Differential operators in curvilinear coordinates **February 6, 2026**

Chapters 10 and 11 of Blackstock's *Fundamental of Physical Acoustics* begin with the acoustic wave equation in spherical and cylindrical coordinates, respectively [1, pp. 335, 386]. But where do the expressions of the Laplacian in spherical and cylindrical coordinates come from? Let us derive $\boldsymbol{\nabla}$, $\boldsymbol{\nabla}\cdot$, and ∇^2 in cylindrical and spherical coordinates.* This worksheet is based on Secs. 10.8 and 10.9 of Ref. 2. For another treatment, see Ref. 3.

Example Cartesian coordinates (x, y, z) are related to cylindrical coordinates (r, θ, z) by

$$x = r\cos\theta \tag{1a}$$

$$y = r\sin\theta \tag{1b}$$

$$z = z. \tag{1c}$$

Use Eqs. (1) to calculate the differentials dx, dy, and dz in cylindrical coordinates:

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \tag{2a}$$

$$dy = \tag{2b}$$

$$dz = \tag{2c}$$

Use Eqs. (2) to calculate

$$ds^2 = d\mathbf{s}\cdot d\mathbf{s} = dx^2 + dy^2 + dz^2 = \tag{3}$$

*I was originally planning on also deriving $\boldsymbol{\nabla}\times$, but we will not have time for it. I can send the derivation via email to those who are interested.

Since the coordinates are orthogonal, the vector $d\mathbf{s}$ can be identified from Eq. (3) as

$$d\mathbf{s} = \mathbf{e}_r dr + \mathbf{e}_\theta r d\theta + \mathbf{e}_z dz, \tag{4}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ are the cylindrical unit vectors. Insert Eqs. (2) into

$$d\mathbf{s} = \mathbf{e}_x dx + \mathbf{e}_y dy + \mathbf{e}_z dz, \tag{5}$$

and compare the result to Eq. (4) to show that

$$\mathbf{e}_r = \mathbf{e}_x \cos\theta + \mathbf{e}_y \sin\theta \equiv \mathbf{a}_r \tag{6a}$$

$$r\mathbf{e}_\theta = -\mathbf{e}_x r \sin\theta + \mathbf{e}_y r \cos\theta \equiv \mathbf{a}_\theta \tag{6b}$$

$$\mathbf{e}_z = \mathbf{e}_z \equiv \mathbf{a}_z. \tag{6c}$$

Formulas for calculating $d\mathbf{s}$ and the basis vectors $\mathbf{a}_n$

Given three coordinates x_1, x_2, and x_3, the general form of Eq. (4) is

$$\begin{aligned} d\mathbf{s} &= \mathbf{e}_x dx + \mathbf{e}_y dy + \mathbf{e}_z dz \\ &= \mathbf{e}_x \frac{\partial x}{\partial x_n} dx_n + \mathbf{e}_y \frac{\partial y}{\partial x_n} dx_n + \mathbf{e}_x \frac{\partial z}{\partial x_n} dx_n \\ &= \mathbf{a}_1 dx_1 + \mathbf{a}_2 dx_2 + \mathbf{a}_3 dx_3 = \mathbf{a}_n dx_n \end{aligned} \tag{7}$$

where the general form of Eqs. (6) is

$$\mathbf{a}_n = \mathbf{e}_x \frac{\partial x}{\partial x_n} + \mathbf{e}_y \frac{\partial y}{\partial x_n} + \mathbf{e}_z \frac{\partial z}{\partial x_n} \tag{8}$$

Define $g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j$. The general form of Eq. (3) is then

$$ds^2 = \begin{pmatrix} dx_1 & dx_2 & dx_3 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}. \tag{9}$$

In index notation, Eq. (9) reads $ds^2 = g_{ij} dx_i dx_j$, and if the coordinate system is orthogonal, Eqs. (7) and (9) become

$$d\mathbf{s} = \mathbf{e}_1 h_1 dx_1 + \mathbf{e}_2 h_2 dx_2 + \mathbf{e}_3 h_3 dx_3 \tag{10}$$

and

$$ds^2 = \begin{pmatrix} dx_1 & dx_2 & dx_3 \end{pmatrix} \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}, \tag{11}$$

respectively, where $h_1 = \sqrt{g_{11}}$, $h_2 = \sqrt{g_{22}}$, and $h_3 = \sqrt{g_{33}}$ are the “scale factors.”

