Introduction to

Elastic Wave Propagation

A. Bedford & D. S. Drumheller

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Contents

Introduction

Earthquakes are detected and studied by measuring the waves they create. Waves are transmitted through the Earth to detect oil and gas deposits and to study the Earth's geological structure. Properties of materials are determined by measuring the behavior of waves transmitted through them. In recent years, elastic waves transmitted through the human body have been used for medical diagnosis and therapy. Many students and professionals in the various branches of engineering and in physics, geology, mathematics, and the life sciences encounter problems requiring an understanding of elastic waves. In this book we present the basic concepts and methods of the theory of wave propagation in elastic materials in a form suitable for a one-semester course at the advanced undergraduate or graduate level, for self-study, or as a supplement in courses that include material on elastic waves. Because persons with such different backgrounds are interested in elastic waves, we provide a more complete discussion of the theory of elasticity than is usually found in a book of this type, and also an appendix on the theory of functions of a complex variable. While persons who are well versed in mechanics and partial differential equations will have a deeper appreciation for some of the topics we discuss, the basic material in Chapters 1-3 requires no preparation beyond a typical undergraduate calculus sequence.

Chapter 1 covers linear elasticity through the displacement equations of motion. Chapter 2 introduces the one-dimensional wave equation and the D'Alembert solution. We show that one-dimensional motions of a linear elastic material are governed by the one-dimensional wave equation and discuss compressional and shear waves. To provide help in understanding and visualizing elastic waves, we demonstrate that small lateral motions of a stretched string (such as a violin string) are governed by the one-dimensional wave equation and show that the boundary conditions for a string are analogous to the boundary conditions for one-dimensional motions of an elastic material. Using the D'Alembert solution, we analyze several problems, including reflection and transmission at a material interface. We define the characteristics of the one-dimensional wave equation and show how they can be used to obtain solutions. We apply characteristics to layered materials and describe simple numerical algorithms for solving one-dimensional problems in materials with multiple layers.

Chapter 3 is concerned with waves having harmonic (oscillatory) time dependence. We present the solution for reflection at a free boundary and discuss Rayleigh waves. To introduce the concepts of dispersion and propagation modes, we analyze the propagation of sound waves down a two-dimensional channel. We then discuss longitudinal waves in an elastic layer and waves in layered media.

Chapter 4 is devoted to transient waves. We introduce the Fourier and Laplace transforms and illustrate some of their properties by applying them to one-dimensional problems. We introduce the discrete Fourier transform and apply it to several examples using the FFT algorithm. We define the concept of group velocity and use our analyses of layered media from Chapters 2 and 3 to discuss and demonstrate the propagation of transient waves in dispersive media.

Chapter 5 discusses some topics in nonlinear wave propagation. Although few applications involving nonlinear waves in solid materials can be analyzed adequately by assuming elastic behavior, we discuss and demonstrate some of the important concepts using an elastic material as an heuristic example. After presenting the theory of nonlinear elasticity in one dimension, we discuss hyperbolic systems of first-order equations and characteristics and derive the class of exact solutions called simple waves. We introduce singular surface theory and determine the properties of acceleration waves and shock waves. The chapter concludes with a discussion of the steady compressional waves and release waves observed in experimental studies of large-amplitude waves.

Nomenclature

Definitions of frequently-used symbols. The pages on which they first appear are shown in parentheses.

$\mathbf{a}$, a_k	Acceleration (2).
$\mathbf{b}$, b_k	Body force (28).
B_k	Basis function (81)
c_1	Phase velocity (146).
c_p	Plate velocity (151).
c_g	Group velocity (183).
c	Slope of characteristic (221). Constant in Hugoniot relation (244).
$\mathbf{C}$, C_{km}	Mass matrix (84).
$\mathbf{d}$, d_k	Vector of nodal displacements (84).
e_{kmn}	Alternator (3).
E_{mn}	Lagrangian strain (15).
E	Young's modulus (34).
$\mathbf{F}$, F_k	Force (2).
f	Frequency, Hz (112).
F	Deformation gradient (214).
$f^F(\omega)$	Fourier transform of f (164).
$f^L(s)$	Laplace transform of f (159).

f_m^{DF}	Discrete Fourier transform of f_m (172).
$\mathbf{f}$, f_k	Force vector (84).
$H(t)$	Step function (76).
$\mathbf{i}$, i_k	Unit direction vector (1).
i	$\sqrt{-1}$ (113).
j	Nondimensional time (100).
k	Wave number (111).
$\mathbf{K}$, K_{km}	Stiffness matrix (84).
m	Nondimensional time (175).
$\mathbf{n}$	Unit vector to surface (4).
n	Nondimensional position (100).
p	Pressure (25).
r	Magnitude of complex variable (354).
s	Laplace transform variable (159). Constant in Hugoniot relation (244).
S	Surface (4).
t	Time (4).
$\mathbf{t}$, t_k	Traction (24).
T_{km}	Stress tensor (26).
T	Window length of FFT (172). Period (112). Normal stress (215).
$\mathbf{u}$, u_k	Displacement vector (9).
u	Dependent variable (48). Real part (354).
U	Shock wave velocity (243).

$\mathbf{v}$, v_k	Velocity (9).
v	Velocity (93). Imaginary part (354).
V	Volume (4).
$\mathbf{x}$, x_k	Position (9).
$\mathbf{X}$, X_k	Reference position (9).
z	Acoustic impedance (72). Complex variable (353).
α	Compressional wave velocity (55). Phase velocity (111).
β	Shear wave velocity (58).
γ_{km}	Shear strain (18).
δ_{km}	Kronecker delta (3).
Δ	Time increment (100).
ε	Longitudinal strain (15). Negative of the longitudinal strain (214).
ζ	Characteristic coordinate (218). Steady-wave variable (243).
η	D'Alembert variable (49). Characteristic coordinate (218).
θ	Propagation direction (120). Phase of complex variable (354).
θ_c	Critical propagation direction (135).
λ	Lamé constant (34). Wavelength (111).
μ	Lamé constant, shear modulus (34).
ν	Poisson's ratio (34).

ξ	D'Alembert variable (49).
ρ	Density (20).
ρ_0	Density in reference state (21).
σ	Normal stress (25).
τ	Shear stress (25).
ϕ	Helmholtz scalar potential (43).
$\boldsymbol{\psi}$, ψ_k	Helmholtz vector potential (43).
ω	Frequency, rad/s (110).

Chapter 1

Linear Elasticity

To analyze waves in elastic materials, we must derive the equations governing the motions of such materials. In this chapter we show how the motion and deformation of a material are described and discuss the internal forces, or stresses, in the material. We define elastic materials and present the relation between the stresses and the deformation in a linear elastic material. Finally, we introduce the postulates of conservation of mass and balance of momentum and use them to obtain the equations of motion for a linear elastic material.

1.1 Index Notation

Here we introduce a compact and convenient way to express the equations used in elasticity.

Cartesian coordinates and vectors

Consider a cartesian coordinate system with coordinates x, y, z and unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (Fig. 1.1.a). We can express a vector $\mathbf{v}$ in terms of its cartesian components as

$$\mathbf{v} = v_x \, \mathbf{i} + v_y \, \mathbf{j} + v_z \, \mathbf{k}.$$

Let us make a simple change of notation, renaming the coordinates x_1, x_2, x_3 and the unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ (Fig. 1.1.b). The subscripts $1, 2, 3$ are called *indices.* With this notation, we can express the vector $\mathbf{v}$ in terms of its components in

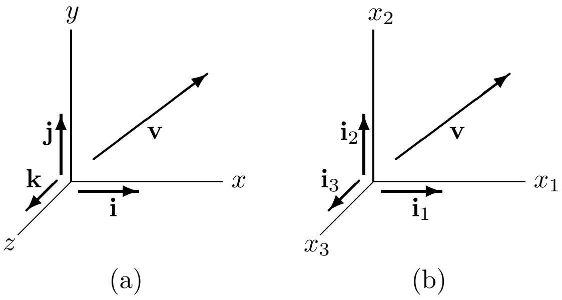


Figure 1.1: A cartesian coordinate system and a vector $\mathbf{v}$. (a) Traditional notation. (b) Index notation.

the compact form

$$\mathbf{v} = v_1\,\mathbf{i}_1 + v_2\,\mathbf{i}_2 + v_3\,\mathbf{i}_3 = \sum_{k=1}^{3} v_k\,\mathbf{i}_k.$$

We can write the vector $\mathbf{v}$ in an even more compact way by using what is called the summation convention. *Whenever an index appears twice in an expression, the expression is assumed to be summed over the range of the index.* With this convention, we can write the vector $\mathbf{v}$ as

$$\mathbf{v} = v_k\,\mathbf{i}_k.$$

Because the index k appears twice in the expression on the right side of this equation, the summation convention tells us that

$$\mathbf{v} = v_k\,\mathbf{i}_k = v_1\,\mathbf{i}_1 + v_2\,\mathbf{i}_2 + v_3\,\mathbf{i}_3.$$

In vector notation, Newton's second law is

$$\mathbf{F} = m\mathbf{a}.$$

In index notation, we can write Newton's second law as the three equations

$$\begin{aligned} F_1 &= ma_1, \\ F_2 &= ma_2, \\ F_3 &= ma_3, \end{aligned}$$

where F_1, F_2, F_3 are the cartesian components of $\mathbf{F}$ and a_1, a_2, a_3 are the cartesian components of $\mathbf{a}$. We can write these three equations in a compact way by using another convention: Whenever an index appears once in each term of an equation, the equation is assumed to hold for each value in the range of that index. With this convention, we write Newton's second law as

$$F_k = ma_k.$$

Because the index k appears once in each term, this equation holds for $k = 1$, $k = 2$, and $k = 3$.

Dot and cross products

The dot and cross products of two vectors are useful definitions from vector analysis. By expressing the vectors in index notation, we can write the dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ as

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (u_k\mathbf{i}_k)\cdot(v_m\mathbf{i}_m)\\ &= u_k v_m(\mathbf{i}_k\cdot\mathbf{i}_m).\end{aligned}$$

The dot product $\mathbf{i}_k\cdot\mathbf{i}_m$ equals one when $k = m$ and equals zero when $k \neq m$. By introducing the *Kronecker delta*

$$\delta_{km} = \begin{cases} 1 & \text{when } k = m, \\ 0 & \text{when } k \neq m, \end{cases}$$

we can write the dot product of the vectors $\mathbf{u}$ and $\mathbf{v}$ as

$$\mathbf{u}\cdot\mathbf{v} = u_k v_m \delta_{km}.$$

We leave it as an exercise to show that

$$u_k v_m \delta_{km} = u_1 v_1 + u_2 v_2 + u_3 v_3,$$

which is the familiar expression for the dot product of the two vectors in terms of their cartesian components.

We can write the cross product of two vectors $\mathbf{u}$ and $\mathbf{v}$ as

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= (u_k\mathbf{i}_k)\times(v_m\mathbf{i}_m)\\ &= u_k v_m(\mathbf{i}_k\times\mathbf{i}_m).\end{aligned}$$

The values of the cross products of the unit vectors are

$$\mathbf{i}_k\times\mathbf{i}_m = e_{kmn}\,\mathbf{i}_n,$$

where the term e_{kmn}, called the *alternator*, is defined by

$$e_{kmn} = \begin{cases} 0 & \text{when any two of the indices } k, m, n \text{ are equal}, \\ 1 & \text{when } kmn = 123 \ \text{ or } \ 312 \ \text{ or } \ 231, \\ -1 & \text{otherwise}. \end{cases}$$

Thus we can write the cross product of the vectors $\mathbf{u}$ and $\mathbf{v}$ as

$$\mathbf{u}\times\mathbf{v} = u_k v_m e_{kmn}\,\mathbf{i}_n.$$

Gradient, divergence, and curl

A *scalar field* is a function that assigns a scalar value to each point of a region in space during some interval of time. The expression for the gradient of a scalar field $\phi = \phi(x_1, x_2, x_3, t)$ in terms of cartesian coordinates is

$$\nabla\phi = \frac{\partial\phi}{\partial x_k}\,\mathbf{i}_k. \tag{1.1}$$

A *vector field* is a function that assigns a vector value to each point of a region in space during some interval of time. The expressions for the divergence and the curl of a vector field $\mathbf{v} = \mathbf{v}(x_1, x_2, x_3, t)$ in terms of cartesian coordinates are

$$\nabla\cdot\mathbf{v} = \frac{\partial v_k}{\partial x_k}, \qquad \nabla\times\mathbf{v} = \frac{\partial v_m}{\partial x_k} e_{kmn}\,\mathbf{i}_n. \tag{1.2}$$

Gauss theorem

We need the Gauss theorem to derive the equations of motion for a material. Here we simply state the theorem and express it in index notation. Consider a smooth, closed surface S with volume V (Fig. 1.2). Let $\mathbf{n}$ denote the unit vector that is perpendicular to S and directed outward. The Gauss theorem for a vector field $\mathbf{v}$ states that

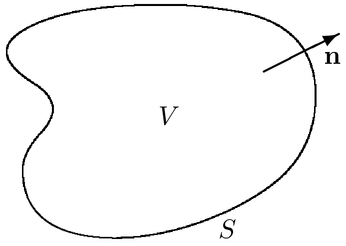


Figure 1.2: A closed surface S with volume V. The unit vector $\mathbf{n}$ is perpendicular to S.

$$\int_V \nabla\cdot\mathbf{v}\,dV = \int_S \mathbf{v}\cdot\mathbf{n}\,dS$$

when the integrands are continuous. In index notation, the Gauss theorem is

$$\int_V \frac{\partial v_k}{\partial x_k}\,dV = \int_S v_k n_k\,dS. \tag{1.3}$$

Exercises

In the following exercises, assume that the range of the indices is 1,2,3.

Exercise 1.1 Show that $\delta_{km} u_k v_m = u_k v_k$.

Discussion—Write the equation explicitly in terms of the numerical values of the indices and show that the right and left sides are equal. The right side of the equation in terms of the numerical values of the indices is $u_k v_k = u_1 v_1 + u_2 v_2 + u_3 v_3$.

Exercise 1.2

(a) Show that $\delta_{km} e_{kmn} = 0$.

(b) Show that $\delta_{km} \delta_{kn} = \delta_{mn}$.

Exercise 1.3 Show that $T_{km} \delta_{mn} = T_{kn}$.

Exercise 1.4 If $T_{km} e_{kmn} = 0$, show that $T_{km} = T_{mk}$.

Exercise 1.5 Consider the equation

$$a_k = \frac{\partial v_k}{\partial t} + \frac{\partial v_k}{\partial x_m} v_m.$$

(a) How many equations result when this equation is written explicitly in terms of the numerical values of the indices?

(b) Write the equations explicitly in terms of the numerical values of the indices.

Exercise 1.6 Consider the equation

$$T_{km} = c_{kmij} E_{ij}.$$

(a) How many equations result when this equation is written explicitly in terms of the numerical values of the indices?

(b) Write the equation for T_{11} explicitly in terms of the numerical values of the indices.

Exercise 1.7 The *del operator* is defined by

$$\begin{aligned} \nabla(\cdot) &= \frac{\partial(\cdot)}{\partial x_1} \mathbf{i}_1 + \frac{\partial(\cdot)}{\partial x_2} \mathbf{i}_2 + \frac{\partial(\cdot)}{\partial x_3} \mathbf{i}_3 \\ &= \frac{\partial(\cdot)}{\partial x_k} \mathbf{i}_k. \end{aligned} \tag{1.4}$$

The expression for the gradient of a scalar field ϕ in terms of cartesian coordinates can be obtained in a formal way by applying the del operator to the scalar field:

$$\nabla\phi = \frac{\partial\phi}{\partial x_k}\,\mathbf{i}_k.$$

(a) By taking the dot product of the del operator with a vector field $\mathbf{v}$, obtain the expression for the divergence $\nabla\cdot\mathbf{v}$ in terms of cartesian coordinates. (b) By taking the cross product of the del operator with a vector field $\mathbf{v}$, obtain the expression for the curl $\nabla\times\mathbf{v}$ in terms of cartesian coordinates.

Exercise 1.8 By expressing the dot and cross products in terms of index notation, show that for any two vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0.$$

Exercise 1.9 Show that

$$u_k v_m e_{kmn}\mathbf{i}_n = \det\begin{bmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

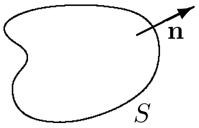
Exercise 1.10 By expressing the divergence and curl in terms of index notation, show that for any vector field $\mathbf{v}$ for which the indicated derivatives exist,

$$\nabla\cdot(\nabla\times\mathbf{v}) = 0.$$

Exercise 1.11 By expressing the operations on the left side of the equation in terms of index notation, show that for any vector field $\mathbf{v}$,

$$\nabla(\nabla\cdot\mathbf{v}) - \nabla\times(\nabla\times\mathbf{v}) = \frac{\partial^2 v_k}{\partial x_m \partial x_m}\,\mathbf{i}_k.$$

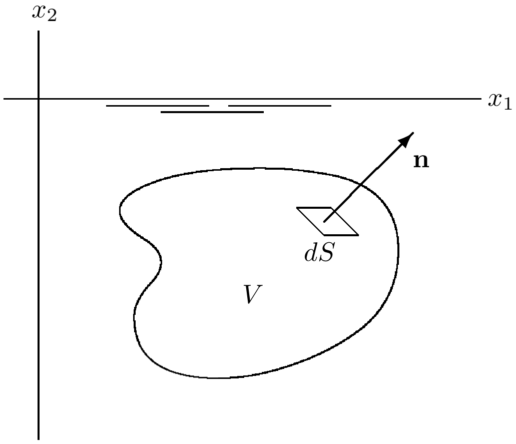
Exercise 1.12



Consider a smooth, closed surface S with outward-directed unit normal vector $\mathbf{n}$. Show that

$$\int_S \mathbf{n}\,dS = 0.$$

EXERCISE 1.13



An object of volume V is immersed in a stationary liquid. The pressure in the liquid is $p_o - \gamma x_2$, where p_o is the pressure at $x_2 = 0$ and γ is the weight per unit volume of the liquid. The force exerted by the pressure on an element dS of the surface of the object is $-(p_o - \gamma x_2)\,\mathbf{n}\,dS$, where $\mathbf{n}$ is an outward-directed unit vector perpendicular to the element dS. Show that the total force exerted on the object by the pressure of the liquid is $\gamma V\,\mathbf{i}_2$. (This result is due to Archimedes, 287-212 BCE).

1.2 Motion

We will explain how to describe a motion of a material and define its velocity and acceleration. The motion is described by specifying the position of the material relative to a *reference state*. Figure 1.3.a shows a sample of a material (imagine a blob of dough) in a reference state and also at a time t when the material has moved and deformed relative to its reference state. Consider a particular point of the material when it is in the reference state. (It's helpful to imagine marking the point with a pen.) As the material moves to its state at time t, the point moves along the path shown.

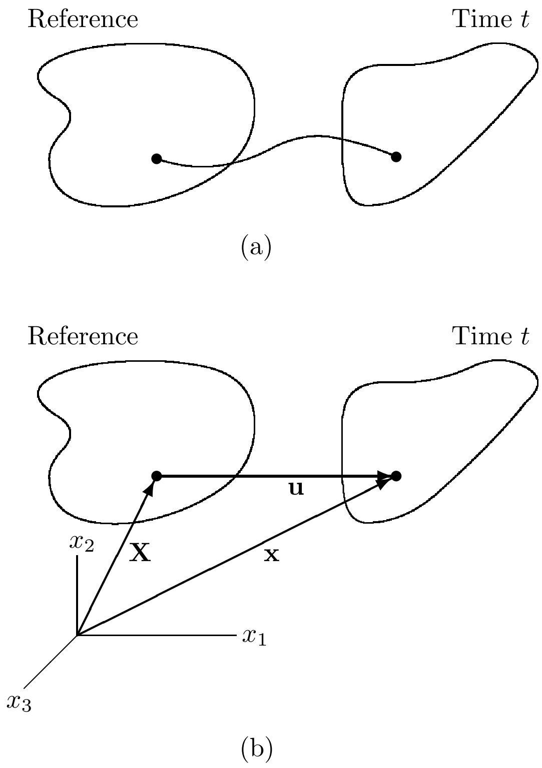


Figure 1.3: (a) A material point in the reference state and at time t. (b) The position vector $\mathbf{X}$ of the point in the reference state and the position vector $\mathbf{x}$ at time t. The vector $\mathbf{u}$ is the displacement.

Figure 1.3.b shows the position vectors of the point relative to the origin of a cartesian coordinate system when the material is in the reference state and at time t. The vector $\mathbf{X}$ is the position vector of the point when the material

is in the reference state and the vector $\mathbf{x}$ is the position vector of the point at time t. The *displacement* $\mathbf{u}$ is the position of the point at time t relative to its position in the reference state:

$$\mathbf{u} = \mathbf{x} - \mathbf{X},$$

or in index notation,

$$u_k = x_k - X_k. \tag{1.5}$$

The motion and inverse motion

To describe the motion of a material, we express the position of a point of the material as a function of the time and the position of that point in the reference state:

$$x_k = \hat{x}_k(X_m, t). \tag{1.6}$$

The term X_m in the argument of this function means that the function can depend on the components X_1, X_2, and X_3 of the vector $\mathbf{X}$. We use a circumflex (ˆ) to indicate that a function depends on the variables $\mathbf{X}$ and t. This function, which gives the position at time t of the point of the material that was at position $\mathbf{X}$ in the reference state, is called the *motion*. The position of a point of the material in the reference state can be specified as a function of its position at time t and the time:

$$X_k = X_k(x_m, t). \tag{1.7}$$

This function is called the *inverse motion*.

