$y'' + xy' + y = 0.$

Solve by series: $y = \sum_{n=0}^{\infty} a_n x^n.$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}.$$

The ODE reads $\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$

$\downarrow$ Shift $\leftarrow 2$ $\qquad \downarrow \sum_{n=1}^{\infty} a_n n x^n$

We want to combine all the sums. $\downarrow$ (1st term is 0)

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Now all the sums have the same indices & all the xs have the same power. Thus

$$\sum_{n=0}^{\infty} \Big[\underbrace{a_{n+2}(n+2)(n+1) + a_n n + a_n}\Big] x^n = 0.$$

Therefore this is 0:

$$a_{n+2}(n+2)(n+1) + a_n(n+1) = 0$$

$$\Longrightarrow \boxed{a_{n+2} = -\frac{a_n}{n+2}} \quad \text{for } n \geq 0.$$

Recursion relation.

Now the recursion relation should be used to identify a series solution.

Because there are 2 solutions (both linearly independent), we have to arbitrary choices: $\begin{cases} a_0 = 1 \text{ and } a_1 = 0 \\ a_0 = 0 \text{ and } a_1 = 1 \end{cases}$.

Start with $a_0 = 1$ and $a_1 = 0$.

Then:

$n=0 \quad a_2 = -\frac{a_0}{0+2} = -\frac{1}{2}$

$n=1 \quad a_3 = -\frac{a_1}{1+2} = 0$

$n=2 \quad a_4 = -\frac{a_2}{2+2} = \frac{1/2}{4} = \frac{1}{2\cdot 4}$

$n=3 \quad a_5 = -\frac{a_3}{3+2} = 0$

$n=4 \quad a_6 = -\frac{a_4}{4+2} = -\frac{1}{2\cdot 4\cdot 6}$

$a_7 = 0$

$a_8 = \frac{1}{2\cdot 4\cdot 6\cdot 8}$

We see the pattern: alternating signs, odds $= 0$, and denominator is $2\cdot 4\cdot 6\cdot 8 \cdots (n-2)$.

Explicitly this pattern can be written as:

$$a_{2k} = \frac{(-1)^k}{2\cdot 4\cdot 6\cdots(2k)}$$ (check above to see that it is true).

Thus the first solution is $\sum_{k=1}^{\infty} \frac{(-1)^k}{2\cdot 4\cdot 6\cdots(2k)} x^{2k} + a_0 = y_1$

The a_0 is included outside the sum because it is not part of the sum (which started at a_2)

Next we let $a_0 = 0$ and $a_1 = 1$:

$n=0 \quad a_2 = -\frac{a_0}{0+2} = 0$

$n=1 \quad a_3 = -\frac{a_1}{1+2} = -\frac{1}{3}$

$n=2 \quad a_4 = -\frac{a_2}{2+2} = 0$

$n=3 \quad a_5 = -\frac{a_3}{3+2} = \frac{1}{3\cdot 5}$

$n=4 \quad a_6 = -\frac{a_4}{4+2} = 0$

$n=5 \quad a_7 = -\frac{1}{3\cdot 5\cdot 7}$

Thus the pattern is: alternating signs, evens $= 0$, and denominator is $1 \cdot 3 \cdot 5 \cdots (n-2)$.

Explicitly, this is $a_{2k+1} = \dfrac{(-1)^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)}$.

The second sol. is thus $y_2 = \sum_{k=1}^{\infty} \dfrac{(-1)^k}{1 \cdot 3 \cdot 5 \cdots (2k+1)} x^{2k+1} + a_1$

again, not in the sum so must appear outside

General solution is simply $y_1 + y_2$.