebraic manipulation only heightens the beauty of the proof.

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[2]page 149

[3]The distinction between metric and nonmetric geometry is deeply physical; Isaacs notes that "no result involving circles could be called nonmetric because a circle is defined as the locus of points of some fixed distance from a given point."

[4]151

3 Brief History of Pappus

Pappus was a Greek mathematician of Alexandria. Sadly, not much of Pappus's life was documented; nearly all we know about him comes from hints of autobiographical information from his works.[5]

Pappus was among the last of the Greek mathematicians active in the late Roman Empire. Historians suggest Pappus was active in ∼4th century C.E. since Pappus quoted Ptolemy (active in the 2nd century C.E.) but was quoted by Proclus (active in the 5th century C.E.).[6]

Interestingly, Pappus was one of the few active mathematicians in a time of relatively little mathematical progress. Much of his work was focused on history and cataloguing previous results. He is known most famously for his *Synagoge*,[7] a compendium of Ancient Greek mathematics. Scholars who have studied the tone and style of his writing have concluded that Pappus was likely a teacher. Pappus focused heavily on polygons, polyhedra, spirals, mechanics, and clever word problems. He was also an astronomer and predicted solar eclipses.[8]

4 Prerequisite Information

4.1 Area of a Triangle

The area A of a triangle of base b and height h is given by

$$A = \frac{1}{2}bh$$

We use I Martin Isaacs's convention of denoting areas using K.

This follows immediately from the definition of area. A rectangle of length l and height h is defined to have an area that equals the product of its length and height: $A = lw$. Any rectangle can be divided along a diagonal into two congruent triangles. Since congruent triangles have equal areas, the area of the rectangle is twice the area of one of the triangles. That is, $A_\Delta = \frac{1}{2}lw \equiv \frac{1}{2}bh$.

4.2 Definition of Cevian

A ***cevian*** *is a line dropped down from a vertex of a triangle to any point on the opposite side. Note that the altitude, median, and angle bisectors are all special cases of cevians.*

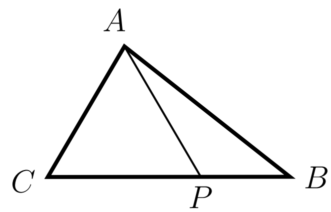
For example, in ΔABC below, line segment AP is a cevian.

[5] Pierre Dedron, J. Itard (1959) Mathematics And Mathematicians, Vol. 1, p.149 (trans. Judith V. Field) (Transworld Student Library, 1974)

[6] Thomas Little Heath (1911). "Porism" . In Chisholm, Hugh (ed.). Encyclopædia Britannica. 22 (11th ed.). Cambridge University Press. pp. 102–103.

[7] "Collection"

[8] Alexander Raymond *Pappus of Alexandria.*



4.3 Definition of Collinearity

Pappus's theorem is a statement about collinearity, so it is important that a semi-formal definition of "line" and "collinear" are provided:

In Euclidean space, a line is a the set of points such that the angle between any point and any other point is $180°$. *Points are* ***collinear*** *if they lie on the same line.*

5 Menelaus's theorem

5.1 Statement

Given ΔABC*, let points P, Q, and R lie on lines BC, CA, and AB, respectively, and assume that none of these points is a vertex of the triangle. Then*[9] *P, Q, and R are collinear if and only if an even number of them*[10] *lie on segments BC, CA, and AB and*

$$\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = 1$$

Since we only need the $\implies$ part of Menelaus's theorem, that is the only part we seek to prove.

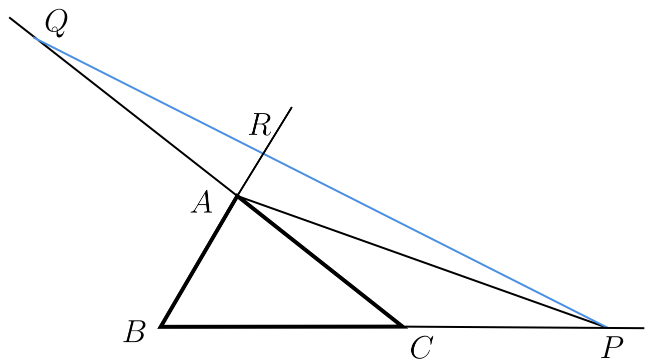
5.2 Proof

Suppose P, Q, and R are collinear (denoted by the blue lines below). We have displayed both possible general configurations below:

