Review for the nonlinear acoustics midterm

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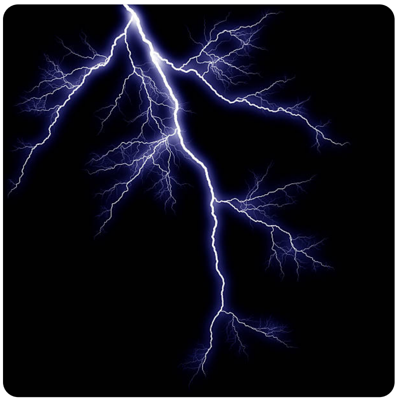

These conceptual and analytical problems, which are based on Dr. Hamilton's lectures, address the major topics of the course thus far. Best wishes on the exam!

1 Gauge functions & linear lossy theory

(a) Use the linearized 1D momentum equation $\rho_0 \partial u/\partial t + \partial p/\partial x = 0$ to show that

$$\frac{u}{c_0} = \frac{p}{\rho_0 c_0^2} = \frac{\rho'}{\rho_0}. \tag{1.1}$$

Hint: Start by integrating the momentum equation,

$$u = -\frac{1}{\rho_0}\int \frac{\partial p}{\partial x} dt,$$

and then let $p = f(t - x/c_0)$. *Note that* $\partial f/\partial x = -\frac{1}{c_0}\partial f/\partial t$.

(b) What is the definition of the acoustic mach number ϵ?

(c) What is the order of

- $a\mathcal{O}(\epsilon^n)$, where a is a constant?

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- $\mathcal{O}(\epsilon^n) + \mathcal{O}(\epsilon^m)$, for $n < m$?
- $\mathcal{O}(\epsilon^n)\mathcal{O}(\epsilon^m)$?
- $[\mathcal{O}(\epsilon^n)]^m$?
- $\nabla^2 p = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}$?
- $\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}$?

(d) What does the "acoustic Stokes number" $\eta = \mu\omega/\rho_0 c_0^2$ characterize? It which governing equation does it arise and aid the ordering of terms? What is the order of μ, the shear viscosity?

(e) How does a term of $\mathcal{O}(\epsilon^2)$ compare to a term of $\mathcal{O}(\eta\epsilon)$?

(f) What is order of the Prandtl number $\mathrm{Pr} = \mu C_P/\kappa$, and what does it characterize? It which governing equation does it manifest? What is the order of κ which appears in the definition of the Prandtl number?

(g) Show that the temperature perturbation T' is $\mathcal{O}(\epsilon)$ by expanding the ideal gas law, $P = R\rho T$, where $P = P_0 + p$, $\rho = \rho_0 + \rho'$, and $T = T_0 + T'$.

(h) Recall the entropy equation,

$$\rho_0(T_0 + T')\frac{\partial s'}{\partial t} = \kappa\nabla^2 T'.$$

What is the order of the entropy perturbation s'?

(i) Derive the attenuation coefficient from the lossy linear wave equation below to $\mathcal{O}(\eta)$. What is the next highest-order term in the attenuation coefficient?

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\delta}{c_0^4}\frac{\partial^3 p}{\partial t^3} \tag{1.2}$$

(j) Derive the progressive wave equation of $\mathcal{O}(\eta\epsilon)$ from the linear lossy wave equation

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\delta}{c_0^4}\frac{\partial^3 p}{\partial t^3} \tag{1.3}$$

Hint: start by transforming coordinates

$$(\eta x, t - x/c_0) \mapsto (x_1, \tau)$$

(k) On a plot of pressure vs. retarded time τ, what is the meaning of "to the left of $\tau = 0$" and "to the right of $\tau = 0$"? What is strange about the solution to the linear lossy progressive wave equation derived in the previous question?

(l) *Coordinate transformation practice:* Write the partial derivatives with respect to the spherical coordinates[1] $\partial/\partial r$, $\partial/\partial\theta$, and $\partial/\partial\psi$ as partial derivatives with respect to Cartesian coordinates x, y, z.