Velocity and acceleration

By using Eqs. (1.5) and (1.6), we can express the displacement as a function of $\mathbf{X}$ and t:

$$u_k = \hat{x}_k(X_m, t) - X_k = \hat{u}_k(X_m, t). \tag{1.8}$$

The velocity at time t of the point of the material that was at position $\mathbf{X}$ in the reference state is defined to be the rate of change of the position of the point:

$$v_k = \frac{\partial}{\partial t}\hat{x}_k(X_m, t) = \frac{\partial}{\partial t}\hat{u}_k(X_m, t) = \hat{v}_k(X_m, t), \tag{1.9}$$

The acceleration at time t of the point of the material that was at position $\mathbf{X}$ in the reference state is defined to be the rate of change of its velocity:

$$a_k = \frac{\partial}{\partial t}\hat{v}_k(X_m, t) = \hat{a}_k(X_m, t).$$

Material and spatial descriptions We have expressed the displacement, velocity, and acceleration of a point of the material in terms of the time and the position of the point in the reference state, that is, in terms of the variables $\mathbf{X}$ and t. By using the inverse motion, Eq. (1.7), we can express these quantities in terms of the time and the position of the point at time t, that is, in terms of the variables $\mathbf{x}$ and t. For example, we can write the displacement as

$$u_k = \hat{u}_k(X_m(x_n, t), t) = u_k(x_n, t). \tag{1.10}$$

Thus the displacement $\mathbf{u}$ can be expressed either by the function $\hat{u}_k(X_m, t)$ or by the function $u_k(x_m, t)$. Expressing a quantity as a function of $\mathbf{X}$ and t is called the *Lagrangian*, or *material* description. Expressing a quantity as a function of $\mathbf{x}$ and t is called the *Eulerian*, or *spatial* description. As Eq. (1.10) indicates, the value of a function does not depend on which description is used. However, when we take derivatives of a function it is essential to indicate clearly whether it is expressed in the material or spatial description. We use a hat ($\hat{}$) to signify that a function is expressed in the material description.

In Eq. (1.9), the velocity is expressed in terms of the time derivative of the material description of the displacement. Let us consider how we can express the velocity in terms of the spatial description of the displacement. Substituting the motion, Eq. (1.6), into the spatial description of the displacement, we obtain the equation

$$u_k = u_k(\hat{x}_n(X_m, t), t).$$

To determine the velocity, we take the time derivative of this expression with $\mathbf{X}$ held fixed:

$$\begin{aligned} v_k &= \frac{\partial u_k}{\partial t} + \frac{\partial u_k}{\partial x_n}\frac{\partial \hat{x}_n}{\partial t} \\ &= \frac{\partial u_k}{\partial t} + \frac{\partial u_k}{\partial x_n} v_n. \end{aligned} \tag{1.11}$$

This equation contains the velocity on the right side. By replacing the index k by n, it can be written as

$$v_n = \frac{\partial u_n}{\partial t} + \frac{\partial u_n}{\partial x_m} v_m.$$

We use this expression to replace the term v_n on the right side of Eq. (1.11), obtaining the equation

$$v_k = \frac{\partial u_k}{\partial t} + \frac{\partial u_k}{\partial x_n}\left(\frac{\partial u_n}{\partial t} + \frac{\partial u_n}{\partial x_m} v_m\right).$$

We can now use Eq. (1.11) to replace the term v_m on the right side of this equation. Continuing in this way, we obtain the series

$$v_k = \frac{\partial u_k}{\partial t} + \frac{\partial u_k}{\partial x_n}\frac{\partial u_n}{\partial t} + \frac{\partial u_k}{\partial x_n}\frac{\partial u_n}{\partial x_m}\frac{\partial u_m}{\partial t} + \cdots \tag{1.12}$$

This series gives the velocity in terms of derivatives of the spatial description of the displacement.

We leave it as an exercise to show that the acceleration is given in terms of derivatives of the spatial description of the velocity by

$$a_k = \frac{\partial v_k}{\partial t} + \frac{\partial v_k}{\partial x_n} v_n. \tag{1.13}$$

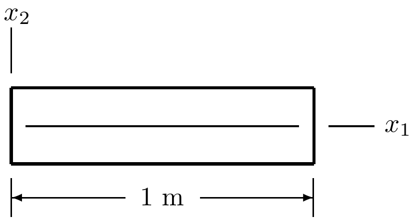
The second term on the right is sometimes called the convective acceleration.

Velocity and acceleration in linear elasticity In *linear elasticity*, derivatives of the displacement are assumed to be "small," meaning that terms containing products of derivatives of the displacement can be neglected. From Eqs. (1.12) and (1.13), we see that in linear elasticity, the velocity and acceleration are related to the spatial description of the displacement by the simple expressions

$$v_k = \frac{\partial u_k}{\partial t}, \qquad a_k = \frac{\partial^2 u_k}{\partial t^2}. \tag{1.14}$$

Exercises

Exercise 1.14



A 1-meter bar of material is subjected to the "stretching" motion

$$x_1 = X_1(1+t^2), \quad x_2 = X_2, \quad x_3 = X_3.$$

(a) Determine the inverse motion of the bar.
(b) Determine the material description of the displacement.
(c) Determine the spatial description of the displacement.

(d) What is the displacement of a point at the right end of the bar when $t = 2$ seconds?

Answer:

$$\begin{array}{lll}
\text{(a)}\ X_1 = x_1/(1+t^2), & X_2 = x_2, & X_3 = x_3. \\
\text{(b)}\ u_1 = X_1 t^2, & u_2 = 0, & u_3 = 0. \\
\text{(c)}\ u_1 = x_1 t^2/(1+t^2), & u_2 = 0, & u_3 = 0. \\
\text{(d)}\ u_1 = 4 \text{ meters}, & u_2 = 0, & u_3 = 0.
\end{array}$$

Exercise 1.15 Consider the motion of the bar described in Exercise 1.14.

(a) Determine the material description of the velocity.

(b) Determine the spatial description of the velocity.

(c) Determine the material description of the acceleration.

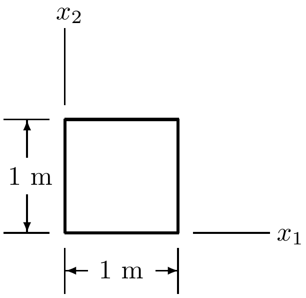
(d) Determine the spatial description of the acceleration by substituting the inverse motion into the result of Part (c).

(e) Determine the spatial description of the acceleration by using Eq. (1.13).

Answer:

$$\begin{array}{clll}
\text{(a)} & v_1 = 2X_1 t, & v_2 = 0, & v_3 = 0. \\
\text{(b)} & v_1 = 2x_1 t/(1+t^2), & v_2 = 0, & v_3 = 0. \\
\text{(c)} & a_1 = 2X_1, & a_2 = 0, & a_3 = 0. \\
\text{(d),(e)} & a_1 = 2x_1/(1+t^2), & a_2 = 0, & a_3 = 0.
\end{array}$$

Exercise 1.16



A 1-meter cube of material is subjected to the "shearing" motion

$$x_1 = X_1 + X_2^2 t^2, \quad x_2 = X_2, \quad x_3 = X_3.$$

(a) Determine the inverse motion of the cube.

(b) Determine the material description of the displacement.

(c) Determine the spatial description of the displacement.

(d) What is the displacement of a point at the upper right edge of the cube when $t = 2$ seconds?

Answer:

$$\begin{array}{llll} \text{(a)} & X_1 = x_1 - x_2^2 t^2, & X_2 = x_2, & X_3 = x_3. \\ \text{(b)} & u_1 = X_2^2 t^2, & u_2 = 0, & u_3 = 0. \\ \text{(c)} & u_1 = x_2^2 t^2, & u_2 = 0, & u_3 = 0. \\ \text{(d)} & u_1 = 4 \text{ meters}, & u_2 = 0, & u_3 = 0. \end{array}$$

Exercise 1.17 Consider the motion of the cube described in Exercise 1.16.

(a) Determine the material description of the velocity.

(b) Determine the spatial description of the velocity.

(c) Determine the material description of the acceleration.

(d) Determine the spatial description of the acceleration by substituting the inverse motion into the result of Part (c).

(e) Determine the spatial description of the acceleration by using Eq. (1.13).

Answer:

$$\begin{array}{llll} \text{(a)} & v_1 = 2X_2^2 t, & v_2 = 0, & v_3 = 0. \\ \text{(b)} & v_1 = 2x_2^2 t, & v_2 = 0, & v_3 = 0. \\ \text{(c)} & a_1 = 2X_2^2, & a_2 = 0, & a_3 = 0. \\ \text{(d),(e)} & a_1 = 2x_2^2, & a_2 = 0, & a_3 = 0. \end{array}$$

Exercise 1.18 Show that the acceleration is given in terms of derivatives of the spatial description of the velocity by

$$a_k = \frac{\partial v_k}{\partial t} + \frac{\partial v_k}{\partial x_n} v_n.$$

Discussion—Begin by substituting the motion,Eq. (1.6), into the spatial description of the velocity to obtain the equation

$$v_k = v_k(\hat{x}_n(X_m, t), t).$$

Then take the time derivative of this expression with **X** held fixed.

1.3 Deformation

We will discuss the deformation of a material relative to a reference state and show that the deformation can be expressed in terms of the displacement field of the material. We also show how changes in the volume and density of the material are related to the displacement field.

Strain tensor

Suppose that a sample of material is in a reference state, and consider two neighboring points of the material with positions $\mathbf{X}$ and $\mathbf{X} + d\mathbf{X}$ (Fig. 1.4). At time t, these two points will have some positions $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$, as shown in the figure.

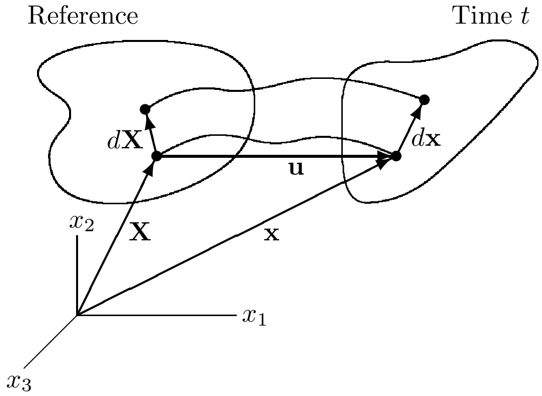


Figure 1.4: Two neighboring points of the material in the reference state and at time t.

In terms of the motion, Eq. (1.6), the vector $d\mathbf{x}$ is related to the vector $d\mathbf{X}$ by

$$dx_k = \frac{\partial \hat{x}_k}{\partial X_m} dX_m. \tag{1.15}$$

We denote the magnitudes of the vectors $d\mathbf{x}$ and $d\mathbf{X}$ by

$$|d\mathbf{x}| = ds, \qquad |d\mathbf{X}| = dS.$$

These magnitudes can be written in terms of the dot products of the vectors:

$$\begin{aligned} dS^2 &= dX_k dX_k, \\ ds^2 &= dx_k dx_k = \frac{\partial \hat{x}_k}{\partial X_m} dX_m \frac{\partial \hat{x}_k}{\partial X_n} dX_n. \end{aligned}$$

The difference between the quantities ds^2 and dS^2 is

$$ds^2 - dS^2 = \left(\frac{\partial \hat{x}_k}{\partial X_m} \frac{\partial \hat{x}_k}{\partial X_n} - \delta_{mn} \right) dX_m dX_n.$$

By using Eq. (1.5), we can write this equation in terms of the displacement $\mathbf{u}$:

$$ds^2 - dS^2 = \left(\frac{\partial \hat{u}_m}{\partial X_n} + \frac{\partial \hat{u}_n}{\partial X_m} + \frac{\partial \hat{u}_k}{\partial X_m} \frac{\partial \hat{u}_k}{\partial X_n} \right) dX_m dX_n.$$

We can write this equation as

$$ds^2 - dS^2 = 2E_{mn} dX_m dX_n, \tag{1.16}$$

where

$$E_{mn} = \frac{1}{2} \left(\frac{\partial \hat{u}_m}{\partial X_n} + \frac{\partial \hat{u}_n}{\partial X_m} + \frac{\partial \hat{u}_k}{\partial X_m} \frac{\partial \hat{u}_k}{\partial X_n} \right) \tag{1.17}$$

The term E_{mn} is called the *Lagrangian strain tensor.* This quantity is a measure of the deformation of the material relative to its reference state. If it is known at a given point, we can choose any line element $d\mathbf{X}$ at that point in the reference state and use Eq. (1.16) to determine its length ds at time t.

Recall that in linear elasticity it is assumed that derivatives of the displacement are small. We leave it as an exercise to show that in linear elasticity,

$$\frac{\partial \hat{u}_k}{\partial X_m} = \frac{\partial u_k}{\partial x_m}. \tag{1.18}$$

Therefore in linear elasticity, the strain tensor is

$$E_{mn} = \frac{1}{2} \left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} \right), \tag{1.19}$$

which is called the *linear strain tensor.*

Longitudinal strain

The *longitudinal strain* is a measure of the change in length of a line element in a material. Here we show how the longitudinal strain in an arbitrary direction can be expressed in terms of the strain tensor.

The longitudinal strain ε of a line element relative to a reference state is defined to be its change in length divided by its length in the reference state. Thus the longitudinal strain of the line element $d\mathbf{X}$ is

$$\varepsilon = \frac{|d\mathbf{x}| - |d\mathbf{X}|}{|d\mathbf{X}|} = \frac{ds - dS}{dS}. \tag{1.20}$$

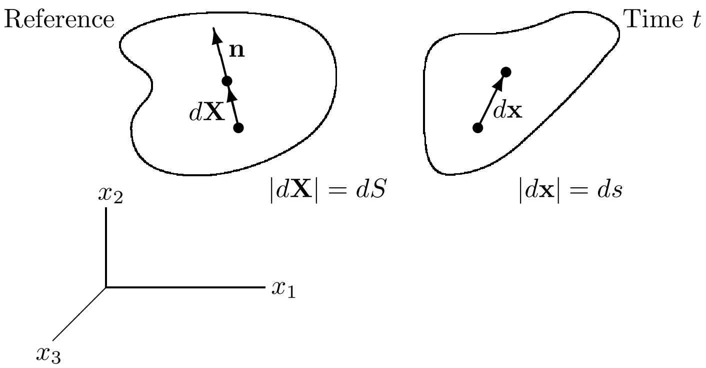


Figure 1.5: The vectors $d\mathbf{X}$ and $d\mathbf{x}$ and a unit vector $\mathbf{n}$ in the direction of $d\mathbf{X}$.

Solving this equation for ds and substituting the result into Eq. (1.16), we obtain the relation

$$(2\varepsilon + \varepsilon^2) dS^2 = 2E_{mn} dX_m dX_n. \tag{1.21}$$

Let $\mathbf{n}$ be a unit vector in the direction of the vector $d\mathbf{X}$ (Fig. 1.5). We can write $d\mathbf{X}$ as

$$d\mathbf{X} = dS\,\mathbf{n}, \quad \text{or} \quad dX_m = dS\,n_m.$$

We substitute this relation into Eq. (1.21) and write the resulting equation in the form

$$2\varepsilon + \varepsilon^2 = 2E_{mn} n_m n_n. \tag{1.22}$$

When the Lagrangian strain tensor E_{mn} is known at a point in the material, we can solve this equation for the longitudinal strain ε of the line element that is parallel to the unit vector $\mathbf{n}$ in the reference state.

Equation (1.22) becomes very simple in linear elasticity. If we assume that ε is small, the quadratic term can be neglected and we obtain

$$\varepsilon = E_{mn} n_m n_n. \tag{1.23}$$

We can determine the longitudinal strain of the line element of the material that is parallel to the x_1 axis in the reference state by setting $n_1 = 1$, $n_2 = 0$, and $n_3 = 0$. The result is $\varepsilon = E_{11}$. Thus in linear elasticity, the component E_{11} of the linear strain tensor is equal to the longitudinal strain in the x_1 axis direction. Similarly, the components E_{22} and E_{33} are the longitudinal strains in the x_2 and x_3 axis directions.

Shear strain

The *shear strain* is a measure of the change in the angle between two line elements in a material relative to a reference state. Here we show how the shear strains at a point in the material that measure the changes in angle between line elements that are parallel to the coordinate axes can be expressed in terms of the strain tensor at that point.

Figure 1.6 shows a point of the material at position $\mathbf{X}$ in the reference state and two neighboring points at positions $\mathbf{X} + d\mathbf{X}_{(1)}$ and $\mathbf{X} + d\mathbf{X}_{(2)}$, where we define $d\mathbf{X}_{(1)}$ and $d\mathbf{X}_{(2)}$ by

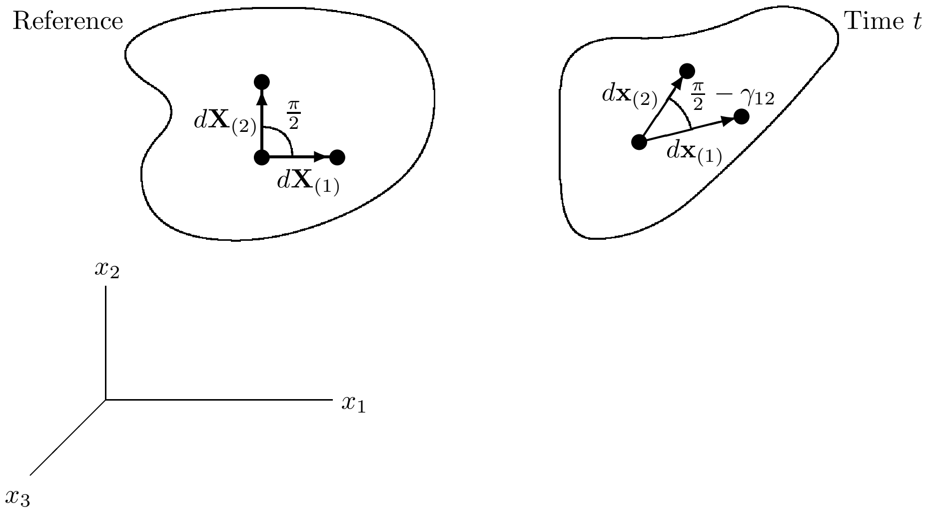


Figure 1.6: A point of the material and two neighboring points in the reference state and at time t.

$$d\mathbf{X}_{(1)} = dS\,\mathbf{i}_1, \quad d\mathbf{X}_{(2)} = dS\,\mathbf{i}_2.$$

(We use parentheses around the subscripts to emphasize that they are not indices.) At time t, the two neighboring points will be at some positions $\mathbf{x} + d\mathbf{x}_{(1)}$ and $\mathbf{x} + d\mathbf{x}_{(2)}$. From Eq. (1.15), we see that

$$d\mathbf{x}_{(1)} = \frac{\partial \hat{x}_k}{\partial X_1} dS\,\mathbf{i}_k, \qquad d\mathbf{x}_{(2)} = \frac{\partial \hat{x}_k}{\partial X_2} dS\,\mathbf{i}_k. \tag{1.24}$$

The angle between the vectors $d\mathbf{X}_{(1)}$ and $d\mathbf{X}_{(2)}$ is $\pi/2$ radians. Let us denote the angle between the vectors $d\mathbf{x}_{(1)}$ and $d\mathbf{x}_{(2)}$ by $\pi/2 - \gamma_{12}$. The angle γ_{12} is a measure of the change in the right angle between the line elements $d\mathbf{X}_{(1)}$

and $d\mathbf{X}_{(2)}$ at time t. This angle is called the shear strain associated with the line elements $d\mathbf{X}_{(1)}$ and $d\mathbf{X}_{(2)}$.

We can obtain an expression relating the shear strain γ_{12} to the displacement by using the definition of the dot product of the vectors $d\mathbf{x}_{(1)}$ and $d\mathbf{x}_{(2)}$:

$$d\mathbf{x}_{(1)} \cdot d\mathbf{x}_{(2)} = |d\mathbf{x}_{(1)}||d\mathbf{x}_{(2)}| \cos(\pi/2 - \gamma_{12}). \tag{1.25}$$

From Eqs. (1.24), we see that the dot product of $d\mathbf{x}_{(1)}$ and $d\mathbf{x}_{(2)}$ is given by

$$d\mathbf{x}_{(1)} \cdot d\mathbf{x}_{(2)} = \frac{\partial \hat{x}_k}{\partial X_1}\frac{\partial \hat{x}_k}{\partial X_2} dS^2.$$

By using Eq. (1.5), we can write this expression in terms of the displacement:

$$\begin{aligned} d\mathbf{x}_{(1)} \cdot d\mathbf{x}_{(2)} &= \left(\frac{\partial \hat{u}_1}{\partial X_2} + \frac{\partial \hat{u}_2}{\partial X_1} + \frac{\partial \hat{u}_k}{\partial X_1}\frac{\partial \hat{u}_k}{\partial X_2} \right) dS^2 \\ &= 2E_{12}\, dS^2, \end{aligned} \tag{1.26}$$

where we have used the definition of the Lagrangian strain tensor, Eq. (1.17).

Let us denote the magnitudes of the vectors $d\mathbf{x}_{(1)}$ and $d\mathbf{x}_{(2)}$ by $|d\mathbf{x}_{(1)}| = ds_{(1)}$, $|d\mathbf{x}_2| = ds_{(2)}$. From Eq. (1.16),

$$ds^2_{(1)} = (1 + 2E_{11})dS^2, \quad ds^2_{(2)} = (1 + 2E_{22})dS^2.$$

We substitute these expressions and Eq. (1.26) into Eq. (1.25), obtaining the relation

$$2E_{12} = (1 + 2E_{11})^{1/2}(1 + 2E_{22})^{1/2} \cos(\pi/2 - \gamma_{12}). \tag{1.27}$$

We can use this equation to determine the shear strain γ_{12} when the displacement field is known. The term γ_{12} measures the shear strain between line elements parallel to the x_1 and x_2 axes. The corresponding relations for the shear strain between line elements parallel to the x_2 and x_3 axes and for the shear strain between line elements parallel to the x_1 and x_3 axes are

$$\begin{aligned} 2E_{23} &= (1 + 2E_{22})^{1/2}(1 + 2E_{33})^{1/2} \cos(\pi/2 - \gamma_{23}), \\ 2E_{13} &= (1 + 2E_{11})^{1/2}(1 + 2E_{33})^{1/2} \cos(\pi/2 - \gamma_{13}). \end{aligned}$$

In linear elasticity, the terms E_{mn} are small. We leave it as an exercise to show that in linear elasticity these expressions for the shear strains reduce to the simple relations

$$\gamma_{12} = 2E_{12}, \quad \gamma_{23} = 2E_{23}, \quad \gamma_{13} = 2E_{13}.$$

Thus in linear elasticity, the shear strains associated with line elements parallel to the coordinate axes are directly related to elements of the linear strain tensor.