Example Given Eqs. (6), calculate the scale factors in cylindrical coordinates.

Example Use Eqs. (7)–(11) to obtain $d\mathbf{s}$, $\mathbf{a}_n$, $\mathbf{e}_n$, g_{ij}, and ds^2 for spherical coordinates, which are related to Cartesian coordinates by

$$x = r\sin\theta\cos\phi \tag{12}$$
$$y = r\sin\theta\sin\phi \tag{13}$$
$$z = r\cos\theta. \tag{14}$$

Answers:

$$\begin{aligned}
d\mathbf{s} &= \mathbf{a}_r dr + \mathbf{a}_\theta d\theta + \mathbf{a}_\phi d\phi \\
\mathbf{a}_r &= \sin\theta\cos\phi\mathbf{e}_x + \sin\theta\sin\phi\mathbf{e}_y + \cos\theta\mathbf{e}_z = \mathbf{e}_r \\
\mathbf{a}_\theta &= r\cos\theta\cos\phi\mathbf{e}_x + r\cos\theta\sin\phi\mathbf{e}_y - r\sin\theta\mathbf{e}_z = r\mathbf{e}_\theta \\
\mathbf{a}_\phi &= -r\sin\theta\sin\phi\mathbf{e}_x + r\sin\theta\cos\phi\mathbf{e}_y = r\sin\theta\mathbf{e}_\phi \\
g_{11} &= 1, \quad g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta, \quad g_{ij} = 0 \text{ for } i \neq j \\
ds^2 &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2
\end{aligned}$$

Example Qualitatively, a partial derivative involves letting one variable vary while keeping the other variables constant. Consider cylindrical coordinates (r, θ, z). From Eq. (4), a variation in r with θ and z held constant means that $ds = dr$. Similarly, holding r and z constant shows that $ds = r d\theta$, and holding the r and θ constant shows that $ds = dz$. The r component of the gradient of a function u in cylindrical coordinates therefore equals $\mathbf{e}_r \partial u/\partial r$. The θ component equals $\mathbf{e}_\theta (1/r)\partial u/\partial\theta$, and the z component equals $\mathbf{e}_z \partial u/\partial z$, i.e.,

$$\boldsymbol{\nabla} u = \mathbf{e}_r \frac{\partial u}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial u}{\partial \theta} + \mathbf{e}_z \frac{\partial u}{\partial z}. \tag{15}$$

Formula for gradient in general orthogonal coordinates

Consider a function $u = u(x_1, x_2, x_3)$ of the general orthogonal coordinates x_1, x_2, and x_3. The magnitude of the component of $\boldsymbol{\nabla} u$ in the direction of x_1 (holding x_2 and x_3 constant) is du/ds, where [from Eq. (10)] $ds = h_1 dx_1$. Thus $du/ds = (1/h_1)\partial u/\partial x_1$. Similarly, the components of $\boldsymbol{\nabla} u$ in the x_2 and x_3 directions are $du/ds = (1/h_2)\partial u/\partial x_2$ and $du/ds = (1/h_3)\partial u/\partial x_3$, respectively. The gradient of $\mathbf{u}$ in general orthogonal coordinates is therefore

$$\begin{aligned}\boldsymbol{\nabla} u &= \mathbf{e}_1 \frac{1}{h_1}\frac{\partial u}{\partial x_1} + \mathbf{e}_2 \frac{1}{h_2}\frac{\partial u}{\partial x_2} + \mathbf{e}_3 \frac{1}{h_3}\frac{\partial u}{\partial x_3} \\ &= \sum_{i=1}^{3} \mathbf{e}_i \frac{1}{h_i}\frac{\partial u}{\partial x_i}. \end{aligned} \tag{16}$$

Example Use Eq. (16) to write the gradient of a function $u = u(r, \theta, \phi)$ in spherical coordinates (r, θ, ϕ). Note from the previous page that $h_1 = \sqrt{g_{11}} = 1$, $h_2 = \sqrt{g_{22}} = r$, and $h_3 = \sqrt{g_{33}} = r \sin\theta$.

