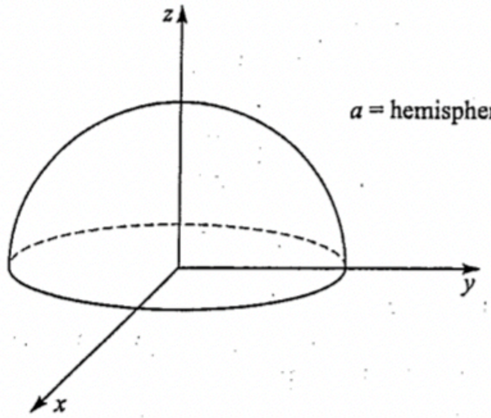
Problem from *Fundamentals of Physical Acoustics* by D. T. Blackstock.

10–13. A certain sound field inside a hollow hemisphere (radius a, all surfaces rigid) has the property that the pressure is zero along the z axis. Find the lowest two eigenfrequencies for this field, and identify their corresponding eigenfunctions ϕ_{lmn}.



First approach. In the original coordinate system of the problem, the boundary conditions are

$$\left.\frac{\partial p}{\partial r}\right|_{r=a} = 0 \qquad (i)$$

$$\left.\frac{\partial p}{\partial \theta}\right|_{\theta=\pi/2} = 0 \qquad (ii)$$

$$p(\theta = 0) = 0\,. \qquad (iii)$$

The Neumann functions are thrown out because they diverge at $r = 0$, which is a point in space in which the sound is finite, so the form of solution to the Helmholtz equation is

$$R(r)\Theta(\theta)\Psi(\psi) = j_n(kr)P_n^m(\cos\theta)\begin{Bmatrix}\cos m\psi \\ \sin m\psi\end{Bmatrix}\,. \qquad (1)$$

Applying boundary condition (i) on r gives $j_n'(ka) = 0$, or

$$k_{nl} = \frac{x_{nl}'}{a}$$

where x_{nl}' is the lth root of the nth order derivative of the spherical Bessel function given by table 10.1 in Blackstock's book. Next, applying boundary condition (iii) gives $P_n^m(\cos 0) = 0$, or $P_n^m(1) = 0$. Looking at a table of the associated Legendre functions shows that all the Legendre polynomials satisfy this property *except for* $m = 0$. But $m = 0, 1, 2, \ldots n$. Therefore $m \neq 0$. Finally, applying boundary condition (ii) gives

$$\frac{d}{d\theta}P_n^m[\cos(\pi/2)] = 0\,. \qquad (2)$$

Evaluating equation (2) is what makes this problem difficult, because no relation is given in Blackstock's book (and some of the Associated Legendre functions listed on page 348 are off by a factor of -1). The left-hand column below are the associated Legendre functions of $\cos\theta$ as provided by *Wikipedia*.

$$
\begin{aligned}
P_1^1(\cos\theta) &= -\sin\theta & &\Longrightarrow & \frac{d}{d\theta}P_1^1(\cos\theta) &= -\cos\theta \\
P_2^1(\cos\theta) &= -3\cos\theta\sin\theta & &\Longrightarrow & \frac{d}{d\theta}P_2^1(\cos\theta) &= 3\cos 2\theta \\
P_2^2(\cos\theta) &= 3\sin^2\theta & &\Longrightarrow & \frac{d}{d\theta}P_2^2(\cos\theta) &= 6\sin\theta\cos\theta \\
P_3^1(\cos\theta) &= -\frac{3}{2}(5\cos^2\theta-1)\sin\theta & &\Longrightarrow & \frac{d}{d\theta}P_3^1(\cos\theta) &= 15[\cos\theta\sin^2\theta-\cos(\theta)/2]+3\cos(\theta)/2 \\
P_3^2(\cos\theta) &= 15\cos\theta\sin^2\theta & &\Longrightarrow & \frac{d}{d\theta}P_3^2(\cos\theta) &= -15(\sin^3\theta-2\cos^2\theta\sin\theta) \\
P_3^3(\cos\theta) &= -15\sin^3\theta & &\Longrightarrow & \frac{d}{d\theta}P_3^3(\cos\theta) &= -45\sin^2\theta\cos\theta
\end{aligned}
$$

It can be seen from the column on the right-hand side that

$$
\begin{aligned}
\frac{d}{d\theta}P_n^m[\cos(\pi/2)] &= 0, \quad m+n = \text{ even} \\
&\neq 0, \quad m+n = \text{ odd}
\end{aligned}
$$

Equation (1) thus becomes

$$
p_{nlm} = j_n(k_{nl}r)P_n^m(\cos\theta)\begin{Bmatrix}\cos m\psi \\ \sin m\psi\end{Bmatrix}, \qquad \text{where} \quad m = 1, 2\ldots \quad \text{and } m+n = \text{ even}
$$

The lowest two nonzero eigenfrequencies are found to be proportional to the underlined values of $x'_{nl} = x'_{11}$ and $x'_{nl} = x'_{21}$ below,