Changes in volume and density

We will show how volume elements and the density of a material change as it deforms. We do so by making use of a result from vector analysis. Figure 1.7 shows a parallelepiped with vectors **u**, **v**, and **w** along its edges. The volume of the parallelepiped is given in terms of the three vectors by the expression

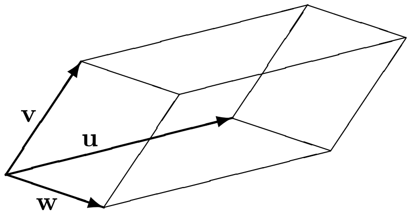


Figure 1.7: A parallelepiped with vectors **u**, **v**, and **w** along its edges.

$$\text{Volume } = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}. \tag{1.28}$$

Now consider an element of volume dV_0 of the reference state of a sample of material with the vectors

$$d\mathbf{X}_{(1)} = dS\,\mathbf{i}_1, \quad d\mathbf{X}_{(2)} = dS\,\mathbf{i}_2, \quad d\mathbf{X}_{(3)} = dS\,\mathbf{i}_3$$

along its edges (Fig. 1.8). At time t, the material occupying the volume dV_0 in the reference state will occupy some volume dV. From Eq. (1.15), we see that the vectors $d\mathbf{x}_{(1)}$, $d\mathbf{x}_{(2)}$, and $d\mathbf{x}_{(3)}$ along the sides of the volume dV are given in terms of the motion by

$$d\mathbf{x}_{(1)} = \frac{\partial \hat{x}_k}{\partial X_1} dS\,\mathbf{i}_k, \quad d\mathbf{x}_{(2)} = \frac{\partial \hat{x}_k}{\partial X_2} dS\,\mathbf{i}_k, \quad d\mathbf{x}_{(3)} = \frac{\partial \hat{x}_k}{\partial X_3} dS\,\mathbf{i}_k.$$

From Eq. (1.28) we see that the volume dV is

$$dV = d\mathbf{x}_{(1)}\cdot(d\mathbf{x}_{(2)}\times d\mathbf{x}_{(3)}) = \det\begin{bmatrix} \dfrac{\partial \hat{x}_1}{\partial X_1} dS & \dfrac{\partial \hat{x}_2}{\partial X_1} dS & \dfrac{\partial \hat{x}_3}{\partial X_1} dS \\[2ex] \dfrac{\partial \hat{x}_1}{\partial X_2} dS & \dfrac{\partial \hat{x}_2}{\partial X_2} dS & \dfrac{\partial \hat{x}_3}{\partial X_2} dS \\[2ex] \dfrac{\partial \hat{x}_1}{\partial X_3} dS & \dfrac{\partial \hat{x}_2}{\partial X_3} dS & \dfrac{\partial \hat{x}_3}{\partial X_3} dS \end{bmatrix}.$$

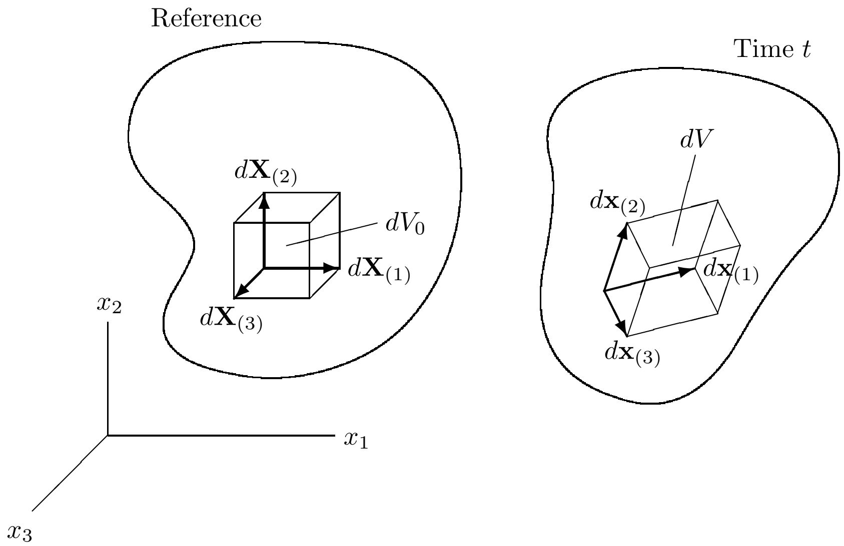


Figure 1.8: The material occupying volume dV_0 in the reference state and a volume dV at time t.

Because the volume $dV_0 = dS^3$, we can write this result as

$$dV = \det \begin{bmatrix} \dfrac{\partial \hat{x}_1}{\partial X_1} & \dfrac{\partial \hat{x}_2}{\partial X_1} & \dfrac{\partial \hat{x}_3}{\partial X_1} \\[2ex] \dfrac{\partial \hat{x}_1}{\partial X_2} & \dfrac{\partial \hat{x}_2}{\partial X_2} & \dfrac{\partial \hat{x}_3}{\partial X_2} \\[2ex] \dfrac{\partial \hat{x}_1}{\partial X_3} & \dfrac{\partial \hat{x}_2}{\partial X_3} & \dfrac{\partial \hat{x}_3}{\partial X_3} \end{bmatrix} dV_0.$$

By using Eq. (1.5), we can write this relation in terms of the displacement field:

$$dV = \det \begin{bmatrix} 1 + \dfrac{\partial \hat{u}_1}{\partial X_1} & \dfrac{\partial \hat{u}_2}{\partial X_1} & \dfrac{\partial \hat{u}_3}{\partial X_1} \\[2ex] \dfrac{\partial \hat{u}_1}{\partial X_2} & 1 + \dfrac{\partial \hat{u}_2}{\partial X_2} & \dfrac{\partial \hat{u}_3}{\partial X_2} \\[2ex] \dfrac{\partial \hat{u}_1}{\partial X_3} & \dfrac{\partial \hat{u}_2}{\partial X_3} & 1 + \dfrac{\partial \hat{u}_3}{\partial X_3} \end{bmatrix} dV_0. \qquad (1.29)$$

When the displacement field is known, we can use this equation to determine the volume dV at time t of an element of the material that occupied a volume dV_0 in the reference state.

The density ρ of a material is defined such that the mass of an element of volume dV of the material at time t is

$$\text{mass } = \rho\, dV.$$

We denote the density of the material in the reference state by ρ_0. Conservation of mass of the material requires that

$$\rho_0\, dV_0 = \rho\, dV. \tag{1.30}$$

From this relation and Eq. (1.29), we obtain an equation for the density of the material ρ at time t in terms of the displacement field:

$$\frac{\rho_0}{\rho} = \det \begin{bmatrix} 1 + \dfrac{\partial \hat{u}_1}{\partial X_1} & \dfrac{\partial \hat{u}_2}{\partial X_1} & \dfrac{\partial \hat{u}_3}{\partial X_1} \\ \dfrac{\partial \hat{u}_1}{\partial X_2} & 1 + \dfrac{\partial \hat{u}_2}{\partial X_2} & \dfrac{\partial \hat{u}_3}{\partial X_2} \\ \dfrac{\partial \hat{u}_1}{\partial X_3} & \dfrac{\partial \hat{u}_2}{\partial X_3} & 1 + \dfrac{\partial \hat{u}_3}{\partial X_3} \end{bmatrix}. \tag{1.31}$$

In linear elasticity, products of derivatives of the displacement are neglected. Expanding the determinant in Eq. (1.29) and using Eq. (1.18), we find that in linear elasticity the volume dV at time t is given by the simple relation

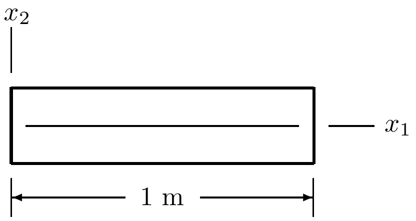
$$dV = \left(1 + \frac{\partial u_k}{\partial x_k}\right) dV_0 = (1 + E_{kk})\ dV_0. \tag{1.32}$$

From this result and Eq. (1.30), it is easy to show that in linear elasticity, the density of the material at time t is related to the displacement field by

$$\rho = (1 - E_{kk})\,\rho_0 = \left(1 - \frac{\partial u_k}{\partial x_k}\right)\rho_0. \tag{1.33}$$

Exercises

Exercise 1.19



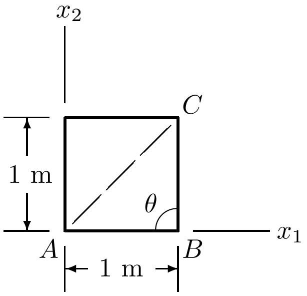
A 1-meter bar of material is subjected to the "stretching" motion

$$x_1 = X_1(1+t^2), \quad x_2 = X_2, \quad x_3 = X_3.$$

(a) Determine the Lagrangian strain tensor of the bar at time t.
(b) Use the motion to determine the length of the bar when $t = 2$ seconds.
(c) Use the Lagrangian strain tensor to determine the length of the bar when $t = 2$ seconds.

Answer: (a) The only nonzero term is $E_{11} = t^2 + \frac{1}{2}t^4$. (b) 5 meters

Exercise 1.20



An object consists of a 1-meter cube in the reference state. At time t, the value of the linear strain tensor at each point of the object is

$$[E_{km}] = \begin{bmatrix} 0.001 & -0.001 & 0 \\ -0.001 & 0.002 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) What is the length of the edge AB at time t?
(b) What is the length of the diagonal AC at time t?
(c) What is the angle θ at time t?

Answer: (a) 1.001 m (b) $1.0005\sqrt{2}$ m (c) $(\pi/2 - 0.002)$ rad

Exercise 1.21 Show that in linear elasticity,

$$\frac{\partial \hat{u}_k}{\partial X_m} = \frac{\partial u_k}{\partial x_m}.$$

Discussion—Use the chain rule to write

$$\frac{\partial \hat{u}_k}{\partial X_m} = \frac{\partial u_k}{\partial x_n}\frac{\partial \hat{x}_n}{\partial X_m},$$

and use Eq. (1.5).

Exercise 1.22 Consider Eq. (1.27):

$$2E_{12} = (1 + 2E_{11})^{1/2}(1 + 2E_{22})^{1/2} \cos(\pi/2 - \gamma_{12}).$$

In linear elasticity, the terms E_{11}, E_{22}, E_{12}, and the shear strain γ_{12} are small. Show that in linear elasticity this relation reduces to

$$\gamma_{12} = 2E_{12}.$$

Exercise 1.23 By using Eqs. (1.30) and (1.32), show that in linear elasticity the density ρ is related to the density ρ_0 in the reference state by

$$\rho = \rho_0(1 - E_{kk}).$$

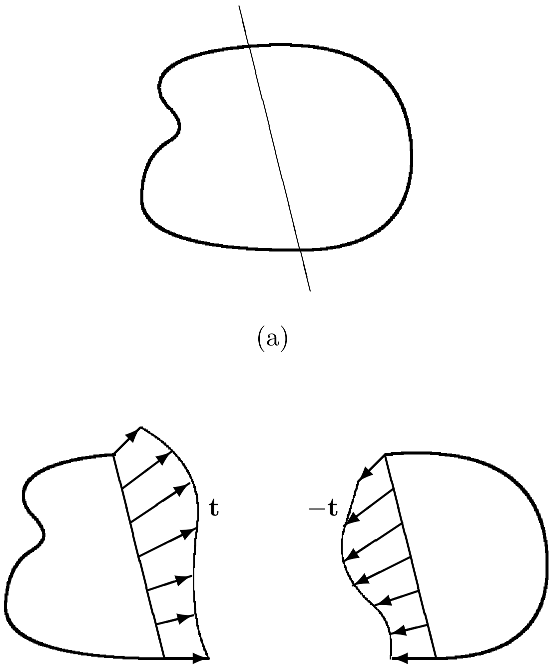
Discussion—Remember that in linear elasticity it is assumed that derivatives of the displacement are small.

1.4 Stress

Here we discuss the internal forces, or stresses, in materials and define the stress tensor.

Traction

Suppose that we divide a sample of material by an imaginary plane (Fig. 1.9.a) and draw the free-body diagrams of the resulting parts (Fig. 1.9.b). The material on the right may exert forces on the material on the left at the surface defined by the plane. We can represent those forces by introducing a function $\mathbf{t}$, the *traction*, defined such that the force exerted on an element dS of the surface is



(a)

(b)

Figure 1.9: (a) A plane passing through a sample of material. (b) The free-body diagrams of the parts of the material on each side of the plane.

$$\text{force } = \mathbf{t}\, dS.$$

Because an equal and opposite force is exerted on the element dS of the material on the right by the material on the left, the traction acting on the material on the right at the surface defined by the plane is $-\mathbf{t}$ (Fig. 1.9.b).

The traction $\mathbf{t}$ is a vector-valued function. We can resolve it into a component normal to the surface, the *normal stress* σ, and a component tangent to the surface, the *shear stress* τ (Fig. 1.10). If $\mathbf{n}$ is a unit vector that is normal to the surface and points outward, the normal stress $\sigma = \mathbf{t} \cdot \mathbf{n}$ and the shear stress τ is the magnitude of the vector $\mathbf{t} - \sigma\mathbf{n}$. As a simple example, if the material is a fluid at rest, the normal stress $\sigma = -p$, where p is the pressure, and the shear stress $\tau = 0$.

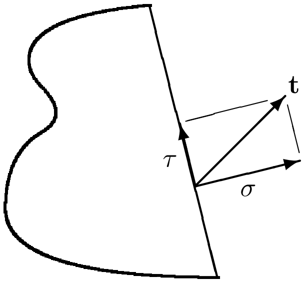


Figure 1.10: The traction $\mathbf{t}$ resolved into the normal stress σ and shear stress τ.

Stress tensor

Consider a point P of a sample of material at time t. Let us introduce a cartesian coordinate system and pass a plane through point P perpendicular to the x_1 axis (Fig. 1.11.a). The free-body diagram of the material on the side of the plane toward the negative x_1 direction is shown in Fig. 1.11.b. Let the traction vector acting on the plane at point P be $\mathbf{t}_{(1)}$. The component of $\mathbf{t}_{(1)}$ normal to the plane—that is, the normal stress acting on the plane at point P—is called T_{11}. The component of $\mathbf{t}_{(1)}$ tangential to the plane (that is, the shear stress acting on the plane at point P) is decomposed into a component T_{12} in the x_2 direction and a component T_{13} in the x_3 direction.

Next, we pass a plane through point P perpendicular to the x_2 axis. The resulting free-body diagram is shown in Fig. 1.11.c. The traction vector $\mathbf{t}_{(2)}$ acting on the plane at point P is decomposed into the normal stress T_{22} and shear stress components T_{21} and T_{23}. In the same way, we pass a plane through point P perpendicular to the x_3 axis and decompose the traction vector $\mathbf{t}_{(3)}$ acting on the plane at point P into the normal stress T_{33} and shear stress

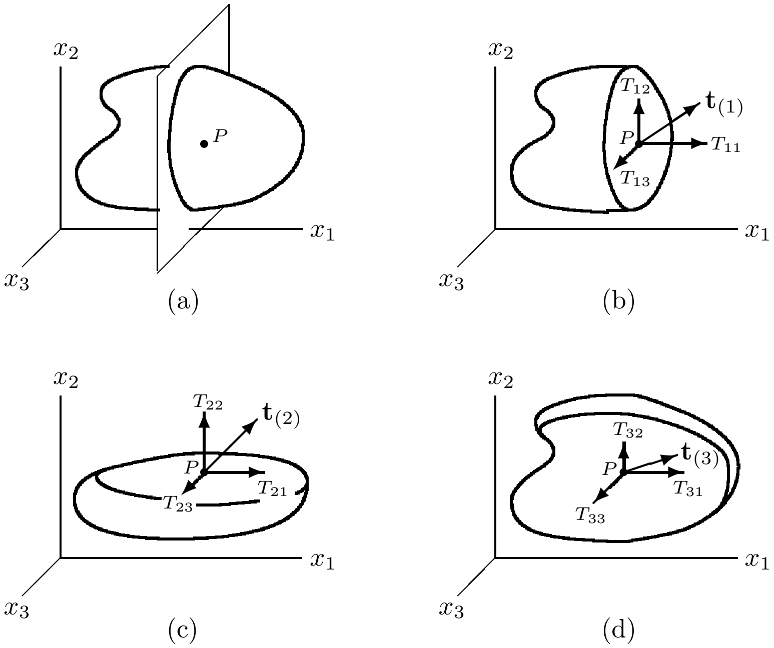


Figure 1.11: A point P of a sample of material and free-body diagrams obtained by passing planes through P perpendicular to the x_1, x_2, and x_3 axes.

components T_{31} and T_{32} (Fig. 1.11.d).

We see that when the components of the *stress tensor*

$$[T_{km}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

are known at a point P, the traction vectors acting on three mutually perpendicular planes through the point P are known. Note that the component T_{km} is the stress acting on the plane perpendicular to the x_k axis in the x_m direction. An exercise later in this chapter will show that the components of the stress tensor are symmetric: $T_{km} = T_{mk}$.

Tetrahedron argument

We have shown that when the components of the stress tensor are known at a point, we know the traction vectors at that point acting on planes perpendicular to the three coordinate axes. However, suppose that we want to know the traction vector acting on a plane that is not perpendicular to one of the coordinate axes. A derivation called the tetrahedron argument shows that when the components T_{km} are known at a point P, the traction vector acting on any plane through P can be determined.

Consider a point P of a sample of material at time t. Let us introduce a tetrahedral volume with P at one vertex (Fig. 1.12.a). Three of the surfaces

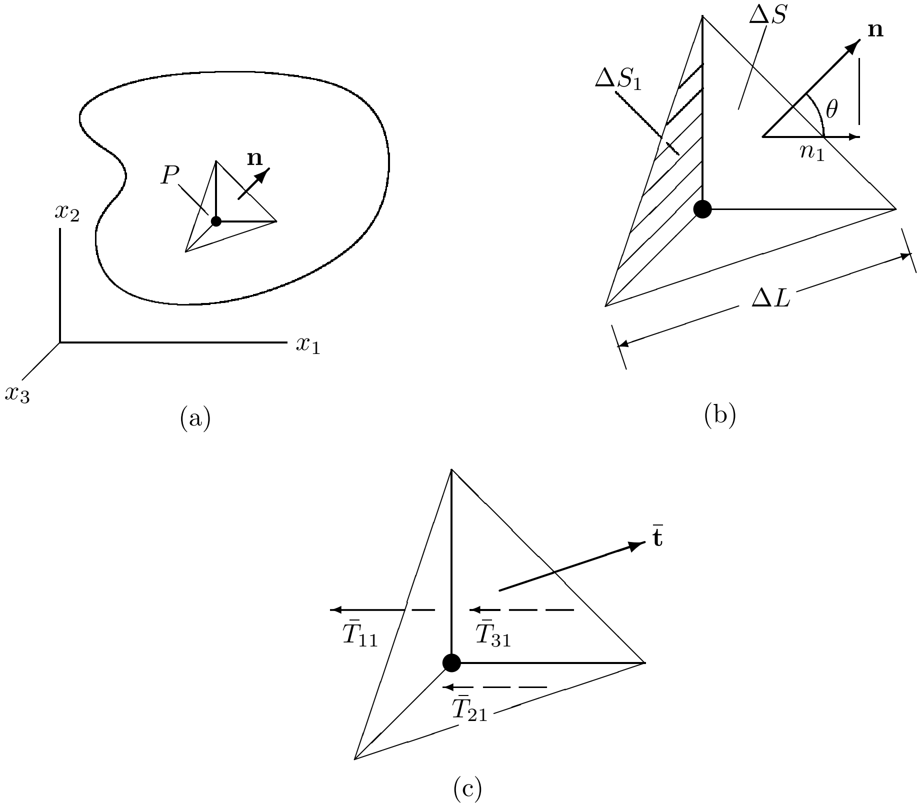


Figure 1.12: (a) A point P of a material and a tetrahedron. (b) The tetrahedron. (c) The stresses acting on the faces.

of the tetrahedron are perpendicular to the coordinate axes, and the point P is at the point where they intersect. The fourth surface of the tetrahedron is perpendicular to a unit vector $\mathbf{n}$. Thus the unit vector $\mathbf{n}$ specifies the orientation of the fourth surface.

Let the area of the surface perpendicular to $\mathbf{n}$ be ΔS, and let the area of the surface perpendicular to the x_1 axis be ΔS_1 (Fig. 1.12.b). If we denote the angle between the unit vector $\mathbf{n}$ and the x_1 axis by θ, the x_1 component of $\mathbf{n}$

is $n_1 = \cos\theta$. Because $\mathbf{n}$ is perpendicular to the surface with area ΔS and the x_1 axis is perpendicular to the surface with area ΔS_1, the angle between the two surfaces is θ. Therefore

$$\begin{aligned} \Delta S_1 &= \Delta S \cos\theta \\ &= \Delta S\, n_1. \end{aligned}$$

By using the same argument for the surfaces perpendicular to the x_2 and x_3 axes, we find that

$$\Delta S_k = \Delta S\, n_k, \quad k = 1, 2, 3. \tag{1.34}$$

Figure 1.12.c shows a free-body diagram of the material contained in the tetrahedron. To obtain the result we seek we must write Newton's second law for this material. The vector $\bar{\mathbf{t}}$ shown in Fig. 1.12.c is the average value of the traction vector acting on the surface with area ΔS, defined by

$$\Delta S\, \bar{\mathbf{t}} = \int_{\Delta S} \mathbf{t}\, dS.$$

The x_1 component of the force exerted on the material by the average traction $\bar{\mathbf{t}}$ is $\bar{t}_1 \Delta S$. Figure 1.12.c also shows the stress components acting on the other surfaces of the tetrahedron that exert forces in the x_1 direction. These terms are also average values; for example, $\bar{T}_{11}$ is defined by

$$\Delta S_1\, \bar{T}_{11} = \int_{\Delta S_1} T_{11}\, dS.$$

In addition to the surface tractions, we must consider *body forces* distributed over the volume of the material, such as its weight. Let $\bar{\mathbf{b}}$ denote the average value of the force per unit volume acting on the material. Thus the force exerted on the material in the x_1 direction by body forces is $\bar{b}_1 \Delta V$, where ΔV is the volume of the tetrahedron.