Answer:

$$\boldsymbol{\nabla} u = \mathbf{e}_r \frac{\partial u}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial u}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin\theta}\frac{\partial u}{\partial \phi}$$

A prerequisite proof We need to first prove that

$$\nabla \cdot \left(\frac{\mathbf{e}_3}{h_1 h_2}\right) = 0, \quad \nabla \cdot \left(\frac{\mathbf{e}_2}{h_1 h_3}\right) = 0, \quad \nabla \cdot \left(\frac{\mathbf{e}_1}{h_2 h_3}\right) = 0. \tag{17}$$

Recall the scale factors are not constants (they cannot be removed from the divergences). Letting $u = x_1$ in Eq. (16) shows that

$$\nabla x_1 = \mathbf{e}_1 / h_1 \tag{18}$$

because $\partial x_1 / \partial x_2 = \partial x_1 / \partial x_3 = 0$. Similarly,

$$\nabla x_2 = \mathbf{e}_2 / h_2, \quad \nabla x_3 = \mathbf{e}_3 / h_3. \tag{19}$$

Since the coordinates are assumed to be orthogonal (and right-handed),

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 \tag{20}$$

Writing Eq. (20) in terms of Eqs. (18) and (19) yields

$$\nabla x_1 \times \nabla x_2 = \mathbf{e}_3 / (h_1 h_2). \tag{21}$$

The divergence of Eq. (21) is[†]

$$\begin{aligned} \nabla \cdot (\nabla x_1 \times \nabla x_2) &= \nabla x_2 \cdot (\nabla \times \nabla x_1) - \nabla x_1 \cdot (\nabla \times \nabla x_2) \\ &= 0 = \nabla \cdot [\mathbf{e}_3 / (h_1 h_2)]. \end{aligned} \tag{22}$$

where it has been noted in the second equality that $\nabla \times \nabla f = 0$. Equation (22) proves the first of Eqs. (17). The second and third of Eqs. (17) are proved by applying the same procedure to $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ and $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$.

Formula for divergence in general orthogonal coordinates

To calculate the divergence of

$$\mathbf{v} = \mathbf{e}_1 v_1 + \mathbf{e}_2 v_2 + \mathbf{e}_3 v_3 \tag{23}$$

in general orthogonal coordinates x_1, x_2, and x_3, write Eq. (23) as

$$\mathbf{v} = \frac{\mathbf{e}_1}{h_2 h_3}(h_2 h_3 v_1) + \frac{\mathbf{e}_2}{h_1 h_3}(h_1 h_3 v_2) + \frac{\mathbf{e}_3}{h_1 h_2}(h_1 h_2 v_3). \tag{24}$$

To calculate $\nabla \cdot \mathbf{v}$, consider the divergence of the first term of Eq. (24):[a]

$$\nabla \cdot \left[\frac{\mathbf{e}_1}{h_2 h_3}(h_2 h_3 v_1)\right] = \frac{\mathbf{e}_1}{h_2 h_3} \cdot \nabla (h_2 h_3 v_1) + h_2 h_3 v_1 \nabla \cdot \left(\frac{\mathbf{e}_1}{h_2 h_3}\right). \tag{25}$$

[†] $\nabla \cdot (\mathbf{u} \cdot \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$.

Equation (17) shows that the second term on the right-hand side of Eq. (25) vanishes. Noting that $\mathbf{e}_1 \cdot \boldsymbol{\nabla} = (1/h_1)\partial/\partial x_1$ allows Eq. (25) to be written as

$$\boldsymbol{\nabla} \cdot \left[\frac{\mathbf{e}_1}{h_2 h_3}(h_2 h_3 v_1)\right] = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x_1}(h_2 h_3 v_1). \tag{26}$$

Similarly, the divergence of the second and third terms of Eq. (23) can be expressed as

$$\boldsymbol{\nabla} \cdot \left[\frac{\mathbf{e}_2}{h_1 h_3}(h_1 h_3 v_2)\right] = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x_2}(h_1 h_3 v_2), \tag{27}$$

$$\boldsymbol{\nabla} \cdot \left[\frac{\mathbf{e}_3}{h_1 h_2}(h_1 h_2 v_1)\right] = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x_3}(h_1 h_2 v_3), \tag{28}$$

respectively. Combining Eqs. (26) and (27) shows that Eq. (23) equals

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 v_1) + \frac{\partial}{\partial x_2}(h_1 h_3 v_2) + \frac{\partial}{\partial x_3}(h_1 h_2 v_3)\right] \tag{29}$$

[a] The identity $\boldsymbol{\nabla} \cdot (\phi \mathbf{v}) = \mathbf{v} \cdot (\boldsymbol{\nabla}\phi) + \phi \boldsymbol{\nabla} \cdot \mathbf{v}$ has been used.