(m) *Fun (but fictitious) coordinate transformation problem:* The spherically symmetric scalar wave equation for light is[2]

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0\,, \tag{1.4}$$

where c is the speed of light. Meanwhile, Einstein's general theory of relativity can be solved exactly to show that at a radial coordinate r from a non-rotating mass, time dilates as

$$t' = t\sqrt{1 - \frac{2GM}{rc^2}}, \tag{1.5}$$

where t is the time far away from the mass, G is the gravitational constant, and M is the mass of the object [3]. Denote $r_s = 2GM/c^2$ for convenience.[3] Write equation (1.4) in the coordinates $(r, t') = (r, t\sqrt{1 - r_s/r})$. *Hint: follow a procedure similar to problem (j). It helped me to introduce an auxiliary coordinate $r' = r$ to keep the "old" and "new" coordinates straight while performing the coordinate transformation.*

2 Lossless nonlinear theory

(a) Show that the so-called Poisson solution

$$u = g[x - (c+u)t]$$

satisfies the exact lossless nonlinear equation for progressive waves,

$$\frac{\partial u}{\partial t} + (c+u)\frac{\partial u}{\partial x} = 0. \tag{2.1}$$

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[1] θ is polar and ψ is azimuthal, i.e., $x = r\cos\psi\sin\theta$, $y = r\sin\psi\sin\theta$, and $z = r\cos\theta$.

[2] p is used as the wave variable for familiarity.

[3] r_s is the Schwarzschild radius.

Hint: for notational ease, denote $y = x - v(u)t$, where $v(u) = c + u$. Note that $\partial y/\partial x = 1 - v'(u)\, t\, \partial u/\partial x$ and $\partial y/\partial t = -v - v'(u)\, t\, \partial u/\partial t$.

(b) Recalling the adiabatic sound speed $c^2 = \gamma P/\rho$ and the adiabatic gas law, show that

$$\left(\frac{\rho}{\rho_0}\right) = \left(\frac{c}{c_0}\right)^{2/(\gamma-1)} . \tag{2.2}$$

Show that the differential $d\rho$ is

$$d\rho = \frac{2}{\gamma - 1}\rho_0 (c/c_0)^{2/(\gamma-1)} c^{-1}\, dc\,. \tag{2.3}$$

(c) Recall that the quantity λ equals u for a progressive wave, because this condition sets the Riemann invariant J_- (which corresponds to backward-traveling waves) to 0. Noting that $d\lambda = (c/\rho)\, d\rho$, use equation (2.3) to show that

$$c + u = c_0 + \beta u \quad \text{where} \quad \beta = \frac{1}{2}(\gamma + 1) \tag{2.4}$$

(d) Derive the Earnshaw solution for a boundary value problem, in which the time variation $u(0, t)$ at the face of the piston is known.

(e) What is the Earnshaw solution for a sinusoidal boundary condition, $u(0, t) = u_0 \sin \omega t$?[4]

(f) Derive the Earnshaw solution for an initial value problem (homework problem 2-7), in which the spatial variation at the face of the boundary is known, e.g., $u(x, 0)$.

(g) What is the Earnshaw solution for a linear boundary condition, $u(x, 0) = m_0 x$?

3 Lossless nonlinear approximate theory

(a) Reduce the exact progressive nonlinear lossless wave equation is given by equation (2.1), repeated below for convenience,

$$\frac{\partial u}{\partial x} + \frac{1}{c + u}\frac{\partial u}{\partial t} = 0 \tag{2.1}$$

[4]This was not done in the lecture; see page 70 of [2].

to $\mathcal{O}(\epsilon^2)$ in $(x,\tau) = (x, t - x/c_0)$ coordinates. *Hint: Expand* $v(u) = c + u = v(0) + v'(0)u + \mathcal{O}(\epsilon^2) = c_0 + \beta u + \mathcal{O}(\epsilon^2)$ *and then perform a binomial expansion of* $(c_0 + \beta_0 u)^{-1}$. *Answer:*

$$\frac{\partial u}{\partial x} = \frac{\beta}{c_0^2} u \frac{\partial u}{\partial \tau} \tag{3.1}$$

(b) Verify that $u = f(\tau + \beta x u / c_0^2)$ solves equation (3.1).

(c) Use $u = f(\tau + \beta x u / c_0^2)$ to calculate $\bar{x}$, the shock-forming distance. *Hint: Find* $x_{\mathrm{vt}} = x$ *such that* $\partial u / \partial \tau = \infty$ *(the condition for a vertical tangent line). The smallest value of* x_{vt} *is* $\bar{x}$.

(d) What is the shock-forming distance $\bar{x}$ for a sinusoidal source $f(t) = u_0 \sin \omega t$? Express the distance in terms of β, ϵ, and k.

(e) Perform a perturbation solution for

$$\frac{\partial u}{\partial x} = \frac{\beta}{2c_0^2} \frac{\partial u^2}{\partial \tau}$$

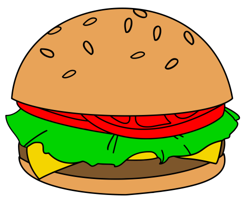
for $u = u_0 \sin \omega \tau$. For what range of $\sigma = x/\bar{x}$ is this solution valid? *Hint: Substitute the power series in* ϵ

$$\frac{u}{c_0} = \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \ldots$$

where $\epsilon \ll 1$ *into the evolution equation, and then match orders on both sides of the equation.*

(f) What problem does the Fubini solution address? For what σ is it valid? To what solution did Blackstock "bridge" the Fubini solution?