The x_1 component of Newton's second law for the tetrahedron is

$$\bar{t}_1 \Delta S - \bar{T}_{11} \Delta S_1 - \bar{T}_{21} \Delta S_2 - \bar{T}_{31} \Delta S_3 + \bar{b}_1 \Delta V = \bar{\rho} \Delta V a_1,$$

where $\bar{\rho}$ is the average density of the material and a_1 is the x_1 component of the acceleration of the center of mass of the material. Dividing this equation by ΔS and using Eq. (1.34), we obtain the expression

$$\bar{t}_1 - \bar{T}_{11} n_1 - \bar{T}_{21} n_2 - \bar{T}_{31} n_3 + \bar{b}_1 \frac{\Delta V}{\Delta S} = \bar{\rho} \frac{\Delta V}{\Delta S} a_1. \tag{1.35}$$

We now evaluate the limit of this equation as the linear dimension of the tetrahedron decreases; that is, as $\Delta L \to 0$ (Fig. 1.12.b). Because ΔS is proportional to $(\Delta L)^2$ and ΔV is proportional to $(\Delta L)^3$, the two terms in Eq. (1.35) containing $\Delta V/\Delta S$ approach zero. The average stress components in Eq. (1.35) approach their values at point P, and the average traction vector $\bar{\mathbf{t}}$ approaches the value of the traction vector $\mathbf{t}$ at point P acting on the plane perpendicular to $\mathbf{n}$. Thus we obtain the equation

$$t_1 = T_{11}n_1 + T_{21}n_2 + T_{31}n_3.$$

By writing the x_2 and x_3 components of Newton's second law for the material in the same way, we find that the components of the traction vector $\mathbf{t}$ are

$$t_k = T_{mk}n_m. \tag{1.36}$$

This important result can be stated as follows: Consider a point P of a material at time t (Fig. 1.13.a), and suppose that we know the components T_{km} of the stress tensor at point P. Let us pass a plane through P and draw a free-body diagram of the material on one side of the plane (Fig. 1.13.b). The unit vector $\mathbf{n}$ is perpendicular to the plane and points outward, that is, it points away from the material. Then the components of the traction vector $\mathbf{t}$ acting on the plane at point P are given by Eq. (1.36).

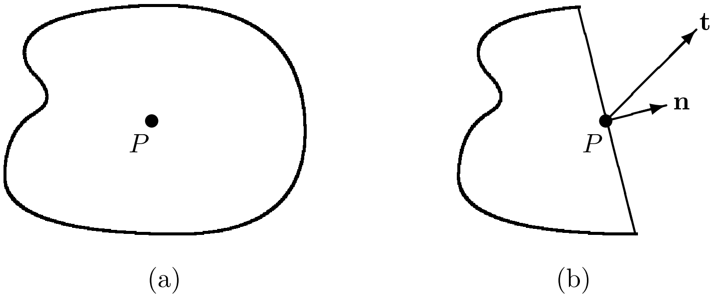
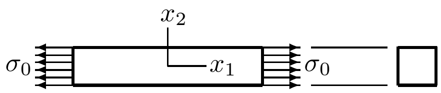


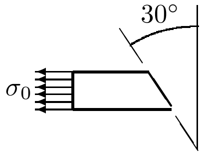
Figure 1.13: (a) A point P. (b) The unit vector $\mathbf{n}$ and the traction vector $\mathbf{t}$.

Exercises

Exercise 1.24



(1)



(2)

A stationary bar is subjected to uniform normal tractions σ_0 at the ends (Fig. 1). As a result, the components of the stress tensor at each point of the material are

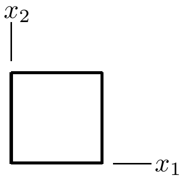
$$[T_{km}] = \left[\begin{array}{ccc} \sigma_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

(a) Determine the magnitudes of the normal and shear stresses acting on the plane shown in Fig. 2 by writing equilibrium equations for the free-body diagram shown.

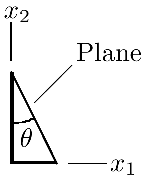
(b) Determine the magnitudes of the normal and shear stresses acting on the plane shown in Fig. 2 by using Eq. (1.36).

Answer: (a),(b) normal stress $= \sigma_0 \cos^2 30°$, shear stress $= \sigma_0 \sin 30° \cos 30°$

Exercise 1.25



(1)



(2)

The components of the stress tensor at each point of the cube of material shown in Fig. 1 are

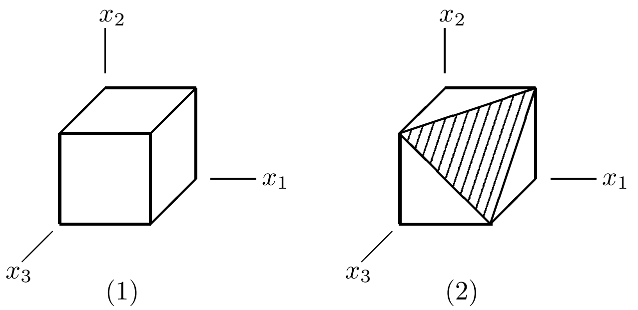
$$[T_{km}] = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix}.$$

Determine the magnitudes of the normal and shear stresses acting on the plane shown in Fig. 2.

Discussion—Assume that the components of the stress tensor are symmetric: $T_{km} = T_{mk}$.

Answer: normal stress $= |T_{11}\cos^2\theta + T_{22}\sin^2\theta + 2T_{12}\sin\theta\cos\theta|$, shear stress $= |T_{12}(\cos^2\theta - \sin^2\theta) - (T_{11} - T_{22})\sin\theta\cos\theta|$.

Exercise 1.26



The components of the stress tensor at each point of the cube of material shown in Fig. 1 are

$$[T_{km}] = \begin{bmatrix} 100 & -100 & 0 \\ -100 & 100 & 0 \\ 0 & 0 & 300 \end{bmatrix} \text{ Pa.}$$

A pascal (Pa) is 1 newton/meter^2. Determine the magnitudes of the normal and shear stresses acting on the cross-hatched plane shown in Fig. 2.

Answer: (a),(b) normal stress $= 100$ Pa, shear stress $= 100\sqrt{2}$ Pa.

1.5 Stress-Strain Relations

We will discuss the relationship between the stresses in an elastic material and its deformation.

Linear elastic materials

An *elastic material* is a model for material behavior based on the assumption that the stress at a point in the material at a time t depends only on the strain at that point at time t. Each component of the stress tensor can be expressed as a function of the components of the strain tensor:

$$T_{km} = T_{km}(E_{ij}).$$

We expand the components T_{km} as power series in terms of the components of strain:

$$T_{km} = b_{km} + c_{kmij}E_{ij} + d_{kmijrs}E_{ij}E_{rs} + \cdots,$$

where the coefficients $b_{km}, c_{kmij}, \cdots$ are constants. In linear elasticity, only the terms up to the first order are retained in this expansion. In addition, we assume that the stress in the material is zero when there is no deformation relative to the reference state, so $b_{km} = 0$ and the relations between the stress and strain components for a linear elastic material reduce to

$$T_{km} = c_{kmij}E_{ij}. \tag{1.37}$$

Isotropic linear elastic materials

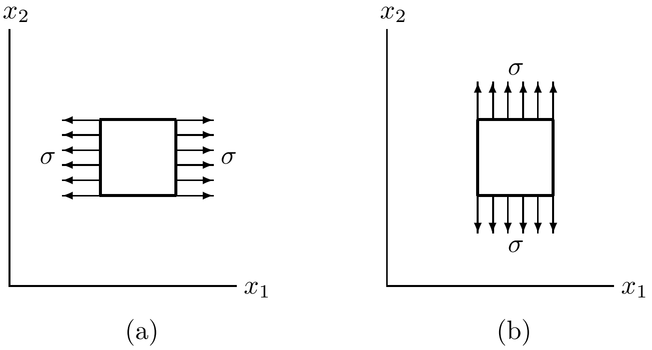
If uniform normal tractions are applied to two opposite faces of a block of wood, the resulting deformation depends on the grain direction of the wood. Clearly, the deformation obtained if the grain direction is perpendicular to the two faces is different from that obtained if the grain direction is parallel to the two faces. The deformation depends on the orientation of the material. A material for which the relationship between stress and deformation is independent of the orientation of the material is called *isotropic.* Wood is an example of a material that is not isotropic, or *anisotropic.*

Equation (1.37) relating the stress and strain components in a linear elastic material can be expressed in a much simpler form in the case of an isotropic linear elastic material. A complete derivation of the relation for an isotropic material is beyond the scope of this chapter, but we can demonstrate the kinds of arguments that are used.

If we solve Eqs. (1.37) for the strain components in terms of the stresses, we can express the results as

$$E_{ij} = h_{ijkm} T_{km}, \tag{1.38}$$

where the h_{ijkm} are constants. Suppose that we subject a block of material to a uniform normal stress σ as shown in Fig. 1.14.a, so that $T_{11} = \sigma$ and the other stress components are zero. From Eq. (1.38), the equation for the strain component E_{11} is



(a) (b)

Figure 1.14: (a) The stress $T_{11} = \sigma$. (b) The stress $T_{22} = \sigma$.

$$E_{11} = h_{1111} T_{11} = h_{1111} \sigma. \tag{1.39}$$

If we apply the stress σ as shown in Fig. 1.14.b instead, the equation for the strain component E_{22} is

$$E_{22} = h_{2222} T_{22} = h_{2222} \sigma. \tag{1.40}$$

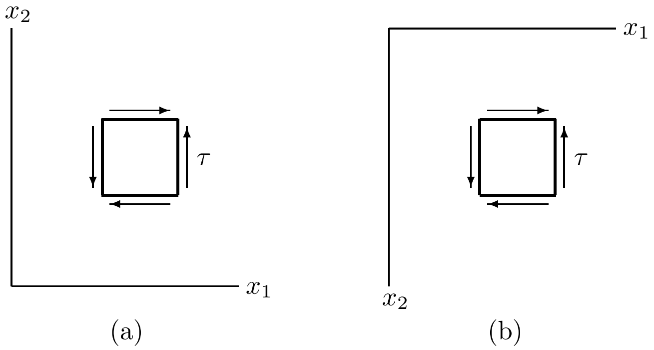
If the material is isotropic, the strain E_{11} in Eq. (1.39) must equal the strain E_{22} in Eq. (1.40), so the constant $h_{2222} = h_{1111}$.

As a second example, suppose we subject the block of material to a uniform shear stress τ as shown in Fig. 1.15.a, so that $T_{12} = \tau$ and all the other stress components are zero. From Eq. (1.38), the strain component E_{11} is

$$\begin{aligned} E_{11} &= h_{1112} T_{12} + h_{1121} T_{21} \\ &= (h_{1112} + h_{1121}) \tau. \end{aligned} \tag{1.41}$$

If the material is isotropic, the constants in Eq. (1.38) must have the same values for any orientation of the coordinate system relative to the material. Let us reorient the coordinate system as shown in Fig. 1.15.b. With this orientation, the stress $T_{12} = T_{21} = -\tau$, and the strain component E_{11} is

$$\begin{aligned} E_{11} &= h_{1112} T_{12} + h_{1121} T_{21} \\ &= -(h_{1112} + h_{1121}) \tau. \end{aligned} \tag{1.42}$$



(a) (b)

Figure 1.15: (a) A uniform shear stress. (b) Reorienting the coordinate system.

Because the x_1 directions are the same in the two cases, the strain E_{11} in Eq. (1.41) must equal the strain E_{11} in Eq. (1.42). Therefore $h_{1112} + h_{1121} = 0$.

By continuing with arguments of this kind, it can be shown that the relations between the stress and strain components, or *stress-strain relations,* for an isotropic linear elastic material must be of the form

$$T_{km} = \lambda \delta_{km} E_{jj} + 2\mu E_{km}, \tag{1.43}$$

where δ_{km} is the Kronecker delta and λ and μ are called the *Lamé constants.* . The constant μ is also called the *shear modulus.*

We see that the stress-strain relations of an isotropic linear elastic material are characterized by only two constants. Two constants called the *Young's modulus E* and the *Poisson's ratio ν* are often used instead of the Lamé constants (see Exercise 1.29). In terms of these constants, the stress-strain relations for an isotropic elastic material are

$$E_{km} = -\frac{\nu}{E} \delta_{km} T_{jj} + \left(\frac{1+\nu}{E} \right) T_{km}. \tag{1.44}$$

Exercises

Exercise 1.27 The Lamé constants of an isotropic material are $\lambda = 1.15(10^{11})$ Pa and $\mu = 0.77(10^{11})$ Pa. The components of the strain tensor at a point P in the material are

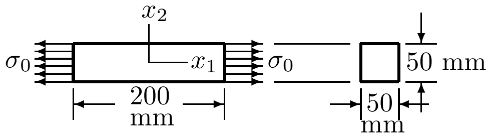
$$[E_{km}] = \begin{bmatrix} 0.001 & -0.001 & 0 \\ -0.001 & 0.001 & 0 \\ 0 & 0 & 0.002 \end{bmatrix}.$$

Determine the components of the stress tensor T_{km} at point P.

Answer:

$$[T_{km}] = \begin{bmatrix} 6.14(10^8) & -1.54(10^8) & 0 \\ -1.54(10^8) & 6.14(10^8) & 0 \\ 0 & 0 & 7.68(10^8) \end{bmatrix} \text{ Pa.}$$

Exercise 1.28



A bar is 200 mm in length and has a square 50 mm × 50 mm cross section in the unloaded state. It consists of isotropic material with Lamé constants $\lambda = 4.5(10^{10})$ Pa and $\mu = 3.0(10^{10})$ Pa. The ends of the bar are subjected to a uniform normal traction $\sigma_0 = 2.0(10^8)$ Pa. As a result, the components of the stress tensor at each point of the material are

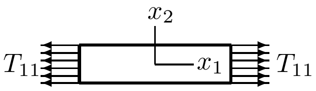
$$[T_{km}] = \begin{bmatrix} 2.0(10^8) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ Pa.}$$

(a) Determine the length of the bar in the loaded state.

(b) Determine the dimensions of the square cross section of the bar in the loaded state.

Answer: (a) 200.513 mm. (b) 49.962 mm×49.962 mm.

Exercise 1.29



The ends of a bar of isotropic material are subjected to a uniform normal traction T_{11}. As a result, the components of the stress tensor at each point of the material are

$$[T_{km}] = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) The ratio E of the stress T_{11} to the longitudinal strain E_{11} in the x_1 direction,

$$E = \frac{T_{11}}{E_{11}},$$

is called the Young's modulus, or *modulus of elasticity* of the material. Show that the Young's modulus is related to the Lamé constants by

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

(b) The ratio

$$\nu = -\frac{E_{22}}{E_{11}}$$

is called the *Poisson's ratio* of the material. Show that the Poisson's ratio is related to the Lamé constants by

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

EXERCISE 1.30 The Young's modulus and Poisson's ratio of an elastic material are defined in Exercise 1.29. Show that the Lamé constants of an isotropic material are given in terms of the Young's modulus and Poisson's ratio of the material by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

Discussion—See Exercise 1.29.

EXERCISE 1.31 Show that the strain components of an isotropic linear elastic material are given in terms of the stress components by

$$E_{km} = \frac{1}{2\mu} T_{km} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \delta_{km} T_{jj}.$$

1.6 Balance and Conservation Equations

We now derive the equations of conservation of mass, balance of linear momentum, and balance of angular momentum for a material. Substituting the stress-strain relation for an isotropic linear elastic material into the equation of balance of linear momentum, we obtain the equations that govern the motion of such materials.

Transport theorem

We need the transport theorem to derive the equations of motion for a material. To derive it, we must introduce the concept of a *material volume*. Let V_0 be a particular volume of a sample of material in the reference state (Fig. 1.16). As the material moves and deforms, the material that was contained in the volume V_0 in the reference state moves and deforms. The volume V occupied by the material as it moves and deforms is called a material volume, since it contains the same material at each time t. It is helpful to imagine "painting" the surface S_0 of the material in the reference state. As the material moves and deforms, the material volume V is the volume within the "painted" surface S.

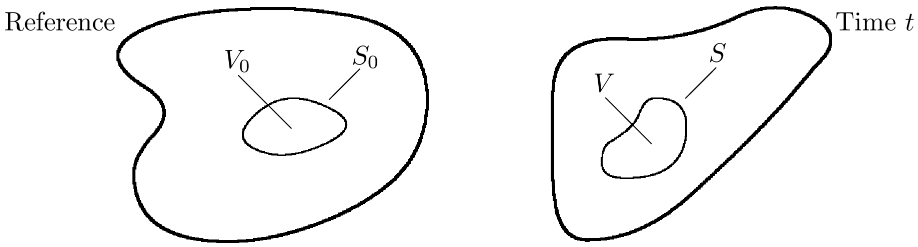


Figure 1.16: A material volume in the reference state and at time t.

Consider the spatial representation of a scalar field $\phi = \phi(\mathbf{x}, t)$. The integral

$$f(t) = \int_V \phi \, dV,$$

where V is a material volume, is a scalar-valued function of time. Let us determine the time derivative of the function $f(t)$. The value of $f(t)$ at time $t + \Delta t$ is

$$f(t + \Delta t) = \int_V \left(\phi + \frac{\partial \phi}{\partial t} \Delta t \right) dV + \int_S \phi (\mathbf{v} \cdot \mathbf{n}) \Delta t \, dS + O(\Delta t^2).$$

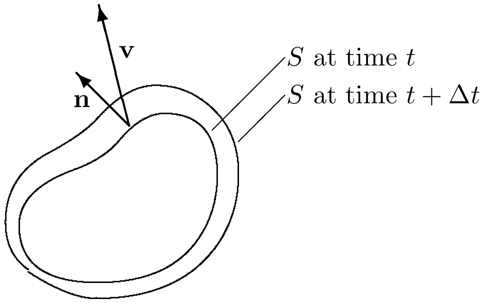


Figure 1.17: The motion of the surface S from t to $t + \Delta t$.

The second term on the right accounts for the motion of the surface S during the interval of time from t to $t + \Delta t$ (Fig. 1.17). The symbol $O(\Delta t^2)$ means "terms of order two or higher in Δt". The time derivative of $f(t)$ is

$$\begin{aligned} \frac{df(t)}{dt} &= \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \\ &= \int_V \frac{\partial \phi}{\partial t} \, dV + \int_S \phi v_k n_k \, dS. \end{aligned}$$

By applying the Gauss theorem to the surface integral in this expression, we obtain

$$\frac{df(t)}{dt} = \frac{d}{dt}\int_V \phi \, dV = \int_V \left[\frac{\partial \phi}{\partial t} + \frac{\partial(\phi v_k)}{\partial x_k} \right] dV. \tag{1.45}$$

This result is the transport theorem. We use it in the following sections to derive the equations describing the motion of a material.

Conservation of mass

Because a material volume contains the same material at each time t, the total mass of the material in a material volume is constant:

$$\frac{d}{dt}\int_V \rho \, dV = 0.$$

By using the transport theorem, we obtain the result

$$\int_V \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_k)}{\partial x_k} \right] dV = 0.$$

This equation must hold for every material volume of the material. If we assume that the integrand is continuous, the equation can be satisfied only if the integrand vanishes at each point:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_k)}{\partial x_k} = 0. \tag{1.46}$$

This is one form of the equation of conservation of mass for a material.

Balance of linear momentum

Newton's second law states that the force acting on a particle is equal to the rate of change of its linear momentum:

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}). \tag{1.47}$$

The equation of balance of linear momentum for a material is obtained by postulating that the rate of change of the total linear momentum of the material contained in a material volume is equal to the total force acting on the material volume:

$$\frac{d}{dt}\int_V \rho\mathbf{v}\,dV = \int_S \mathbf{t}\,dS + \int_V \mathbf{b}\,dV.$$

The first term on the right is the force exerted on the surface S by the traction vector $\mathbf{t}$. The second term on the right is the force exerted on the material by the body force per unit volume $\mathbf{b}$. Writing this equation in index notation and using Eq. (1.36) to express the traction vector in terms of the components of the stress tensor, we obtain

$$\frac{d}{dt}\int_V \rho v_m\,dV = \int_S T_{mk} n_k\,dS + \int_V b_m\,dV. \tag{1.48}$$

We leave it as an exercise to show that by using the Gauss theorem, the transport theorem, and the equation of conservation of mass, this equation can be expressed in the form

$$\int_V \left(\rho a_m - \frac{\partial T_{mk}}{\partial x_k} - b_m\right) dV = 0, \tag{1.49}$$

where

$$a_m = \frac{\partial v_m}{\partial t} + \frac{\partial v_m}{\partial x_k} v_k$$

is the acceleration. Equation (1.49) must hold for every material volume of the material. If we assume that the integrand is continuous, we obtain the equation

$$\rho a_m = \frac{\partial T_{mk}}{\partial x_k} + b_m. \tag{1.50}$$

This is the equation of balance of linear momentum for a material.

Balance of angular momentum

Taking the cross product of Newton's second law for a particle with the position vector $\mathbf{x}$ of the particle, we can write the result as

$$\begin{aligned} \mathbf{x} \times \mathbf{F} &= \mathbf{x} \times \frac{d}{dt}(m\mathbf{v}) \\ &= \frac{d}{dt}(\mathbf{x} \times m\mathbf{v}). \end{aligned}$$

The term on the left is the moment exerted by the force $\mathbf{F}$ about the origin. The term $\mathbf{x} \times m\mathbf{v}$ is called the angular momentum of the particle. Thus this equation states that the moment is equal to the rate of change of the angular momentum.