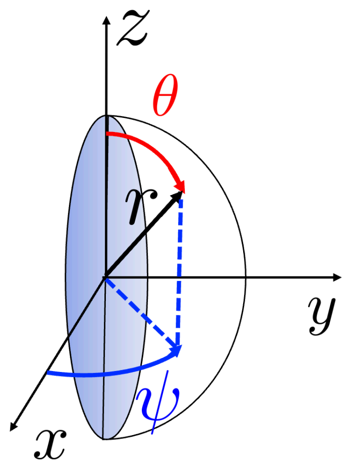
Table 10.1 Tables of Zeros of Spherical Bessel Functions

	Roots $x_{n\ell}$ of $j_n(x)=0$					Roots $x'_{n\ell}$ of $j'_n(x)=0$				
ℓ	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
1	π	4.493	5.763	6.988	8.183	0	2.082	3.342	4.514	5.647
2	2π	7.725	9.095	10.417	11.705	4.493	5.940	7.290	8.578	9.840
3	3π	10.904	12.323	13.698	15.040	7.725	9.206	10.614	11.973	13.296
4	4π	14.066	15.515	16.924	18.301	10.904	12.405	13.846	15.245	16.609
5	5π	17.221	18.689	20.122	21.525	14.066	15.579	17.043	18.468	19.862

i.e., the lowest two eigenfrequencies are

$$
f_{11} = \frac{2.082c_0}{2\pi a}, \quad f_{21} = \frac{3.342c_0}{2\pi a}
$$

Second approach. In this approach, the hemisphere is rotated 90° about the x-axis, as shown below:



In this case, the boundary conditions are

$$\frac{\partial}{\partial r} p(r, \theta, \psi)\bigg|_{r=a} = 0 \qquad (i)$$

$$\frac{\partial}{\partial \psi} p(r, \theta, \psi)\bigg|_{\psi=0} = 0 \qquad (ii)$$

$$\frac{\partial}{\partial \psi} p(r, \theta, \psi)\bigg|_{\psi=\pi} = 0 \qquad (iii)$$

$$p(r, \pi/2, 0) = 0 \qquad (iv)$$

Boundary condition (i) describes curved hemisphere being rigid, and boundary conditions (ii) and (iii) describes the base of the hemisphere (now corresponding to the $y = 0$ plane) being rigid. Boundary condition (iv) corresponds to the requirement that $p = 0$ along axis of symmetry of the hemisphere (now corresponding to the y-axis).

In the solution, the Neumann functions are again tossed because sound cannot be infinitely loud at the origin:

$$R(r)\Theta(\theta)\Psi(\psi) = j_n(kr) P_n^m(\cos\theta) \begin{Bmatrix} \cos(m\psi) \\ \sin(m\psi) \end{Bmatrix}, \qquad (3)$$

Setting boundary condition (i) equal to equation (3) gives the condition

$$j_n'(ka) = 0 \qquad \Longrightarrow \qquad k_{nl} = \frac{x'_{nl}}{a}, \qquad (4)$$

where x'_{nl} is as defined before. Meanwhile, setting boundary condition (ii) equal to equation (3) gives

$$\frac{\partial}{\partial \psi}(A\cos m\psi + B\sin m\psi)\bigg|_{\psi=0} = 0 \qquad \Longrightarrow \qquad B = 0.$$

Next, invoking boundary condition (iii) determines the values of m:

$$\frac{\partial}{\partial\psi}(A\cos m\psi)\bigg|_{\psi=\pi} = 0 \quad\Longrightarrow\quad \sin m\pi = 0 \quad\Longrightarrow\quad m = 0, 1, 2\ldots$$

Finally, invoking boundary condition (iv) gives

$$P_n^m[\cos(\pi/2)]\cos(m\pi/2) = 0$$

Note that $\cos(m\pi/2)$ is ± 1 for m even and 0 for m odd. Therefore, $P_n^m(0) = 0$ by the zero product property. At this juncture, the following footnote in Blackstock's book is noted:

[4]The odd, even properties of $P_n^m(z)$ are worth noting. Equation B–32 shows that P_n^m is an odd function of z if $m+n$ is an odd number, an even function if $m+n$ is an even number. This information is useful, for example, when one analyzes the sound field in a hemispherical enclosure.

$P_n^m(0) = 0$ is guaranteed for an odd function $P_n^m(z)$. Therefore, $m + n =$ odd. The eigenfunctions are therefore given by

$$\begin{aligned} p_{nlm} &= j_n(k_{nl}r)P_n^m(\cos\theta)\left\{\cos(m\psi)\right\}, \\ \text{where}\quad n &= 0, 1, 2\ldots, \\ m &= 0, 1, 2\ldots, n \\ \text{and } m + n &= \text{ odd} \end{aligned}$$

The lowest two nonzero[1]eigenfrequencies are found to be proportional to the underlined values of $x'_{nl} = x'_{11}$ and $x'_{nl} = x'_{21}$ below,

Table 10.1 Tables of Zeros of Spherical Bessel Functions

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[1] Let's talk more about the "dc" solution for corresponding to $x'_{nl} = x'_{01}$