Example Write the divergence of $\mathbf{v}$ in cylindrical coordinates ($h_1 = 1$, $h_2 = r$, and $h_3 = 1$).

Example Write the divergence of $\mathbf{v}$ in spherical coordinates.

Formula for the Laplacian in general orthogonal coordinates

The Laplacian is simply $\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$. Using Eqs. (16) to evaluate the divergence of Eq. (29) yields

$$\begin{aligned}\nabla^2 u &= \boldsymbol{\nabla} \cdot \left[\mathbf{e}_1 \frac{1}{h_1}\frac{\partial u}{\partial x_1} + \mathbf{e}_2 \frac{1}{h_2}\frac{\partial u}{\partial x_2} + \mathbf{e}_3 \frac{1}{h_3}\frac{\partial u}{\partial x_3}\right] \\ &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right] \qquad (30)\end{aligned}$$

Example Write the Laplacian in cylindrical coordinates.

Example Write the Laplacian in spherical coordinates.

Miscellaneous grad skill: Nondimensionalizing equations There are two main reasons to nondimensionalize equations:

1. Dimensionless quantities reveals relationships that are obscured in dimensional equations.
2. Using normalized equations leads to more concise code. It is often unnecessary to enter numerical values of density, sound speed, bulk modulus, etc. in your code.

For example, consider the axial pressure radiated by a baffled circular piston [4, Eq. 5.7.3]:‡

$$p = -2i\rho_0 c_0 u_0 \sin\left[\frac{k(z^2+a^2)^{1/2} - kz}{2}\right] \exp\left\{\frac{ik[z+(z^2+a^2)^{1/2}]}{2}\right\}. \tag{31}$$

Nondimensionalize Eq. (31) by introducing the dimensionless quantities

$$P = p/\rho_0 c_0 u_0, \quad K = ka, \quad Z = z/z_R \tag{32}$$

where $z_R = ka^2/2$ is the Rayleigh distance. Show that the result can be written in the form

$$P(Z) = -2i \sin[\chi_-(Z)] e^{i\chi_+(Z)},$$

where

$$\chi_\pm(Z) = (K/2)\left[\sqrt{1+(KZ/2)^2} \pm KZ/2\right].$$

‡See also Blackstock's result [1, Chap. 13, Eq. (C-4)], but note that the equation contains a typo (see errata).

Miscellaneous grad skill: Email etiquette

- Always start your emails with a greeting, even if it is a one-line reply in a longer conversation. Emails are not texts, and addressing the recipient conveys respect and professional distance.
  - For initial communication with a professional contact, use "Dear (name)."
  - If the tone of the conversation becomes warmer, use "Hello (name)."
  - For friendly conversation, use "Hi (name)."
  - Do not start emails addressed to a person with just "Hi" (and no name). It sounds disrespectful/apathetic. If you are writing to an anonymous recipient (like an office or admin address), use "Good morning," "Good afternoon," or "Good evening."
- Sign off with your first name even if you have an automated signature. It demonstrates respect and feels more personal.
- Avoid contractions (isn't, don't, can't, etc.) in professional emails.
- Be concise! Break up large walls of text into separate paragraphs.

Miscellaneous grad skill: Office/lab etiquette

- Return equipment to the third or fourth floor labs; return books to the sixth floor lab.
- Let's maintain a quiet atmosphere to help each other achieve our academic goals.
- Try to restrict socializing to
  - lunch
  - entering/exiting the lab
  - Friday meetings/research group meetings
  - Student Chapter meeting or after seminar

References

[1] D. T. Blackstock. *Fundamentals of Physical Acoustics*. Wiley, New York, 2000.

[2] M. L. Boas. *Mathematical Methods in the Physical Sciences*. John Wiley & Sons, 3rd edition, 1980.

[3] D. J. Griffiths. *Introduction to Electrodynamics*. Pearson, Upper Saddle River, New Jersey, 3rd edition, 1999.

[4] A. D. Pierce. *Acoustics: An Introduction to Its Physical Principles and Applications*. Springer, Cham, Switzerland, 2019.