The equation of balance of angular momentum for a material is obtained by postulating that the rate of change of the total angular momentum of the material contained in a material volume is equal to the total moment exerted on the material volume:

$$\frac{d}{dt}\int_V \mathbf{x} \times \rho\mathbf{v}\, dV = \int_S \mathbf{x} \times \mathbf{t}\, dS + \int_V \mathbf{x} \times \mathbf{b}\, dV.$$

We leave it as an exercise to show that this postulate implies that the stress tensor is symmetric:

$$T_{km} = T_{mk}.$$

Exercises

Exercise 1.32 By using the Gauss theorem, the transport theorem, and the equation of conservation of mass, show that Eq. (1.48),

$$\frac{d}{dt}\int_V \rho v_m\, dV = \int_S T_{mk} n_k\, dS + \int_V b_m\, dV,$$

can be expressed in the form

$$\int_V \left(\rho a_m - \frac{\partial T_{mk}}{\partial x_k} - b_m \right) dV = 0,$$

where a_m is the acceleration of the material.

Exercise 1.33 Show that the postulate of balance of angular momentum for a material

$$\frac{d}{dt}\int_V \mathbf{x} \times \rho\mathbf{v}\, dV = \int_S \mathbf{x} \times \mathbf{t}\, dS + \int_V \mathbf{x} \times \mathbf{b}\, dV$$

implies that the stress tensor is symmetric:

$$T_{km} = T_{mk}.$$

Exercise 1.34 For a stationary material, the postulate of balance of energy is

$$\frac{d}{dt}\int_V \rho e\, dV = -\int_S q_j n_j\, dS,$$

where V is a material volume with surface S. The term e is the *internal energy* and q_j is the *heat flux vector*. Suppose that the internal energy and the heat flux vector are related to the absolute temperature T of the material by the equations

$$e = cT, \qquad q_j = -k\frac{\partial T}{\partial x_j},$$

where the specific heat c and the thermal conductivity k are constants. Show that the absolute temperature is governed by the heat transfer equation

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c}\frac{\partial^2 T}{\partial x_j \partial x_j}.$$

1.7 Equations of Motion

The balance and conservation equations and the stress-strain relation can be used to derive the equations of motion for a linear elastic material. We begin by obtaining equations of motion that are expressed in terms of the displacement of the material.

Displacement equations of motion

The equation of balance of linear momentum for a material is

$$\rho a_m = \frac{\partial T_{mk}}{\partial x_k}. \tag{1.51}$$

Notice that we do not include the body force term. In many elastic wave propagation problems the body force does not have a significant effect, and we neglect it in the following development.

Recall that in linear elasticity it is assumed that derivatives of the displacement are small. In this case we can express the acceleration in terms of the displacement using Eq. (1.14),

$$a_m = \frac{\partial^2 u_m}{\partial t^2},$$

and express the density in terms of the density in the reference state and the divergence of the displacement using Eq. (1.33):

$$\rho = \rho_0 \left(1 - \frac{\partial u_k}{\partial x_k} \right).$$

From these two equations, we see that in linear elasticity, the product of the density and the acceleration can be expressed as

$$\rho a_m = \rho_0 \frac{\partial^2 u_m}{\partial t^2}. \tag{1.52}$$

The stress-strain relation for an isotropic linear elastic material is given by Eq. (1.43),

$$T_{mk} = \lambda \delta_{mk} E_{jj} + 2\mu E_{mk},$$

and the components of the linear strain tensor are related to the components of the displacement by Eq. (1.19):

$$E_{mk} = \frac{1}{2} \left(\frac{\partial u_m}{\partial x_k} + \frac{\partial u_k}{\partial x_m} \right).$$

Substituting this expression into the stress-strain relation, we obtain the stress components in terms of the components of the displacement:

$$T_{mk} = \lambda \delta_{mk} \frac{\partial u_j}{\partial x_j} + \mu \left(\frac{\partial u_m}{\partial x_k} + \frac{\partial u_k}{\partial x_m} \right).$$

Then by substituting this expression and Eq. (1.52) into the equation of balance of linear momentum, Eq. (1.51), we obtain the equation of balance of linear momentum for an isotropic linear elastic material:

$$\rho_0 \frac{\partial^2 u_m}{\partial t^2} = (\lambda + \mu) \frac{\partial^2 u_k}{\partial x_k \partial x_m} + \mu \frac{\partial^2 u_m}{\partial x_k \partial x_k}. \tag{1.53}$$

By introducing the cartesian unit vectors, we can write this equation as the vector equation

$$\rho_0 \frac{\partial^2 u_m}{\partial t^2} \mathbf{i}_m = (\lambda + \mu) \frac{\partial}{\partial x_m} \left(\frac{\partial u_k}{\partial x_k} \right) \mathbf{i}_m + \mu \frac{\partial^2 u_m}{\partial x_k \partial x_k} \mathbf{i}_m.$$

We can express this equation in vector notation as

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}. \tag{1.54}$$

The second term on the right side is called the *vector Laplacian* of $\mathbf{u}$. It can be written in the form (see Exercise 1.11)

$$\nabla^2 \mathbf{u} = \nabla (\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}). \tag{1.55}$$

Substituting this expression into Eq. (1.54), we obtain

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}). \tag{1.56}$$

Equations (1.53) and (1.56) express the equation of balance of linear momentum for isotropic linear elasticity in index notation and in vector notation. Both forms are useful. One advantage of the vector form is that it is convenient for expressing the equation in terms of various coordinate systems. These equations are called the displacement equations of motion because they are expressed in terms of the displacement of the material.

Helmholtz decomposition

In the Helmholtz decomposition, the displacement field of a material is expressed as the sum of the gradient of a scalar potential ϕ and the curl of a vector potential $\boldsymbol{\psi}$:

$$\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\psi}. \tag{1.57}$$

Substituting this expression into the equation of balance of linear momentum, Eq. (1.56), and using Eq. (1.55), we can write the equation of balance of linear momentum in the form

$$\begin{aligned} \nabla\left[\rho_0\frac{\partial^2\phi}{\partial t^2}-(\lambda+2\mu)\nabla^2\phi\right] & \\ +\nabla\times\left[\rho_0\frac{\partial^2\boldsymbol{\psi}}{\partial t^2}-\mu\nabla^2\boldsymbol{\psi}\right] &= 0. \end{aligned}$$

We see that the equation of balance of linear momentum is satisfied if the potentials ϕ and $\boldsymbol{\psi}$ satisfy the equations

$$\frac{\partial^2\phi}{\partial t^2}=\alpha^2\nabla^2\phi \tag{1.58}$$

and

$$\frac{\partial^2\boldsymbol{\psi}}{\partial t^2}=\beta^2\nabla^2\boldsymbol{\psi}, \tag{1.59}$$

where the constants α and β are defined by

$$\alpha=\left(\frac{\lambda+2\mu}{\rho_0}\right)^{1/2}, \quad \beta=\left(\frac{\mu}{\rho_0}\right)^{1/2}.$$

Equation (1.58) is called the *wave equation.* When it is expressed in terms of cartesian coordinates, Eq. (1.59) becomes three wave equations, one for each component of the vector potential $\boldsymbol{\psi}$. Because these equations are simpler in form than the displacement equations of motion, many problems in elastic wave propagation are approached by first expressing the displacement field in terms of the Helmholtz decomposition.

Acoustic medium

Setting the shear modulus $\mu=0$ in the equations of linear elasticity yields the equations for an elastic medium that does not support shear stresses, or an elastic inviscid fluid. These equations are used to analyze sound propagation or *acoustic* problems, and we therefore refer to an elastic inviscid fluid as an *acoustic medium.* In this case, the equation of balance of linear momentum, Eq. (1.56), reduces to

$$\rho_0\frac{\partial^2\mathbf{u}}{\partial t^2}=\lambda\nabla(\nabla\cdot\mathbf{u}).$$

The displacement vector can be expressed in terms of the scalar potential ϕ,

$$\mathbf{u}=\nabla\phi,$$

and it is easy to show that the equation of balance of linear momentum is satisfied if ϕ satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = \alpha^2 \nabla^2 \phi,$$

where the constant α is defined by

$$\alpha = \left(\frac{\lambda}{\rho_0} \right)^{1/2}.$$

The constant λ is called the *bulk modulus* of the fluid. Thus acoustic problems are governed by a wave equation expressed in terms of one scalar variable.

Exercises

EXERCISE 1.35 By substituting the Helmholtz decomposition $\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi}$ into the equation of balance of linear momentum, Eq. (1.56), show that the latter equation can be written in the form

$$\nabla \left[\rho_0 \frac{\partial^2 \phi}{\partial t^2} - (\lambda + 2\mu)\nabla^2 \phi \right] + \nabla \times \left[\rho_0 \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} - \mu \nabla^2 \boldsymbol{\psi} \right] = 0.$$

EXERCISE 1.36 We define an acoustic medium on page 44. Show that the density ρ of an acoustic medium is governed by the wave equation

$$\frac{\partial^2 \rho}{\partial t^2} = \alpha^2 \nabla^2 \rho.$$

Discussion—See Exercise 1.33.

1.8 Summary

We will briefly summarize the equations that govern the motion of an isotropic linear elastic material.

Displacement equations of motion

The displacement equations of motion for an isotropic linear elastic material in index notation are (Eq. (1.53))

$$\rho_0 \frac{\partial^2 u_m}{\partial t^2} = (\lambda + \mu) \frac{\partial^2 u_k}{\partial x_k \partial x_m} + \mu \frac{\partial^2 u_m}{\partial x_k \partial x_k}. \tag{1.60}$$

They can be expressed in vector notation as (Eq. (1.53))

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}). \tag{1.61}$$

The vector $\mathbf{u}$ is the displacement of the material relative to the reference state. The scalar ρ_0 is the density of the material in the reference state. The constants λ and μ are the Lamé constants of the material. The Lamé constants are related to the Young's modulus E and Poisson's ratio ν of the material by (see Exercise 1.30)

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

Helmholtz decomposition

The Helmholtz decomposition of the displacement vector $\mathbf{u}$ is [Eq. (1.57)]

$$\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\psi}, \tag{1.62}$$

where ϕ is a scalar potential and $\boldsymbol{\psi}$ is a vector potential. The displacement equations of motion are satisfied if the potentials ϕ and $\boldsymbol{\psi}$ satisfy the equations (Eqs. (1.58) and (1.59))

$$\frac{\partial^2 \phi}{\partial t^2} = \alpha^2 \nabla^2 \phi \tag{1.63}$$

and

$$\frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} = \beta^2 \nabla^2 \boldsymbol{\psi}, \tag{1.64}$$

where the constants α and β are defined by

$$\alpha = \left(\frac{\lambda + 2\mu}{\rho_0} \right)^{1/2}, \quad \beta = \left(\frac{\mu}{\rho_0} \right)^{1/2}. \tag{1.65}$$

Strain tensor and density

The components of the linear strain tensor E_{km} are related to the components of the displacement vector by (Eq. (1.19))

$$E_{km} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_m} + \frac{\partial u_m}{\partial x_k}\right). \tag{1.66}$$

The density ρ of the material is related to its density ρ_0 in the reference state by (Eq. (1.33))

$$\rho = \rho_0(1 - E_{jj}) = \rho_0\left(1 - \frac{\partial u_j}{\partial x_j}\right). \tag{1.67}$$

Stress-strain relation

In isotropic linear elasticity, the components of the stress tensor T_{km} are related to the components of the strain tensor by (Eq. (1.43))

$$T_{km} = \lambda\delta_{km}E_{jj} + 2\mu E_{km}, \tag{1.68}$$

where λ and μ are the Lamé constants of the material.

Traction

If a plane is passed through a point p of a sample of material and a free-body diagram is drawn of the material on one side of the plane, the traction $\mathbf{t}$ exerted on the plane at point p (Fig. 1.13) is given in terms of the components of the stress tensor at p by (Eq. (1.36))

$$t_k = T_{km}n_m. \tag{1.69}$$

Chapter 2

One-Dimensional Waves

When you see straight, parallel waves approach a beach, you are observing one-dimensional waves. The heights of the waves depend on time and a single spatial variable, the distance perpendicular to the beach. We begin our discussion of waves in elastic materials with one-dimensional waves because they are simple both from the intuitive and theoretical points of view. After introducing the one-dimensional wave equation and deriving its general solution, we show that this equation governs one-dimensional motions of an elastic material. To aid in visualizing solutions, we demonstrate that the same equation also governs lateral disturbances in a stretched string. We obtain solutions to several important problems, including the reflection of a wave at the interface between two elastic materials. We present examples of the use of characteristics to solve simple one-dimensional problems, then apply them to materials consisting of multiple layers.

2.1 One-Dimensional Wave Equation

The one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \tag{2.1}$$

is a second-order partial differential equation in one dependent variable u and two independent variables t and x. The term α is a constant. We will show that this equation has the remarkable property that its general solution can be obtained by expressing the equation in terms of the independent variables

$\xi = x - \alpha t$ and $\eta = x + \alpha t$. The dependent variable u can be expressed either as a function of the variables x, t or of the variables ξ, η:

$$u = u(x,t) = \tilde{u}(\xi, \eta).$$

By using the chain rule, we can write the partial derivative of u with respect to t as

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial \tilde{u}}{\partial \eta} \\ &= -\alpha \frac{\partial \tilde{u}}{\partial \xi} + \alpha \frac{\partial \tilde{u}}{\partial \eta}.\end{aligned}$$

Then we apply the chain rule again to determine the second partial derivative of u with respect to t in terms of derivatives with respect to ξ and η:

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}\left(-\alpha \frac{\partial \tilde{u}}{\partial \xi} + \alpha \frac{\partial \tilde{u}}{\partial \eta}\right) \\ &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi}\left(-\alpha \frac{\partial \tilde{u}}{\partial \xi} + \alpha \frac{\partial \tilde{u}}{\partial \eta}\right) \\ &\quad + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\left(-\alpha \frac{\partial \tilde{u}}{\partial \xi} + \alpha \frac{\partial \tilde{u}}{\partial \eta}\right) \\ &= \alpha^2 \left(\frac{\partial^2 \tilde{u}}{\partial \xi^2} - 2\frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{u}}{\partial \eta^2}\right).\end{aligned}$$

The second partial derivative of u with respect to x in terms of derivatives with respect to ξ and η can be obtained in the same way. The result is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \tilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{u}}{\partial \eta^2}.$$

Substituting these expressions for the second partial derivatives of u into Eq. (2.1), we obtain the one-dimensional wave equation in the form

$$\frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} = 0.$$

We can integrate this equation twice to obtain its general solution. Integrating with respect to η yields

$$\frac{\partial \tilde{u}}{\partial \xi} = h(\xi),$$

where $h(\xi)$ is an arbitrary function of ξ. Then integrating with respect to ξ gives the solution

$$u = \int h(\xi)\, d\xi + g(\eta),$$

where $g(\eta)$ is an arbitrary function of η. By defining

$$f(\xi) = \int h(\xi)\, d\xi,$$

we write the solution in the form

$$u = f(\xi) + g(\eta), \tag{2.2}$$

where $f(\xi)$ and $g(\eta)$ are arbitrary twice-differentiable functions. This is called the *D'Alembert solution* of the one-dimensional wave equation.

Consider the function $f(\xi) = f(x - \alpha t)$. Suppose that at a particular time t_0, the graph of the function $f(x - \alpha t_0)$ is the curve shown in Fig. 2.1.a. Our objective is to show that as time increases, the graph of the function $f(\xi)$ as a function of x remains unchanged in shape, but translates in the positive x direction with velocity α.

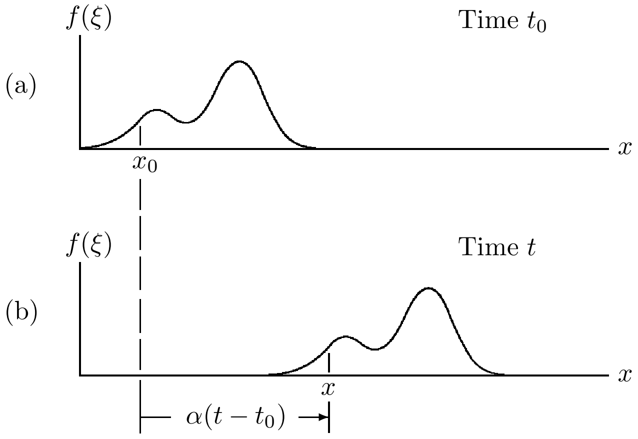


Figure 2.1: The function $f(\xi)$ at time t_0 and at time t.

The value of the function $f(x - \alpha t_0)$ at a particular position x_0 is $f(x_0 - \alpha t_0)$. The graph of $f(\xi)$ at a time $t > t_0$ is shown in Fig. 2.1.b. Its value $f(x - \alpha t)$ at a position x is equal to the value $f(x_0 - \alpha t_0)$ if $x - \alpha t = x_0 - \alpha t_0$, or

$$x = x_0 + \alpha(t - t_0).$$

This expression shows that during the interval of time from t_0 to t, the graph of the function $f(\xi)$ translates a distance $\alpha(t - t_0)$ in the positive x direction. Thus as time increases the graph of the function $f(\xi)$ as a function of x moves in the positive x direction with constant velocity α (Fig. 2.2). By using the

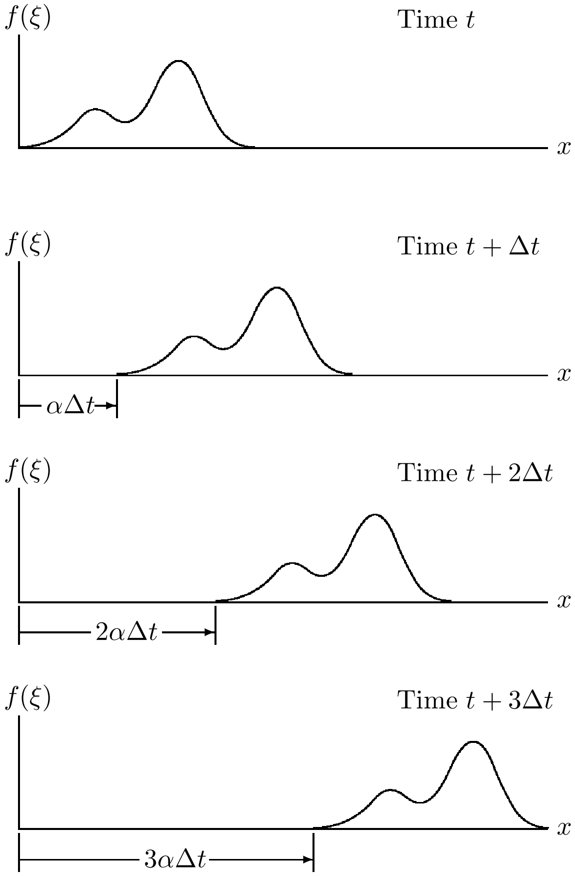


Figure 2.2: The function $f(\xi)$ translates in the positive x direction with constant velocity α.

same argument, we can show that the graph of the function $g(\eta)$ as a function of x translates in the negative x direction with constant velocity α.

The behavior of the functions $f(\xi)$ and $g(\eta)$ agrees with our intuitive notion of a wave. Roughly speaking, a *wave* is a disturbance in a medium that moves or *propagates* through the medium. The functions $f(\xi)$ and $g(\eta)$ describe waves that propagate in the positive and negative x directions with constant velocity or *wave speed* α. This explains why Eq. (2.1) is called the one-dimensional wave equation. In the next section, we show that one-dimensional disturbances in a linear elastic material are governed by this equation.

Exercises

Exercise 2.1 A function $f(x)$ is defined by

$$\begin{array}{rl} x < 0 & f(x) = 0, \\ 0 \leq x \leq 1 & f(x) = \sin(\pi x), \\ x > 1 & f(x) = 0. \end{array}$$

Plot the function $f(\xi) = f(x - \alpha t)$ as a function of x for $t = 0$, $t = 1$ s, and $t = 2$ s if $\alpha = 1$.

Exercise 2.2 Show that as time increases, the graph of the function $g(\eta) = g(x + \alpha t)$ as a function of x translates in the negative x direction with constant velocity α.

Exercise 2.3 By using the chain rule, show that the second partial derivative of $u = u(x, t)$ with respect to x can be expressed in terms of partial derivatives of $u = \tilde{u}(\xi, \eta)$ by

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \tilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{u}}{\partial \eta^2}.$$

Exercise 2.4 Show that the D'Alembert solution $u = f(\xi) + g(\eta)$, where $\xi = x - \alpha t$ and $\eta = x + \alpha t$, is a solution of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}.$$

Discussion—Substitute the solution into the equation and show that the equation is satisfied. Use the chain rule to evaluate the derivatives.

Exercise 2.5 Consider the expression

$$u = Ae^{i(kx - \omega t)}.$$

What conditions must the constants A, k, and ω satisfy in order for this expression to be a solution of the one-dimensional wave equation?

Exercise 2.6 Consider the first-order partial differential equation

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = 0,$$

where α is a constant. By expressing it in terms of the independent variables $\xi = x - \alpha t$ and $\eta = x + \alpha t$, show that its general solution is

$$u = f(\xi),$$

where f is an arbitrary twice-differentiable function.

2.2 One-Dimensional Motions of an Elastic Material

We will describe two kinds of one-dimensional motion that can occur in an elastic material and show that they are governed by the one-dimensional wave equation.

Compressional waves

Figure 2.3.a shows a *half space* of material. The material is assumed to be unbounded to the right of the infinite plane boundary. A cartesian coordinate system is oriented with the positive x_1 direction pointing into the material and the x_2-x_3 plane coincident with the boundary of the half space.