4 Burgers equation



Comic relief

$$\frac{\partial p}{\partial x} - \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial \tau^2} = \frac{\beta p}{\rho_0 c_0^3}\frac{\partial p}{\partial \tau} \tag{4.1}$$

1. Why did the Burgers equation go to the fast food restaurant?
   To get a quadratic meal deal!
2. Why was the Burgers equation feeling down?
   Because it felt like it was getting grilled by all those derivatives!
3. Why did the Burgers equation get stuck in traffic?
   Because of its nonlinear convection term, it just couldn't get past all the other terms!

(a) Why is equation (4.1) accurate to $\mathcal{O}(\epsilon^2)$?

(b) What is an alternate way of writing the right-hand-side of equation (4.1)?

(c) What is the Gol'berg number Γ? What do the limits $\Gamma < 1$ and $\Gamma \gg 1$ correspond to? In which case does a shock form?

(d) Which of the following is a solution to equation (4.1) in the case that $\delta = 0$? Which is a solution in the case that $\beta = 0$?

$$p = f[\tau + (\beta p/\rho_0 c_0^3)x] \qquad p = p_0 \exp[j\omega\tau - (\delta\omega^2/2c_0^3)x] \tag{4.2}$$

(e) Assess second harmonic generation in the Burgers equation, using as as the first approximation

$$p_1 = p_0 e^{-\alpha_1 x} \sin \omega\tau.$$

where $\alpha_1 = \delta\omega^2/2c_0^3$. *Hint: In this case it is more convenient to write the Burgers equation as*

$$\frac{\partial p}{\partial x} - \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial \tau^2} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial (p^2)}{\partial \tau}.$$

(f) The solution to the Burgers equation for a so-called Taylor shock is

$$p = \frac{\Delta p}{2}\left[1 + \tanh\left\{\frac{\beta \Delta p}{2\rho_0 \delta}(t' - t_0)\right\}\right] \tag{4.3}$$

Rewrite equation (4.3) by defining

$$t_{\text{rise}} = \frac{4\rho_0\delta}{\beta\Delta p} \tag{4.4}$$

and use the result to show that

$$t_{\text{rise}}\left(\frac{\partial p}{\partial t'}\right)_{t'=t_0} = \Delta p,$$

(g) In what limit of viscosity δ does the Taylor shock become a step shock? *Hint: A step shock is defined by* $t_{\text{rise}} \to 0$.

(h) What is the name of the transformation that leads to the Fay solution? What equation does the Fay solution satisfy? What are the restrictions on this solution?

(i) Show that $\sigma = x/\bar{x}$ can be written as

$$\sigma = \frac{\beta p_0 k x}{\rho_0 c_0^2} \tag{4.5}$$

by recalling the definition of the shock formation distance and the acoustic mach number.

(j) The Fay solution is

$$p = p_0 \sum_{n=1}^{\infty} B_n(\sigma)\sin n\omega\tau, \quad \text{where} \quad B_n(\sigma) = \frac{2A}{\sinh[nA(1+\sigma)]}.$$

where $A = \alpha\bar{x} = \alpha\rho_0 c_0^2/p_0\beta k$. Show that for $\sigma \gg 1$, the pressure becomes

$$p = \frac{4\rho_0 c_0^2\alpha}{\beta k}\sum_{n=1}^{\infty} e^{-n\alpha x}\sin n\omega\tau\,,$$

where equation (4.5) has been invoked. What is this waveform called? *Hint: think about the color of Dr. Hamilton's hair.*

(k) It was shown in class that for $\Gamma \to \infty$ (equivalently, $A \to 0$), the Fay solution reduces to

$$p = \frac{2p_0}{1+\sigma}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\omega\tau\,. \tag{4.6}$$

Combine equations (4.5) and (4.6) to show that for $\sigma \gg 1$,

$$p = \frac{2\rho_0 c_0^2}{\beta k x} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\omega\tau \,. \tag{4.7}$$

What is remarkable about equation (4.7)?

(l) What is the time-domain version of the Fay solution called?

References

[1] M. F. Hamilton, Lecture notes from Nonlinear Acoustics. University of Texas at Austin, (2023).

[2] M. F. Hamilton and D. T. Blackstock, "Nonlinear Acoustics." Acoustical Society of America, (2008).

[3] "Gravitational time dilation," Wikipedia.