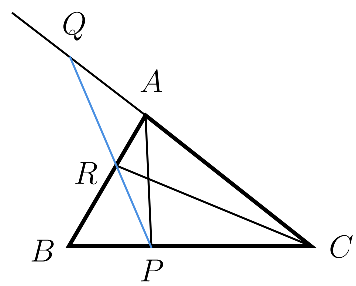
Either zero points are on the sides of the triangle...

[9] Isaacs, Theorem 4.14

[10] Of course, "an even number of them" refers to either zero or two points. The reason Isaacs uses this wording is to contrast the hypothesis of Menelaus's theorem with the hypothesis of Ceva's theorem (which makes a comment on when the number of interior Cevians is odd



...or two points are on the sides of the triangle:



Note that both cevians AP and CR are drawn in the figures above. Since BP and PC are bases of ΔBPR and ΔCPR, and that both of these triangles have the same height. By our discussion in section 4.1, we know that the the ratio of the bases are

$$\frac{BP}{PC} = \frac{K_{BPR}}{K_{CPR}} \tag{1}$$

Similarly, the ratio of AR to RB is the ratio of the areas of triangles ΔAPR and ΔBPR:

$$\frac{AR}{RB} = \frac{K_{APR}}{K_{BPR}} \tag{2}$$

Again, by relating triangles of the same height but different bases,

$$\frac{CQ}{QA} = \frac{K_{CQP}}{K_{AQP}} = \frac{K_{CQR}}{K_{AQR}} \tag{3}$$

Now note that generally if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then[11]

$$\frac{a}{b} = \frac{c-e}{d-f}$$

Applying this principle to (3),

$$\frac{CQ}{QA} = \frac{K_{CQP} - K_{CQR}}{K_{AQP} - K_{AQR}} \tag{4}$$

Now refer to the figures to see that $K_{CQP} = K_{CQR} + K_{CPR}$ and $K_{AQP} = K_{AQR} + K_{APR}$. So (4) becomes

[11]Isaacs calls this the "subtraction principle for ratios."

$$\frac{CQ}{QA} = \frac{K_{CQP} - K_{CQR}}{K_{AQP} - K_{AQR}} = \frac{K_{CPR}}{K_{APR}} \tag{5}$$

Now multiplying equations (2), (1), and (5) (in that order to match the statement of the theorem):

$$\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = \frac{K_{APR}}{K_{BPR}}\frac{K_{BPR}}{K_{CPR}}\frac{K_{CPR}}{K_{APR}} = 1 \tag{6}$$

Thus the $\implies$ part of the Menelaus's theorem (the part we need) is proved.

6 Proof of Pappus's theorem

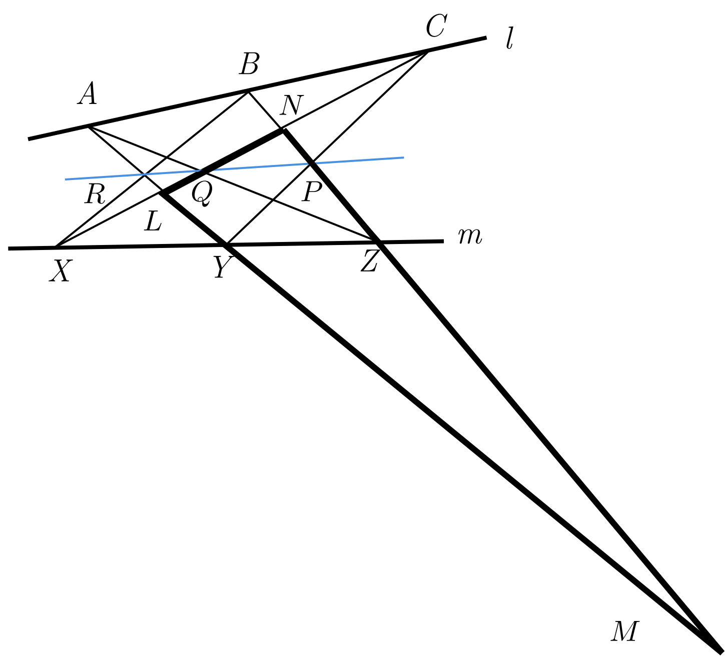
We now have what we need to straightforwardly prove Pappus's theorem. Let us add a triangle to the figure displayed in 1.3.