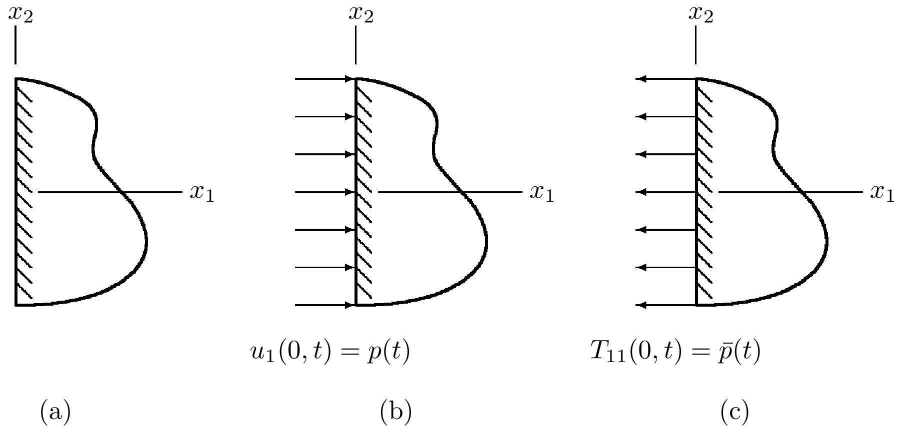


Figure 2.3: (a) A half space of material. (b) Displacement boundary condition. (c) Normal stress boundary condition.

The motion of the material is specified by its displacement field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$. Suppose that the half space of material is initially undisturbed, that is, $\mathbf{u}(\mathbf{x}, t) = 0$ for $t \leq 0$, and we give the boundary a uniform motion in the x_1 direction described by the equation

$$u_1(0,t) = p(t), \tag{2.3}$$

where $p(t)$ is a prescribed function of time that vanishes for $t \leq 0$ (Fig. 2.3.b). This is called a *displacement boundary condition.* The displacement of the boundary is specified as a function of time. The resulting motion of the material is described by the displacement field

$$\begin{aligned} u_1 &= u_1(x_1, t), \\ u_2 &= 0, \\ u_3 &= 0. \end{aligned} \tag{2.4}$$

Because the motion of the boundary in the x_1 direction is uniform—that is, it is the same at each point of the boundary—the resulting motion of the material cannot depend on x_2 or x_3, and the material has no motion in the x_2 or x_3 directions. Each point of the material moves only in the x_1 direction. This is a one-dimensional motion: it depends on only one spatial dimension and the time.

For the one-dimensional motion described by Eq. (2.4), the displacement equation of motion, Eq. (1.60), reduces to

$$\frac{\partial^2 u_1}{\partial t^2} = \alpha^2 \frac{\partial^2 u_1}{\partial x_1^2},$$

where the constant α is defined by Eq. (1.65). We see that one-dimensional motions of a linear elastic material described by Eq. (2.4) are governed by the one-dimensional wave equation with wave speed

$$\alpha = \left(\frac{\lambda + 2\mu}{\rho_0} \right)^{1/2}.$$

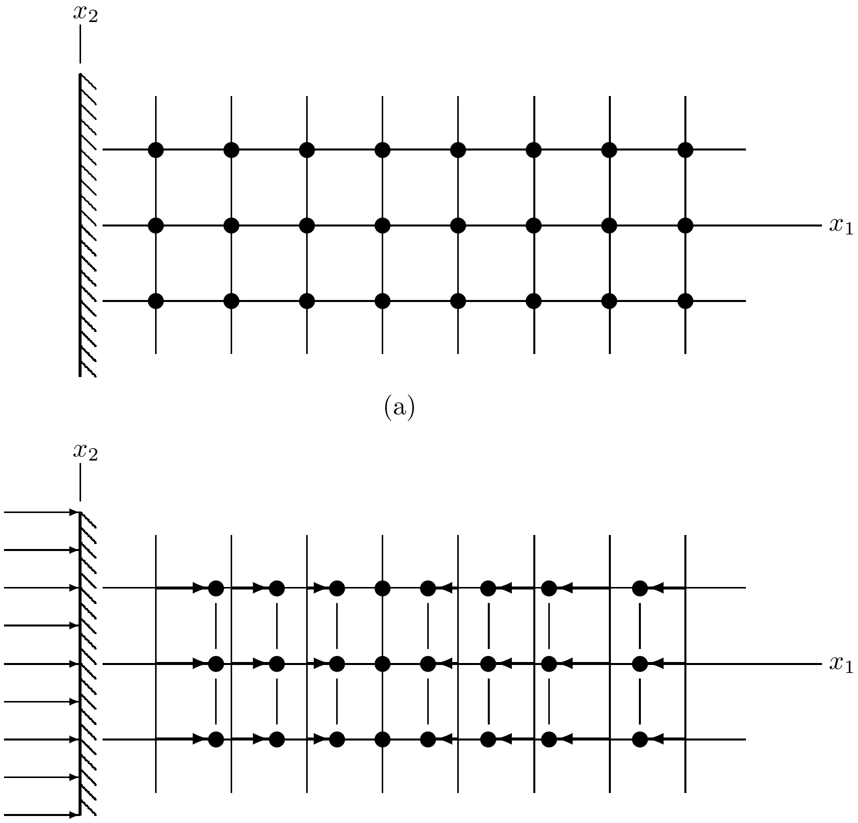
Waves of this type are called *compressional,* or P waves, and α is called the compressional, or P wave speed. Figure 2.4.a shows an array of material points in the undisturbed half space. Figure 2.4.b shows the displacements of the material points at a time t resulting from a compressional wave. The points move only in the x_1 direction, and their motions depend only on x_1 and t. From Eq. (1.67), we see that in a compressional wave described by Eq. (2.4), the density of the material is given by

$$\rho = \rho_0 \left(1 - \frac{\partial u_1}{\partial x_1} \right).$$

Compressional waves described by Eq. (2.4) are also obtained if the boundary of the undisturbed half space is subjected to a uniform normal stress described by the equation

$$T_{11}(0, t) = \bar{p}(t),$$

where $\bar{p}(t)$ is a prescribed function of time that vanishes for $t \leq 0$ (Fig. 2.3.c). This is called a *stress boundary condition.* The stress on the boundary is



(a)

(b)

Figure 2.4: (a) An array of points in the undisturbed half space. (b) Displacements of the points due to a compressional wave.

specified as a function of time. By using the stress-strain relation, Eq. (1.68), this boundary condition can be expressed in terms of the displacement:

$$\frac{\partial u_1}{\partial x_1}(0,t) = p(t),$$

where

$$p(t) = \frac{1}{\lambda + 2\mu}\,\bar{p}(t). \tag{2.5}$$

If the boundary is *free,* meaning that the applied stress is zero, the boundary condition is

$$\frac{\partial u_1}{\partial x_1}(0,t) = 0.$$

Shear waves

Suppose that the half space of material shown in Fig. 2.5.a is initially undisturbed and we give the boundary a uniform motion in the x_2 direction described by the equation

$$u_2(0,t) = p(t), \tag{2.6}$$

where $p(t)$ is a prescribed function of time that vanishes for $t \leq 0$ (Fig. 2.5.b). The resulting motion of the material is described by the displacement field

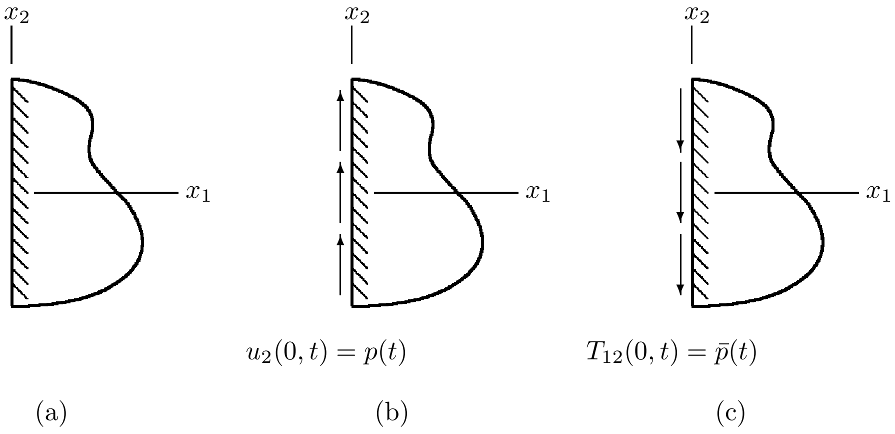


Figure 2.5: (a) A half space of material. (b) Displacement boundary condition. (c) Shear stress boundary condition.

$$\begin{aligned} u_1 &= 0, \\ u_2 &= u_2(x_1,t), \\ u_3 &= 0. \end{aligned} \tag{2.7}$$

Because the motion of the boundary in the x_2 direction is uniform, the resulting motion of the material does not depend on x_2 or x_3, and each point of the material moves only in the x_2 direction.

For one-dimensional motions described by Eq. (2.7), the displacement equation of motion, Eq. (1.60), reduces to

$$\frac{\partial^2 u_2}{\partial t^2} = \beta^2 \frac{\partial^2 u_2}{\partial x_1^2},$$

where the constant β is defined by Eq. (1.65). We see that one-dimensional motions of a linear elastic material described by Eq. (2.7) are governed by the one-dimensional wave equation with wave velocity

$$\beta = \left(\frac{\mu}{\rho_0}\right)^{1/2}. \tag{2.8}$$

Waves of this type are called *shear* waves, or S waves, and β is called the shear wave or S wave speed. Figure 2.6.a shows an array of material points in the undisturbed half space. Figure 2.6.b shows the displacements of the material points at a time t resulting from a shear wave. The points move only in the x_2 direction, and their motions depend only on x_1 and t. From Eq. (1.67), the density of the material in a shear wave described by Eq. (2.7) is given by

$$\rho = \rho_0.$$

Thus the density of the material is unchanged by a one-dimensional shear wave.

Shear waves described by Eq. (2.7) are also obtained if we subject the boundary of the undisturbed half space to a uniform shear stress described by the equation

$$T_{12}(0, t) = \bar{p}(t),$$

where $\bar{p}(t)$ is a prescribed function of time that vanishes for $t \leq 0$ (Fig. 2.5.c). By using the stress-strain relation, this boundary condition can be expressed in terms of the displacement:

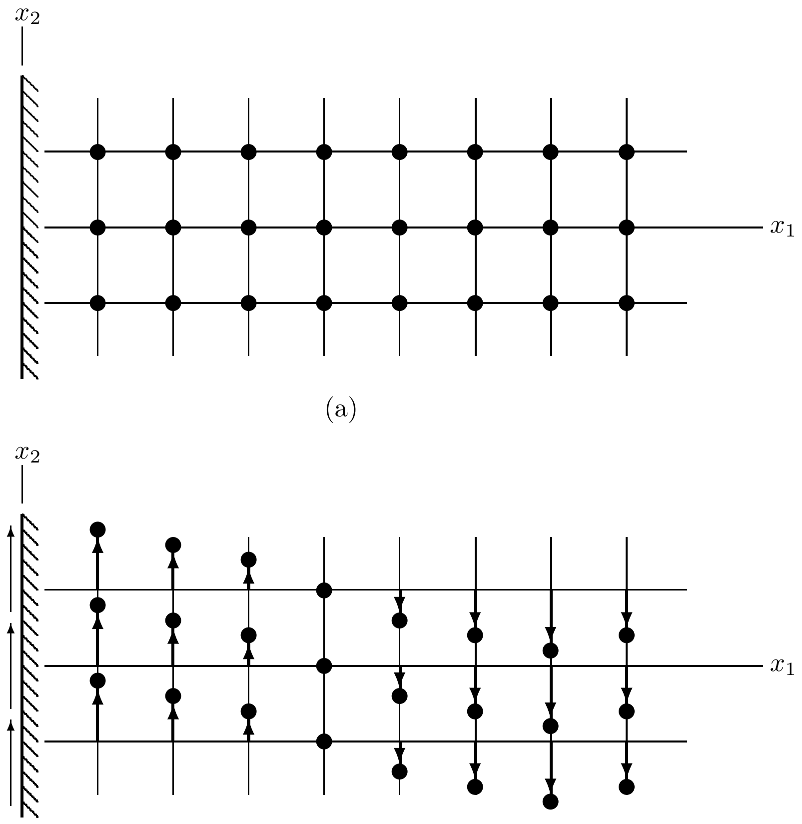
$$\frac{\partial u_2}{\partial x_1}(0, t) = p(t), \tag{2.9}$$

where

$$p(t) = \frac{1}{\mu}\bar{p}(t).$$

If the boundary is free, the boundary condition is

$$\frac{\partial u_2}{\partial x_1}(0, t) = 0.$$



(a)

(b)

Figure 2.6: (a) An array of points in the undisturbed half space. (b) Displacements of the points due to a shear wave.

2.3 Motion of a String

We will show that small lateral motions of a stretched string are governed by the one-dimensional wave equation. This problem has important practical applications, for example in explaining the behaviors of stringed musical instruments. However, the reason we discuss it here is that waves in a string are easier to visualize than one-dimensional compressional and shear waves in an elastic material. In addition to the governing equation being identical, we will show that the boundary conditions discussed in the previous section for one-dimensional waves in an elastic half space have equivalents in the lateral motion of a string. As a result, the motion of a string makes an excellent analog for visualizing one-dimensional waves in elastic materials.

Equation of motion

Consider a string that is stretched between two points (Fig. 2.7.a). We can describe the lateral motion of the string in terms of its lateral displacement $u = u(x,t)$ relative to the equilibrium position, where x is the position (Fig. 2.7.b).

Figure 2.7.c shows a free-body diagram of an element of the string of length ds. We assume the tension T of the string to be uniform. The angle θ is the angle between the centerline of the string and the x axis. Newton's second law for the lateral motion of the string is

$$\sin\left(\theta + \frac{\partial\theta}{\partial x}\,dx\right) - T\sin\theta = (\rho_L\,ds)\frac{\partial^2 u}{\partial t^2},$$

where ρ_L is the mass per unit length of the string. Expanding the first term on the left side in a Taylor series, we obtain the equation

$$T\frac{\partial\theta}{\partial x}\,dx = \rho_L\frac{\partial^2 u}{\partial t^2}\,ds. \tag{2.10}$$

We can write the length of the element ds in terms of dx:

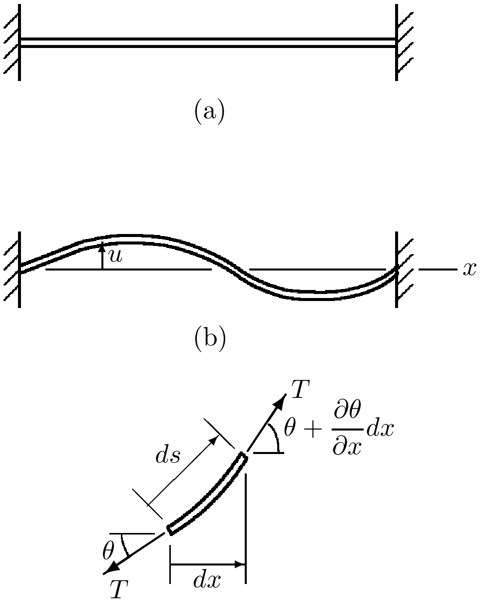
$$ds = (dx^2 + du^2)^{1/2} = \left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right]^{1/2} dx.$$

If we assume that the term $\partial u/\partial x$ (the slope of the string relative to the x axis) is small, then $ds = dx$ and $\theta = \partial u/\partial x$. Therefore we can write Eq. (2.10) in the form

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2\frac{\partial^2 u}{\partial x^2}, \tag{2.11}$$

where

$$\alpha = \left(\frac{T}{\rho_L}\right)^{1/2}.$$



(a)

(b)

(c)

Figure 2.7: (a) A stretched string. (b) A lateral motion. u is the lateral displacement. (c) Free-body diagram of an element of the string.

So small lateral motions of a stretched string are governed by the one-dimensional wave equation. The wave velocity α depends on the tension and mass of the string.

Boundary conditions

When a string is fixed at the ends as shown in Fig. 2.7.a, the boundary condition at the fixed ends is simply that the displacement $u = 0$. Suppose that the string is supported at $x = 0$ so that it can move in the lateral direction (Fig. 2.8.a). If the lateral displacement of the string at $x = 0$ is specified, the boundary condition is

$$u(0, t) = p(t),$$

where $p(t)$ is a prescribed function of time (Fig. 2.8.b). This boundary condition is identical in form to Eqs. (2.3) and (2.6).

Suppose that a prescribed lateral force $F(t)$ is applied to the string at $x = 0$ (Fig. 2.8.c). As shown in Fig. 2.8.d, the force $F(t)$ is related to the tension in

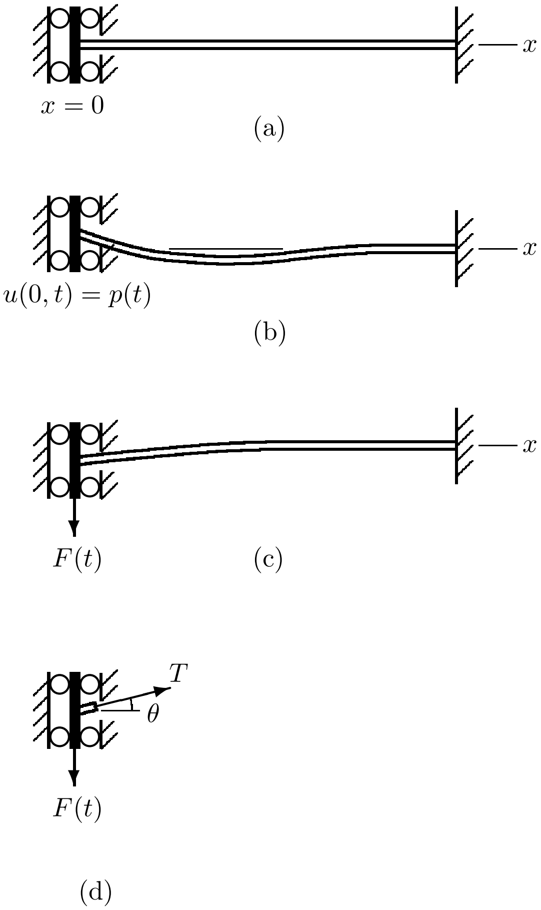


Figure 2.8: (a) A string that can move in the lateral direction at $x = 0$. (b) A displacement boundary condition. (c) A force boundary condition. (d) Free-body diagram of the left end of the string.

the string by $F(t) = T\sin\theta$. (We neglect the mass of the support.) Because we assume θ to be small,

$$F(t) = T\theta = T\frac{\partial u}{\partial x}.$$

Therefore the boundary condition at $x = 0$ is

$$\frac{\partial u}{\partial x}(0, t) = p(t),$$

where $p(t) = F(t)/T$. This boundary condition is identical in form to Eqs. (2.5) and (2.9). If the end of the string is "free," that is, $F(t) = 0$, the boundary condition at $x = 0$ is

$$\frac{\partial u}{\partial x}(0, t) = 0.$$

Thus the slope of the string is zero at the "free" end.

2.4 One-Dimensional Solutions

We now apply the D'Alembert solution to some important problems in one-dimensional wave propagation. We state the problems in terms of compressional wave (P wave) propagation in an elastic material. However, as we pointed out in Sections 2.2 and 2.3, they can also be interpreted in terms of shear wave (S wave) propagation in an elastic material or in terms of the propagation of disturbances in the lateral displacement of a stretched string.

Initial-value problem

Suppose that at $t = 0$, the x_1 components of the displacement and velocity of an unbounded elastic material are prescribed:

$$\begin{aligned} u_1(x_1, 0) &= p(x_1), \\ \frac{\partial u_1}{\partial t}(x_1, 0) &= q(x_1), \end{aligned} \tag{2.12}$$

where $p(x_1)$ and $q(x_1)$ are given functions. Let us further assume that the displacements and velocities of the material in the x_2 and x_3 directions are zero at $t = 0$. These initial conditions cause one-dimensional motion of the material described by the displacement field given in Eq. (2.4), and the motion is governed by the one-dimensional wave equation

$$\frac{\partial^2 u_1}{\partial t^2} = \alpha^2 \frac{\partial^2 u_1}{\partial x_1^2} \tag{2.13}$$

with wave velocity

$$\alpha = \left(\frac{\lambda + 2\mu}{\rho_0} \right)^{1/2}.$$

Although we have stated this problem in terms of one-dimensional motion of an elastic material, it is easier to visualize when it is stated in terms of the lateral motion of a string: at $t = 0$, the lateral displacement and the lateral velocity of an unbounded string are prescribed as functions of position.

To obtain the solution, we write the displacement $u_1(x_1, t)$ in terms of the D'Alembert solution:

$$u_1(x_1, t) = f(\xi) + g(\eta) = f(x_1 - \alpha t) + g(x_1 + \alpha t). \tag{2.14}$$

This solution satisfies the one-dimensional wave equation. Our objective is to determine functions $f(\xi)$ and $g(\eta)$ that satisfy the initial conditions.

We can obtain an expression for the velocity from Eq. (2.14) by using the chain rule:

$$\begin{aligned} \frac{\partial u_1}{\partial t}(x_1, t) &= \frac{df(\xi)}{d\xi} \frac{\partial \xi}{\partial t} + \frac{dg(\eta)}{d\eta} \frac{\partial \eta}{\partial t} \\ &= -\alpha \frac{df(\xi)}{d\xi} + \alpha \frac{dg(\eta)}{d\eta}. \end{aligned}$$

By setting $t = 0$ in this equation and in Eq. (2.14), we can write the initial conditions, Eq. (2.12), as

$$f(x_1) + g(x_1) = p(x_1), \tag{2.15}$$

$$-\alpha \frac{df(x_1)}{dx_1} + \alpha \frac{dg(x_1)}{dx_1} = q(x_1). \tag{2.16}$$

We integrate the second equation from 0 to x_1, obtaining the equation

$$-f(x_1) + g(x_1) = \frac{1}{\alpha} \int_0^{x_1} q(\bar{x})\, d\bar{x} - f(0) + g(0), \tag{2.17}$$

where $\bar{x}$ is an integration variable. Now we can solve Eqs. (2.15) and (2.17) for the functions $f(x_1)$ and $g(x_1)$:

$$\begin{aligned} f(x_1) &= \frac{1}{2} p(x_1) - \frac{1}{2\alpha} \int_0^{x_1} q(\bar{x})\, d\bar{x} + \frac{1}{2} f(0) - \frac{1}{2} g(0), \\ g(x_1) &= \frac{1}{2} p(x_1) + \frac{1}{2\alpha} \int_0^{x_1} q(\bar{x})\, d\bar{x} - \frac{1}{2} f(0) + \frac{1}{2} g(0). \end{aligned}$$

Substituting these expressions into Eq. (2.14), we obtain the solution of the initial-value problem in terms of the initial conditions:

$$u(x_1, t) = \frac{1}{2} p(x_1 - \alpha t) + \frac{1}{2} p(x_1 + \alpha t) + \frac{1}{2\alpha} \int_{x_1 - \alpha t}^{x_1 + \alpha t} q(\bar{x})\, d\bar{x}. \tag{2.18}$$

This solution is easy to interpret when the initial velocity is zero: $q(x_1) = 0$. In this case, the solution consists of the sum of forward and rearward propagating waves that have the same form as the initial displacement distribution $p(x_1)$ but have one-half its amplitude:

$$u(x_1, t) = \frac{1}{2}p(x_1 - \alpha t) + \frac{1}{2}p(x_1 + \alpha t). \tag{2.19}$$

The displacement distribution at any value of time can be constructed by superimposing the forward and rearward propagating waves. Figure 2.9 shows an initial displacement distribution $p(x_1)$ [the solid line] and the resulting displacement distribution at several values of time. Observe that for times such that $t\alpha \geq 30$, the displacement distribution consists of forward and rearward propagating waves having the same form as the initial distribution but with one-half its amplitude.

