Finite element replication of acoustic Dirac-like cone and double zero refractive index

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Introduction

- Theoretical interest: can a material with $\rho = 0$ and $\chi = 0$ be achieved?
- Analogous to a phenomenon studied in condensed matter physics
- Overcome the diffraction limit in acoustics experiments and applications
- Collimate sound in the linear regime

Example

- One limitation is that the DZ AMM operates at only one frequency
- Fortunately, many acoustical applications are realized in the narrow band, including transformation acoustics, wavefront and dispersion engineering, phase matching, ultrasound medical imaging, and underwater communication.
- An example of a DZ AMM is a 2D acoustic lens that refracts sound based on the AMM's macroscopic geometry.

Constraints on χ and ρ

- The phase speed of a material is given by $c_{\text{phase}} = \pm(\chi\rho)^{-1/2}$, where χ is the compressibility and ρ is the density
- Engineering materials that explore the limits of c_{phase} thus requires knowledge of these parameters.

$$p(\boldsymbol{x}, t) = p \exp j(\omega t - \boldsymbol{k} \cdot \boldsymbol{x}) \tag{1}$$

$$\frac{|p|^2}{2\left|c_{\text{phase}}\right|^2} \left(\frac{\omega}{\rho}\right) > 0 \tag{2}$$

- Since $|p|^2$, $\left|c_{\text{phase}}\right|$, and ω are positive quantities, $\rho > 0$
- Since $c_{\text{phase}} = (\chi\rho)^{-1/2}$ must be a real quantity, $\chi > 0$, too.
- Further noting that $c_{\text{phase}} \to \infty$ as $\chi\rho$ vanishes, $\chi, \rho \to 0^+$

Another possibility...

- χ (ρ) vanishes while ρ (χ) is finite

For plane progressive waves, $Z = \rho c$. Substituting $c = (\chi\rho)^{-1/2}$,

$$Z = \sqrt{\frac{\rho}{\chi}}$$

- Impedance is infinite (zero) when χ (ρ) vanishes while ρ (χ) is finite

$$\begin{aligned} T &= \frac{2Z}{Z + Z_{\mathrm{wg}}} \\ &= \begin{cases} 2, & Z \to \infty \implies R = -1 \\ 0, & Z \to 0 \implies R = 1 \end{cases} \end{aligned}$$

- So for single-zero metamaterials, the incident sound is totally reflected, either out-of-phase (when χ vanishes and ρ is finite) or in-phase (when χ is finite and ρ vanishes).

Inspiration from condensed matter physics

$$\mu\epsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \nabla^2 \boldsymbol{E} = 0 \tag{3}$$

Using Hamilton's formulation,

$$\frac{1}{m_{\beta\alpha}} = \frac{1}{\hbar^2}\frac{\partial^2 E(q)}{\partial \boldsymbol{q}_\beta \partial \boldsymbol{q}_\alpha} \tag{4}$$

- For the dispersion relation $E(q)$ to be linear, $\frac{\partial E(q)}{\partial \boldsymbol{q}}$ is a constant, so the effective mass goes to 0.
- Relativistic constraint of infinite phase speeds
- Quantum mechanical constraint of $\hbar \neq 0$
- Photonic system featuring linear dispersions is governed by a massless, relativistic, and quantum mechanical equation

Massless Dirac Equation

To first order,

$$\begin{pmatrix} 0 & -jv_{\text{group}}\left(\frac{\partial}{\partial x} - j\frac{\partial}{\partial y}\right) \\ -jv_{\text{group}}\left(\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right) & 0 \end{pmatrix} \begin{pmatrix} E_{z,1} \\ E_{z,2} \end{pmatrix} = (\omega - \omega_D) \begin{pmatrix} E_{z,1} \\ E_{z,2} \end{pmatrix} \tag{5}$$

The eigenvalues are

$$\omega - \omega_D = \pm v_{\text{group}} k(\omega) + \mathcal{O}(k^2) \tag{6}$$

Comparison between condensed matter physics and acoustics

	Condensed matter physics	**Acoustics**
Classical wave equation	$\mu\epsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \nabla^2 \boldsymbol{E} = 0$	$\chi\rho\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0$
Double-zero parameters	μ, ϵ	χ, ρ
Dispersion relation	$E(q)$	$\omega(k)$
Number of eigenvalues	2	3
Effectively massless	✓	✓
Relativistic	✓	✗
Quantum mechanical	✓	✗

Multiple Scattering Theory

From Mie scattering...

$$\begin{pmatrix} i\left(\frac{\partial A_1}{\partial \omega}(\omega-\omega_D)+B\delta k^2\right) & -C_1\delta k e^{i\phi_k} & C_2\delta k^2 \\ C_1\delta k e^{i\phi_k} & i\left(\frac{\partial A_1}{\partial \omega}(\omega-\omega_D)+B(\omega)\delta k^2\right) & -C_1\delta k e^{i\phi_k} \\ -C_2^*\delta k^2 & C_1\delta k e^{-i\phi_k} & i\left(\frac{\partial A_1}{\partial \omega}(\omega-\omega_D)+B\delta k^2\right) \end{pmatrix} \begin{pmatrix} b_{-1} \\ b_0 \\ b_1 \end{pmatrix} = \mathbf{0} \tag{7}$$