Call the intersection of lines AY and XC point L.

Call the intersection of BZ and AY point M

Call the intersection of lines XC and BZ point N

The triangle formed by the vertices L, M, and N is drawn below in **bold**. It is this ΔLMN to which we will apply Menelaus's theorem (section 5) with the intention of showing that P, Q, and R are collinear.



Menelaus's theorem suggests that we should compute the Cevian product

$$\frac{LR}{RM}\frac{MP}{PN}\frac{NQ}{QL} \tag{7}$$

and show that this equals 1.

Since (by hypothesis) A, B, and C are collinear and respectively lie on LM, MN, and NL, Menelaus's theorem says

$$\frac{LA}{AM}\frac{MB}{BN}\frac{NC}{CL} = 1 \tag{8}$$

Also, since X, Y, and Z are collinear,

$$\frac{LY}{YM}\frac{MZ}{ZN}\frac{NX}{XL} = 1 \tag{9}$$

Note than there are quite a few triples of collinear points, all giving Cevian products equal to 1. Since R, B, and X are by hypothesis collinear, for example,

$$\frac{LR}{RM}\frac{MB}{BN}\frac{NX}{XL} = 1 \tag{10}$$

...and since A, Q, and Z are collinear,

$$\frac{LA}{AM}\frac{MZ}{ZN}\frac{NQ}{QL} = 1 \tag{11}$$

...and finally, since since C, P, and Y are collinear,

$$\frac{LY}{YM}\frac{MP}{PN}\frac{NC}{CL} = 1 \tag{12}$$

Here comes the algebraic convenience of ratios equaling 1. Since $1 \times 1 = 1$, we are at liberty to multiply and equate whichever ratios we wish. Let's multiply equations giving

$$\begin{aligned}\frac{LR}{RM}\frac{MB}{BN}\frac{NX}{XL} \times \frac{LA}{AM}\frac{MZ}{ZN}\frac{NQ}{QL} \times \frac{LY}{YM}\frac{MP}{PN}\frac{NC}{CL} &= 1 \\ &= \frac{LA}{AM}\frac{MB}{BN}\frac{NC}{CL} \times \frac{LY}{YM}\frac{MZ}{ZN}\frac{NX}{XL}\end{aligned}$$

But all six fractions on the right-hand-side also appear on the left-hand-side, cancelling to give

$$\frac{LR}{RM}\frac{MP}{PN}\frac{NQ}{QL} = 1$$

This is exactly what we wanted: by Menelaus's theorem, we have shown that points R, P, and Q are collinear.

QED

7 Closing remarks & acknowledgments

We hope that this presentation of Pappus's theorem has been clear, enjoyable, and comprehensive. We urge listeners to ask questions via the email addresses provided or using the YouTube comments section.

This project was fulfilling as it served as an intersection between deductive reasoning, geometric understanding, algebraic manipulation, and physical intuition, topics that are "taken for granted" in many college math and physics courses.

As students who are motivated to pursue physics at a higher level upon graduation from UTD, Chirag and Tucker's successful completion of this project has helped them add to their portfolios and has demonstrated their ability to work under a supervisor along with the guidance of peers. This project has also strengthened their LaTeX, After Effects, and general presentation skills, which will be central to their future endeavors. Chirag would like to admit that he abandoned his original plans of learning the documentation for the enticing `tkz-euclide` package and resorted to using `mathcha` upon the suggestion of TA Scott Goodson.

We thank Dr. Akbar, who has taught us the prerequisite geometry to understand this project. Dr. Akbar helped us narrow our list of topics from six (all of which were quite interesting and worthy of study) and kept us on track to completing the project on time.

We also thank I Martin Isaacs for his enjoyable text, *Geometry for College Students*.[12]

[12] I Martin Isaacs, *Geometry for College Students*. American Mathematical Society. 2001.