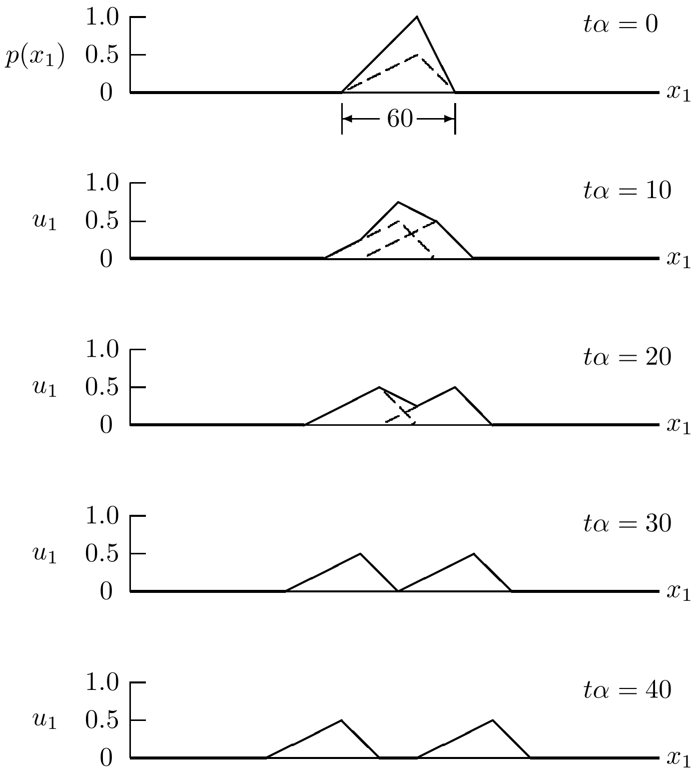


Figure 2.9: An initial displacement distribution and the displacement distributions at subsequent times.

The solution given by Eq. (2.18) is not as easy to interpret when the initial velocity is not zero, because the solution involves the integral of the initial velocity distribution. At a given position x_1 and time t, the solution depends on the initial velocity distribution over the interval from $x_1 - \alpha t$ to $x_1 + \alpha t$. In Fig. 2.10 we illustrate an example in which the initial displacement distribution is zero and the initial velocity is constant over an interval of the x_1 axis and zero everywhere else. (To visualize this initial condition, imagine hitting a stretched string with a board 2 units wide.) The resulting displacement distribution is shown at several values of time. For times such that $t\alpha \geq 1$, the solution consists of forward and rearward propagating "ramp" waves with a uniform displacement between them.

Initial-boundary value problems

Suppose that a half space of elastic material is initially undisturbed, that is, $\mathbf{u}(\mathbf{x}, t) = 0$ for $t \leq 0$, and we subject the boundary to the displacement boundary condition

$$u_1(0, t) = p(t), \tag{2.20}$$

where $p(t)$ is a prescribed function of time that vanishes for $t \leq 0$ (Fig. 2.11). This is called an initial-boundary value problem: the initial state of the material and the displacement at the boundary of the half space are prescribed.

We can determine the solution by expressing the displacement in terms of the D'Alembert solution. We expect the imposed motion of the boundary to give rise to a wave propagating in the positive x_1 direction but not a wave propagating in the negative x_1 direction. Therefore we express the displacement in terms of the part of the D'Alembert solution that represents a forward propagating wave. To do so, it is convenient to redefine the variable ξ by

$$\xi = -\frac{1}{\alpha}(x_1 - \alpha t) = t - \frac{x_1}{\alpha}. \tag{2.21}$$

Thus we assume that

$$u_1(x_1, t) = f(\xi) = f\left(t - \frac{x_1}{\alpha}\right).$$

By setting $x_1 = 0$ in this expression, we can write the displacement boundary condition as

$$f(t) = p(t).$$

We see that the function $f(t)$ is equal to the prescribed function $p(t)$, so the solution for the displacement field is

$$u_1(x_1, t) = f(\xi) = p\left(t - \frac{x_1}{\alpha}\right).$$

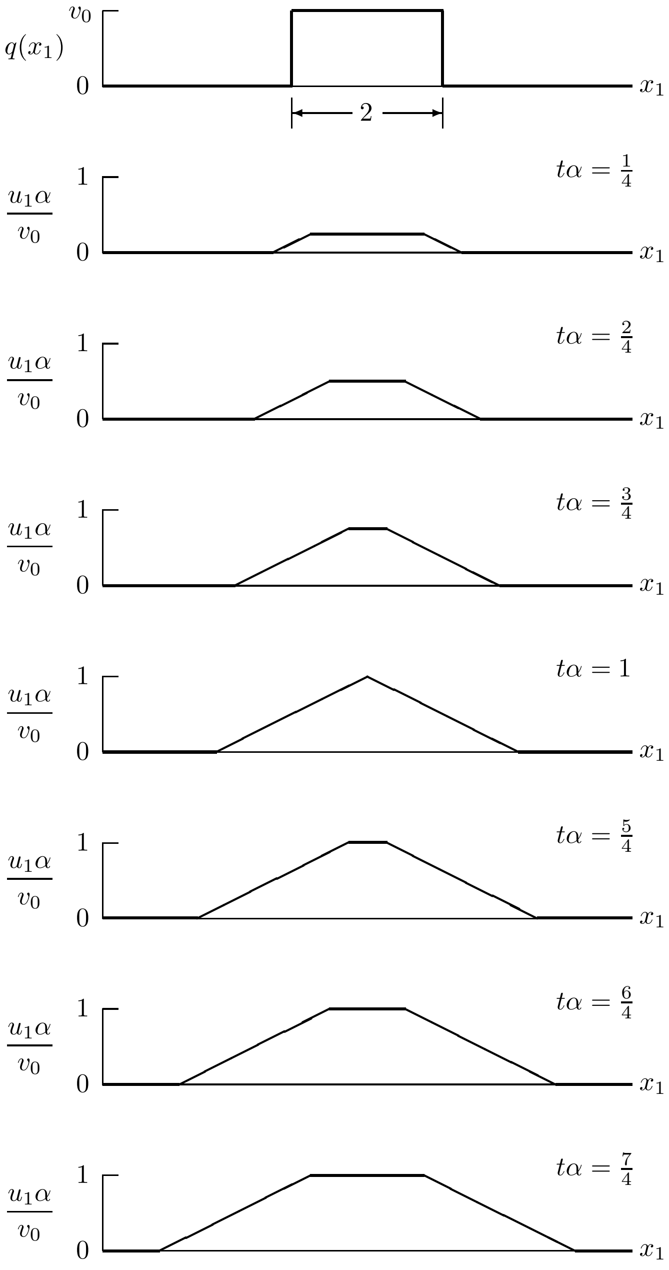


Figure 2.10: An initial velocity distribution and the displacement distribution at subsequent times.

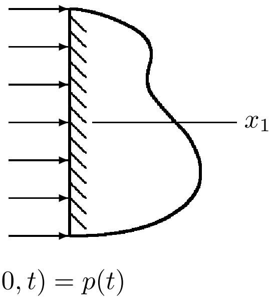


Figure 2.11: Half space subjected to a displacement boundary condition.

Because the function $p(t)$ vanishes for $t \leq 0$, note that $u_1 = 0$ when $t \leq x_1/\alpha$. Figure 2.12.a shows a particular function $p(t)$. The resulting displacement field at time t_0 is plotted as a function of x_1 in Fig. 2.12.b.

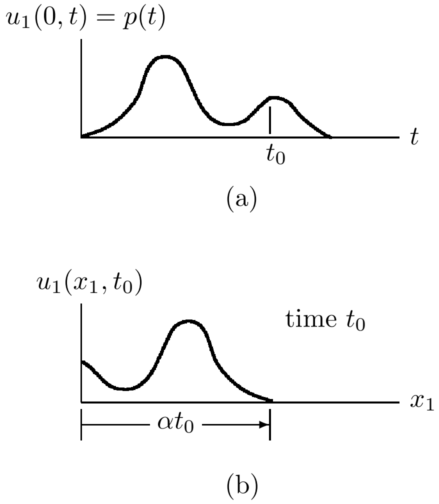


Figure 2.12: (a) A displacement boundary condition. (b) The resulting displacement field at time t_0.

Suppose that we subject the boundary of the half space to a normal stress boundary condition

$$T_{11}(0,t) = \bar{p}(t), \tag{2.22}$$

where $\bar{p}(t)$ is a prescribed function of time that vanishes for $t \leq 0$ (Fig. 2.13). The stress component T_{11} is related to the displacement field by

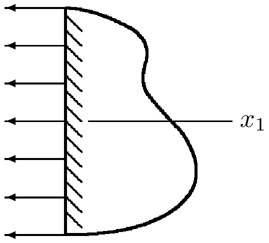


Figure 2.13: Half space subjected to a stress boundary condition.

$$T_{11} = (\lambda + 2\mu)E_{11} = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1}, \tag{2.23}$$

so we can write the boundary condition as

$$\frac{\partial u_1}{\partial x_1}(0,t) = p(t), \tag{2.24}$$

where

$$p(t) = \frac{\bar{p}(t)}{\lambda + 2\mu}.$$

Let us assume that the displacement field consists of a forward propagating wave:

$$u_1(x_1,t) = f(\xi) = f\left(t - \frac{x_1}{\alpha}\right).$$

The partial derivative of this expression with respect to x_1 is

$$\frac{\partial u_1}{\partial x_1}(x_1,t) = \frac{df(\xi)}{d\xi}\frac{\partial \xi}{\partial x_1} = -\frac{1}{\alpha}\frac{df(\xi)}{d\xi}.$$

By setting $x_1 = 0$ in this expression, we can write the boundary condition as

$$\frac{df(t)}{dt} = -\alpha p(t). \tag{2.25}$$

Integrating this equation from 0 to t yields the function $f(t)$ in terms of $p(t)$:

$$f(t) = f(0) - \alpha \int_0^t p(\bar{t})\, d\bar{t},$$

where $\bar{t}$ is an integration variable. Thus the displacement field is

$$u_1(x_1,t) = f(\xi) = f(0) - \alpha \int_0^{t - x_1/\alpha} p(\bar{t})\, d\bar{t}.$$

The initial condition is violated unless the constant $f(0) = 0$. Therefore the solution is

$$u_1(x_1, t) = -\alpha \int_0^{t - x_1/\alpha} p(\bar{t})\, d\bar{t}. \tag{2.26}$$

This problem can be approached in a simpler way. We take the partial derivative of the one-dimensional wave equation with respect to x_1 and write the result as

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_1}{\partial x_1}\right) = \alpha^2 \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial u_1}{\partial x_1}\right). \tag{2.27}$$

This equation shows that the strain $\partial u_1/\partial x_1$ is governed by the same one-dimensional wave equation that governs the displacement. Therefore we can write a D'Alembert solution for the strain field:

$$\frac{\partial u_1}{\partial x_1}(x_1, t) = h(\xi) = h\left(t - \frac{x_1}{\alpha}\right).$$

By setting $x_1 = 0$ in this expression, we can write the boundary condition as

$$h(t) = p(t).$$

We see that the function $h(t)$ is equal to the prescribed function $p(t)$, so the solution for the strain field is

$$\frac{\partial u_1}{\partial x_1}(x_1, t) = h(\xi) = p\left(t - \frac{x_1}{\alpha}\right).$$

Although this approach is simpler, it results in the solution for the strain field instead of the displacement field.

Reflection and transmission at an interface

Suppose that half spaces of two different elastic materials are bonded together as shown in Fig. 2.14. Let the left half space be material L and the right half space material R. The interface between the two materials is at $x_1 = 0$. Let us assume that there is a prescribed *incident* wave in the left half space propagating in the positive x_1 direction (Fig. 2.15.a). The interaction of the wave with the interface between the two materials gives rise to a *reflected* wave propagating in the negative x_1 direction in material L and a *transmitted* wave propagating in the positive x_1 direction in material R (Fig. 2.15.b).

We can write the displacement field in the material L as

$$u_1^L = p(\xi_L) + g(\eta_L) = p\left(t - \frac{x_1}{\alpha_L}\right) + g\left(t + \frac{x_1}{\alpha_L}\right), \tag{2.28}$$

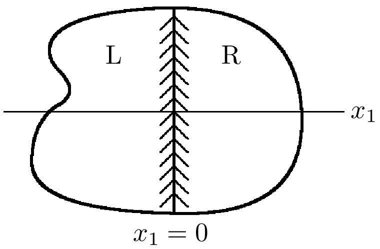


Figure 2.14: Bonded half spaces of two elastic materials.

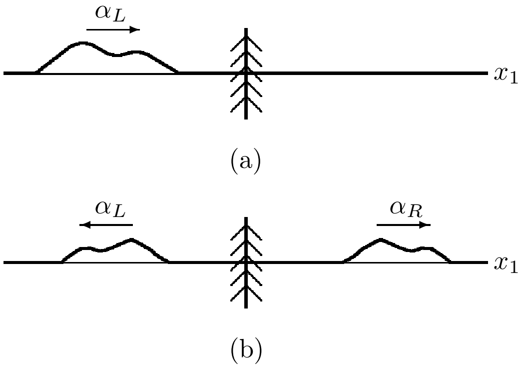


Figure 2.15: (a) The incident wave. (b) Reflected and transmitted waves.

where the function p is the incident wave and the function g is the reflected wave. The displacement field in material R is the transmitted wave

$$u_1^R = h(\xi_R) = h\left(t - \frac{x_1}{\alpha_R}\right). \tag{2.29}$$

Our objective is to determine the reflected and transmitted waves in terms of the incident wave, that is, we want to determine the functions g and h in terms of the function p.

At the bonded interface the displacements of the two materials are equal,

$$u_1^L(0,t) = u_1^R(0,t), \tag{2.30}$$

and the normal stresses are equal,

$$T_{11}^L(0,t) = T_{11}^R(0,t).$$

By using Eq. (2.23), we can write the relation between the stresses as

$$\rho_L \alpha_L^2 \frac{\partial u_1^L}{\partial x_1}(0,t) = \rho_R \alpha_R^2 \frac{\partial u_1^R}{\partial x_1}(0,t), \tag{2.31}$$

where ρ_L and ρ_R are the densities of the two materials in the reference state.

The partial derivatives of Eqs. (2.28) and (2.29) with respect to x_1 are

$$\begin{aligned}
\frac{\partial u_1^L}{\partial x_1} &= \frac{dp(\xi_L)}{d\xi_L}\frac{\partial \xi_L}{\partial x_1} + \frac{dg(\eta_L)}{d\eta_L}\frac{\partial \eta_L}{\partial x_1} \\
&= -\frac{1}{\alpha_L}\frac{dp(\xi_L)}{d\xi_L} + \frac{1}{\alpha_L}\frac{dg(\eta_L)}{d\eta_L}, \\
\frac{\partial u_1^R}{\partial x_1} &= \frac{dh(\xi_R)}{d\xi_R}\frac{\partial \xi_R}{\partial x_1} = -\frac{1}{\alpha_R}\frac{dh(\xi_R)}{d\xi_R}.
\end{aligned}$$

Using these two equations and Eqs. (2.28) and (2.29), we can write the displacement condition, Eq. (2.30), and the stress condition, Eq. (2.31), in the forms

$$p(t) + g(t) = h(t), \tag{2.32}$$

$$-\frac{dp(t)}{dt} + \frac{dg(t)}{dt} = -K\frac{dh(t)}{dt}, \tag{2.33}$$

where we define

$$K = \frac{z_R}{z_L} = \frac{\rho_R \alpha_R}{\rho_L \alpha_L}.$$

The term $z = \rho_0 \alpha$ is called the *acoustic impedance* of a material. Integrating Eq. (2.33), we obtain the equation

$$-p(t) + g(t) = -Kh(t) + C,$$

where C is an integration constant. Solving this equation and Eq. (2.32) for the functions $g(t)$ and $h(t)$, we determine the reflected and transmitted waves in terms of the incident wave:

$$g\left(t + \frac{x_1}{\alpha_L}\right) = \left(\frac{1-K}{1+K}\right) p\left(t + \frac{x_1}{\alpha_L}\right), \tag{2.34}$$

$$h\left(t - \frac{x_1}{\alpha_R}\right) = \left(\frac{2}{1+K}\right) p\left(t - \frac{x_1}{\alpha_R}\right). \tag{2.35}$$

We see that the reflection and transmission of a compressional wave at an interface between two elastic materials is characterized by the ratio of acoustic impedances $K = z_R/z_L$. If the two materials have the same acoustic impedance, there is no reflected wave and the transmitted wave is identical to the incident wave.

Reflection at a rigid boundary

Suppose that the right half space in Fig. 2.14 is rigid. That is, the left half space is bonded to a fixed support at $x_1 = 0$. Let us assume that there is a prescribed incident wave in the left half space propagating in the positive x_1 direction. We can determine the wave reflected from the boundary from Eq. (2.34) by letting the acoustic impedance $z_R \to \infty$, which means that $K \to \infty$. The reflected wave is

$$g\left(t + \frac{x_1}{\alpha_L}\right) = -p\left(t + \frac{x_1}{\alpha_L}\right).$$

This result has a simple interpretation. Consider the incident wave shown in Fig. 2.16.a. The reflected wave is identical in form to the incident wave and is of opposite sign. It can be visualized as shown in Fig. 2.16.b. As the incident wave reaches the boundary, the solution is obtained by superimposing the incident and reflected waves (Figs. 2.16.c and 2.16.d). Notice that the displacement is zero at the boundary. After the incident wave has "passed through" the boundary, only the reflected wave remains (Fig. 2.16.e).

Reflection at a free boundary

Now suppose that the left half space in Fig. 2.14 has a free boundary at $x_1 = 0$, and let us assume that there is a prescribed incident wave in the left half space propagating in the positive x_1 direction. We can determine the wave reflected from the boundary from Eq. (2.34) by letting the acoustic impedance $z_R \to 0$, which means that $K \to 0$. The reflected wave is

$$g\left(t + \frac{x_1}{\alpha_L}\right) = p\left(t + \frac{x_1}{\alpha_L}\right).$$

Consider the wave approaching a free boundary shown in Fig. 2.17.a. The reflected wave can be visualized as shown in Fig. 2.17.b. The solution as the incident wave reaches the boundary is shown in Figs. 2.17.c and 2.17.d. Notice that the slope is zero at the boundary. After the incident wave has "passed through" the boundary, only the reflected wave remains (Fig. 2.17.e).

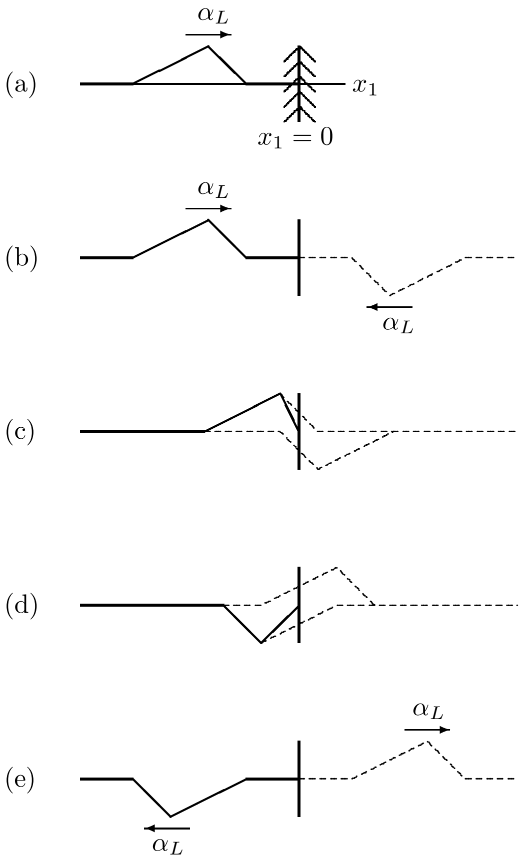


Figure 2.16: Reflection of a wave at a rigid boundary.

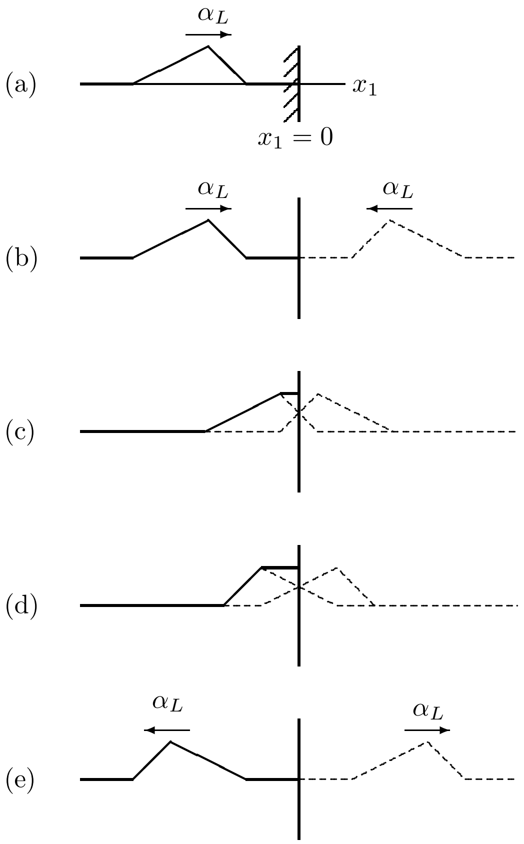


Figure 2.17: Reflection of a wave at a free boundary.