The characteristic equation is

$$-\left(\frac{\partial A_1}{\partial \omega}\delta\omega + B\delta k^2\right)^2 \left(\frac{\partial A_0}{\partial \omega}\delta\omega + B\delta k^2\right) +$$

$$+2\left(\frac{\partial A_1}{\partial \omega} + B\delta k^2\right) C_1^2\delta k^2 + \left(\frac{\partial A_1}{\partial \omega}\delta\omega + B\delta k^2\right)|C_2|^2\,\delta k^4 - 2\ \mathrm{Im}\ (C_2^* e^2 i\phi_k) C_1^2 \delta k^4 = 0$$

Solving the cubic characteristic equation…

$$\omega_1 - \omega_D = 0 + \mathcal{O}(\delta k^2)$$
$$\omega_{2,3} - \omega_D = \pm v_{\text{group}} \delta k + \mathcal{O}(\delta k^2)$$

- Dispersion corresponding to the dipolar modes is of the same form as equation (6)
- Accidental degeneracy of the monopolar and dipolar modes forces these dispersions to be linear and cross at a point

Dimensions of the double-zero acoustic metamaterial

Parameter	Dimension (mm)
Diameter of cylindrical scatterers	16
Waveguide thickness	10
Height of cylindrical scatterers	14.5
Square unit cell side length	30

- The required x, y dimensions can be found from Mie scattering equations.
- The z dimensions are fairly arbitrary, per some constraints.

Constraints on z Dimensions

The thickness of the waveguide and height of the cylindrical scatterers is arbitrary, as long as the chosen thickness and height of the scatterers are such that the operating frequency lies within only one waveguide mode.

$$f_{\text{cut-on, 1}} = \frac{c_0}{2h} = \begin{cases} 17150 \text{ Hz}, & h = 10 \text{ mm} \\ 12250 \text{ Hz}, & h = 14 \text{ mm} \end{cases}$$

$$f_{\text{cut-on, 2}} = \frac{c_0}{h} = \begin{cases} 34300 \text{ Hz}, & h = 10 \text{ mm} \\ 24500 \text{ Hz}, & h = 14 \text{ mm} \end{cases}$$

The FEM sweeps 0 - 20 kHz and therefore operates in the first waveguide mode.

Building the geometry...

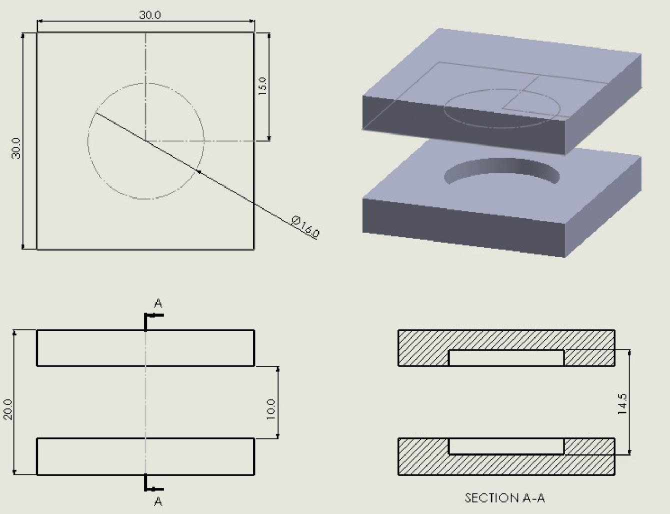
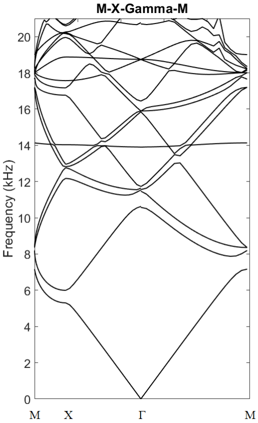
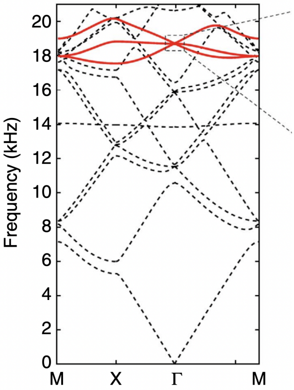


Figure: The geometry of the unit cell, created in SolidWorks and then imported to COMSOL

Band diagram



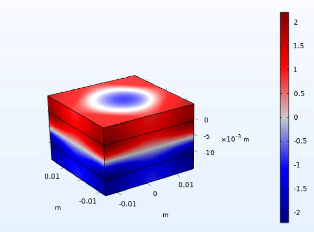
(a) Our band diagram



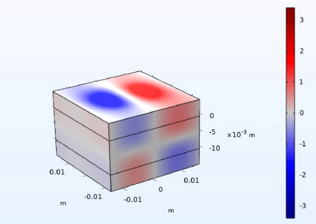
(b) Fig 2a in Dubois et al.

Figure: Comparison of our band diagram to that of Dubois et al.

Monopole and dipole degeneracy



(a) Monopolar pressure field



(b) Dipolar pressure field

Figure: Pressure fields calculated in COMSOL at the Dirac point of the degenerate monopolar and dipolar modes of the lowest-order waveguide mode. These figures match Dubois et al.'s figures 2c and 2d

Conclusion

- DZ AMMs have infinite phase speeds and are impedance-matched
- The FEM successfully illustrated the Dirac-like cone for $1.87 * 10^4$ Hz at the BZ center.
- The Dirac-like cone is analogous to the Dirac cone, which is predicted by the massless, relativistic, quantum mechanical equation.
- DZ technology have many potential narrow-band applications.

Supplemental Notes & References

Questions/Discussion