Exercises

EXERCISE 2.7 A particular type of steel has Lamé constants $\lambda = 1.15 \times 10^{11}$ Pa and $\mu = 0.77 \times 10^{11}$ Pa and density $\rho_0 = 7800$ kg/m^3. Determine (a) the compressional wave velocity α; (b) the shear wave velocity β.

Answer: (a) $\alpha = 5.87$ km/s. (b) $\beta = 3.14$ km/s.

EXERCISE 2.8 Consider the steel described in Exercise 2.7. A compressional wave propagates through the material. The displacement field is

$$u_1 = 0.001 \sin[2(x_1 - \alpha t)] \text{ m}, \quad u_2 = 0, \quad u_3 = 0.$$

Determine the maximum normal stress T_{11} caused by the wave.

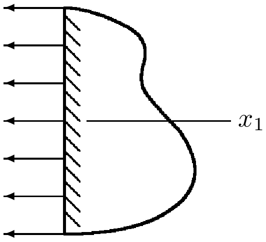
Answer: 538 MPa.

EXERCISE 2.9 Suppose that at $t = 0$ the velocity of an unbounded elastic material is zero and its displacement field is described by $u_1(x_1, 0) = p(x_1)$, where

$$\begin{aligned} x_1 < 0 &\quad p(x_1) = 0, \\ 0 \le x_1 \le 1 &\quad p(x_1) = A \sin(\pi x_1), \\ x_1 > 1 &\quad p(x_1) = 0, \end{aligned}$$

where A is a constant. Plot the displacement field as a function of x_1 when $t = 0$, $t = 1/(2\alpha)$, and $t = 1/\alpha$.

EXERCISE 2.10



$T_{11}(0, t) = T_0 H(t)$

Suppose that a half space of elastic material is initially undisturbed and the boundary is subjected to a uniform normal stress $T_{11} = T_0\, H(t)$, where T_0 is a constant and the Heaviside, or step function $H(t)$ is defined by

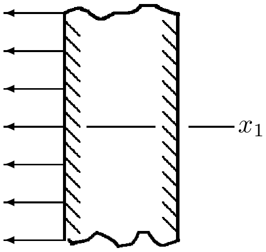
$$H(t) = \begin{cases} 0 & \text{when} \quad t \le 0, \\ 1 & \text{when} \quad t > 0. \end{cases}$$

(a) Show that the resulting displacement field in the material is

$$u_1 = -\frac{\alpha T_0}{\lambda + 2\mu}\left(t - \frac{x_1}{\alpha}\right) H\left(t - \frac{x_1}{\alpha}\right).$$

(b) Assume that $T_0/(\lambda + 2\mu) = 1$. Plot the displacement field as a function of x_1 when $t = 1/\alpha$, $t = 2/\alpha$, and $t = 3/\alpha$.

Exercise 2.11



$T_{11}(0,t) = T_0 H(t)$

Consider a plate of elastic material of thickness L. Assume that the plate is infinite in extent in the x_2 and x_3 directions and that the right boundary is free of stress. Suppose that the plate is initially undisturbed and the left boundary is subjected to a uniform normal stress $T_{11} = T_0\, H(t)$, where T_0 is a constant and $H(t)$ is the step function defined in Exercise 2.10. This boundary condition gives rise to a wave propagating in the positive x_1 direction. When $t = L/\alpha$, the wave reaches the right boundary and causes a reflected wave. Show that from $t = L/\alpha$ until $t = 2L/\alpha$ the displacement field in the plate is

$$u_1 = -\frac{\alpha T_0}{\lambda + 2\mu}\left[\left(t - \frac{x_1}{\alpha}\right) H\left(t - \frac{x_1}{\alpha}\right) + \left(t + \frac{x_1}{\alpha} - \frac{2L}{\alpha}\right) H\left(t + \frac{x_1}{\alpha} - \frac{2L}{\alpha}\right)\right].$$

Exercise 2.12 Confirm that the solution given by Eq. (2.18) satisfies the initial conditions, Eq. (2.12).

Exercise 2.13 Confirm that the solution given by Eq. (2.26) satisfies the boundary condition, Eq. (2.24).

2.5 The Finite Element Method

The finite element method is the most flexible and widely used technique for obtaining approximate solutions to problems in fluid and solid mechanics, including problems involving the propagation of waves. In this section we apply finite elements to one-dimensional wave propagation in a layer of elastic material. This simple example permits us to introduce the terminology and concepts used in finite-element solutions of more complex problems. In particular, readers familiar with applications of finite elements to static problems will recognize how the approach we describe can be extended to two- and three-dimensional problems in elastic wave propagation. Persons interested in wave propagation who are new to finite elements may be motivated to seek further background. Our discussion is based on Becker, *et al.* (1981) and Hughes (2000).

One-dimensional waves in a layer

Figure 2.18 shows a layer of isotropic elastic material with thickness H that is unbounded in the x_2 and x_3 directions. The layer is fixed at the right boundary $x_1 = H$. We assume that the material is initially stationary, and at $t = 0$ is subjected to the normal stress boundary condition

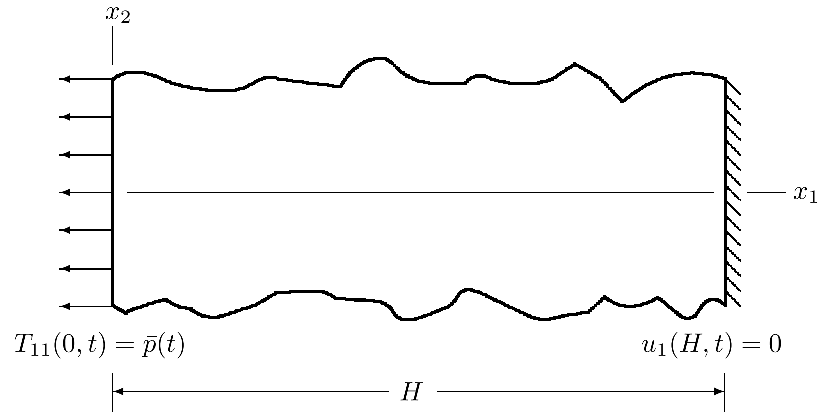


Figure 2.18: A layer subjected to a uniform normal stress boundary condition at the left boundary and fixed at the right boundary.

$$T_{11}(0,t) = \bar{p}(t), \tag{2.36}$$

where $\bar{p}(t)$ is a prescribed function of time. The stress component T_{11} is related to the displacement field by

$$T_{11} = (\lambda + 2\mu)E_{11} = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1}, \tag{2.37}$$

so the conditions at the left and right boundaries of the layer can be written

$$(\lambda + 2\mu)\frac{\partial u_1}{\partial x_1}(0, t) = \bar{p}(t), \tag{2.38}$$

$$u_1(H, t) = 0. \tag{2.39}$$

The uniform stress boundary condition will give rise to one-dimensional waves in the layer governed by the wave equation (2.2),

$$\frac{\partial^2 u_1}{\partial t^2} = \alpha^2 \frac{\partial^2 u_1}{\partial x_1^2}, \tag{2.40}$$

where

$$\alpha^2 = \frac{\lambda + 2\mu}{\rho_0}. \tag{2.41}$$

Our objective is to determine the displacement field $u_1(x_1, t)$ in the layer by solving Eq. (2.40) subject to the boundary conditions (2.38) and (2.39). For example, at a particular time t the displacement field might be as shown in Fig. 2.19. To apply the finite element method, we divide the layer into N parts (elements) of equal width $h = H/N$ and seek an approximation to the displacement field that is linear within each element (Fig. 2.20). That is, the

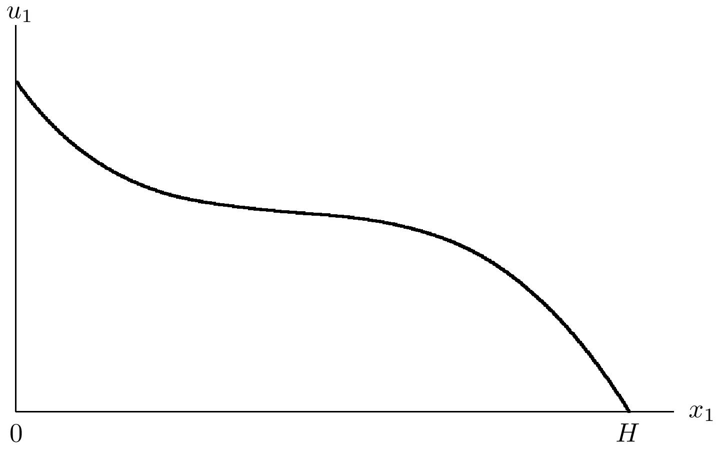


Figure 2.19: Distribution of the displacement in the layer at a time t.

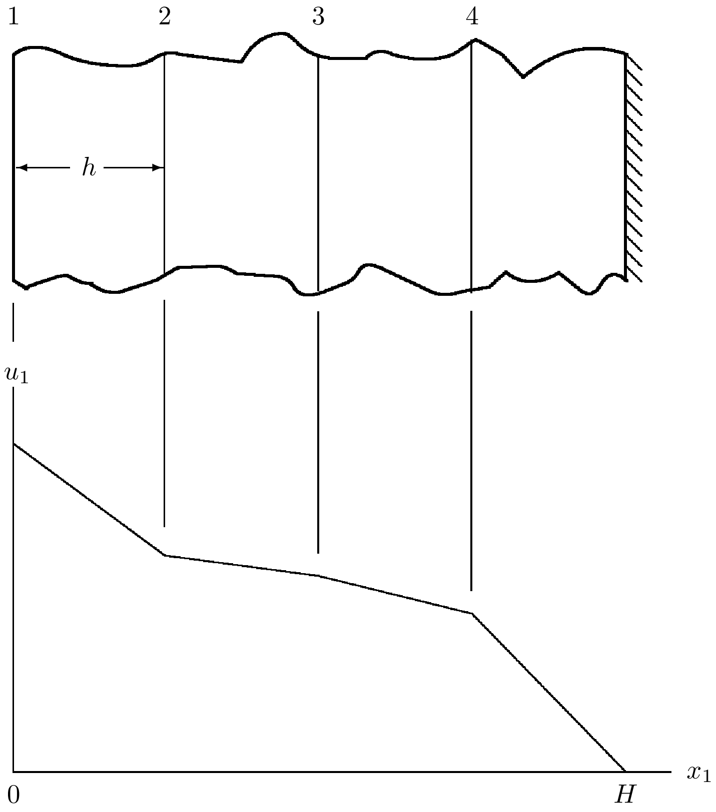


Figure 2.20: Approximate solution for the displacement in the layer with $N = 4$.

displacement field is approximated by a continuous, piecewise-linear function. In the example shown, $N = 4$. The boundaries 1, 2, 3, and 4 are called the *nodes*. Before explaining how to determine the approximate displacement field in terms of a piecewise linear function, we must discuss how such functions are expressed in the finite element method.

Basis functions

A key step in the finite element method is the introduction of *basis functions* that describe the displacement of the material within the elements. For our one-dimensional example, the most commonly used basis functions are shown in Fig. (2.21). The functions $B_1, B_2, \ldots, B_N$ depend upon x_1 but not time. The displacement $u_1(x_1, t)$ is expressed in terms of these functions as

$$u_1 = \sum_{i=1}^{N} d_m B_m = d_m B_m, \qquad 0 \leq x_1 \leq H, \tag{2.42}$$

where the coefficients d_m depend upon time but not x_1. Because of the way the basis functions are defined, at a given time t Eq. (2.42) describes a continuous, piecewise-linear function of x_1. Furthermore, the coefficients d_m *are the values of the displacement at the nodes.* Any piecewise-linear displacement distribution in the domain $0 \leq x_1 \leq H$ that vanishes at $x_1 = H$ can be represented in this way. The displacement distribution in Fig. 2.20 is described by

$$u_1 = 4.35B_1 + 2.87B_2 + 2.60B_3 + 2.10B_4, \tag{2.43}$$

as shown in Fig. 2.22. We see that obtaining an approximate solution for the displacement field $u_1(x_1, t)$ requires determining the coefficients d_m as functions of time.

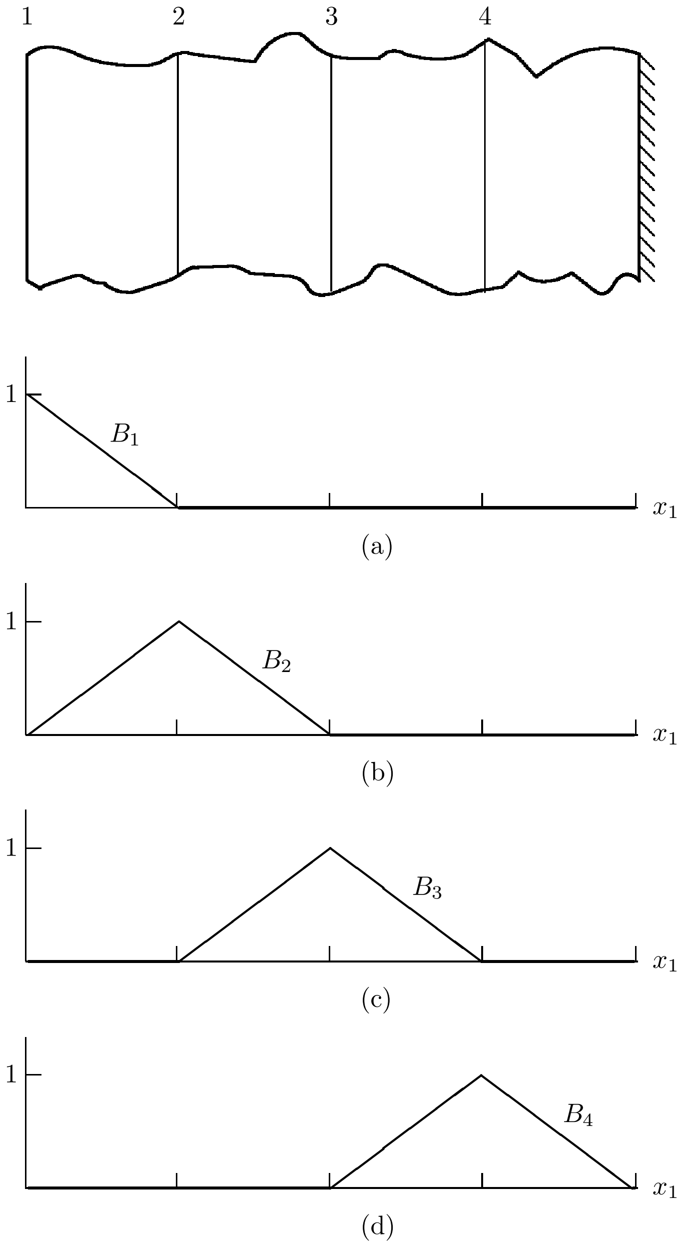
The finite element solution

Let $w(x_1)$ be a function of x_1 that equals zero at the fixed end of the layer: $w(H) = 0$. We multiply Eq. (2.40) by $w(x_1)$ and integrate the resulting equation with respect to x_1 from 0 to H:

$$\rho_0 \int_0^H \frac{\partial^2 u_1}{\partial t^2} w \, dx_1 = (\lambda + 2\mu) \int_0^H \frac{\partial^2 u_1}{\partial x_1^2} w \, dx_1. \tag{2.44}$$

Using the result

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} w\right) = \frac{\partial^2 u_1}{\partial x_1^2} w + \frac{\partial u_1}{\partial x_1}\frac{dw}{dx_1}, \tag{2.45}$$



(a)

(b)

(c)

(d)

Figure 2.21: Basis functions for the displacement in the layers.

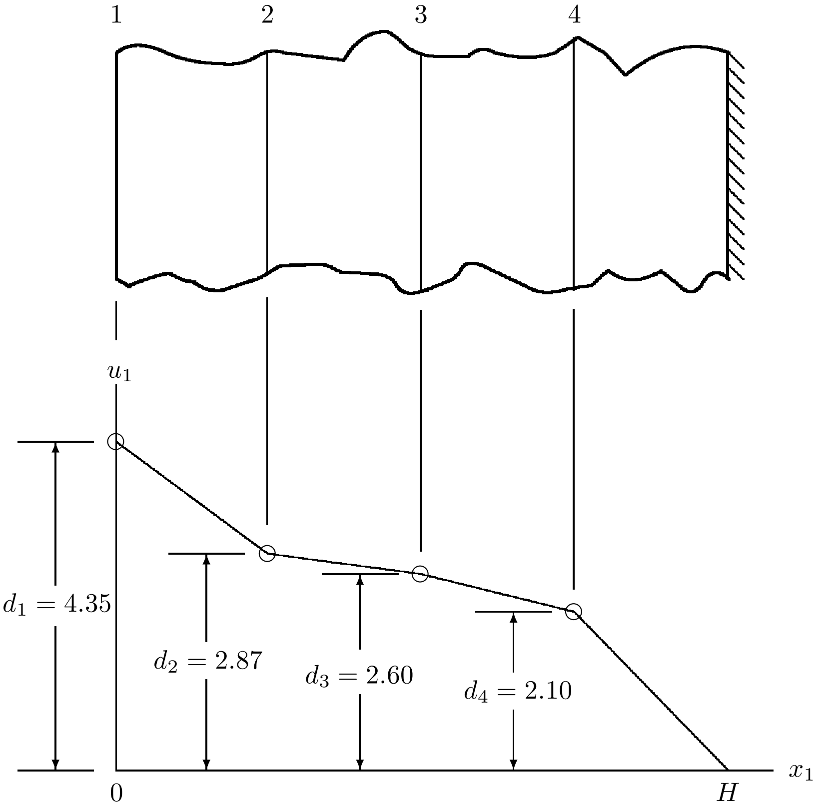


Figure 2.22: Displacement distribution described by Eq. (2.43).

we integrate the right side of Eq. (2.44) by parts, obtaining

$$\rho_0 \int_0^H \frac{\partial^2 u_1}{\partial t^2} w \, dx_1 = (\lambda + 2\mu) \left[\frac{\partial u_1}{\partial x_1} w \right]_0^H - (\lambda + 2\mu) \int_0^H \frac{\partial u_1}{\partial x_1} \frac{dw}{dx_1} \, dx_1. \tag{2.46}$$

Using the imposed condition $w(H) = 0$ and the boundary condition (2.38), this becomes

$$\rho_0 \int_0^H \frac{\partial^2 u_1}{\partial t^2} w \, dx_1 = -\bar{p}(t) w(0) - (\lambda + 2\mu) \int_0^H \frac{\partial u_1}{\partial x_1} \frac{dw}{dx_1} \, dx_1. \tag{2.47}$$

This equation, together with the boundary conditions (2.38) and (2.39), is called the *weak*, or *variational*, formulation of this boundary-value problem. Instead of imposing a condition at each point of the domain of the solution, as is done by the differential equation (2.40), the weak form imposes a condition on integrals over the entire domain of the solution. Notice that we are implicitly assuming that the displacement distribution u_1 and the function $w(x_1)$ are such that the integrals in Eq. (2.47) exist.

To solve Eq. (2.47) using finite elements, the displacement field is expressed in terms of the basis functions we introduced:

$$u_1 = d_m B_m. \tag{2.48}$$

The function $w(x_1)$ is also expressed in terms of the basis functions,

$$w(x_1) = c_m B_m, \tag{2.49}$$

where the c_m are *arbitrary constants*. (This representation of $w(x_1)$ ensures that $w(H) = 0$.) Substituting the expressions (2.48) and (2.49) into Eq. (2.47) yields

$$\rho_0 \int_0^H \frac{d^2 d_m}{dt^2} B_m c_k B_k \, dx_1 = -\bar{p}(t) c_k \delta_{k1} - (\lambda + 2\mu) \int_0^H d_m \frac{dB_m}{dx_1} c_k \frac{dB_k}{dx_1} \, dx_1. \tag{2.50}$$

The Kronecker delta in the first term on the right arises because the only basis function that is not equal to zero at $x_1 = 0$ is B_1. Therefore $w(0) = c_k B_k(0) = c_k \delta_{k1}$. Using (2.41), Eq. (2.50) can be written

$$\left(C_{km} \frac{d^2 d_m}{dt^2} + K_{km} d_m - f_k \right) c_k = 0, \tag{2.51}$$

where

$$\begin{aligned} C_{km} &= \rho_0 \int_0^H B_k B_m \, dx_1, \\ K_{km} &= (\lambda + 2\mu) \int_0^H \frac{dB_k}{dx_1} \frac{dB_m}{dx_1} \, dx_1, \\ f_k &= -\bar{p}(t) \delta_{k1}. \end{aligned} \tag{2.52}$$

The terms C_{km} and K_{km} are constants that can be evaluated by integration of the basis functions and their derivatives. Because the coefficients c_k can be chosen arbitrarily, the expression in parentheses in Eq. (2.51) must vanish for each value of k, resulting in the equations

$$C_{km}\frac{d^2 d_m}{dt^2} + K_{km} d_m = f_k, \quad k = 1, 2, \ldots, N, \tag{2.53}$$

or

$$\mathbf{C}\frac{d^2}{dt^2}\mathbf{d} + \mathbf{K}\mathbf{d} = \mathbf{f}. \tag{2.54}$$

The matrix of the linear transformation **C** is called the *mass matrix* for this problem, the matrix of **K** is the *stiffness matrix*, and **f** is the *force vector*.[1] Completing the solution for the vector of nodal displacements **d** requires numerical integration of Eq. (2.54). Before carrying out that step, we briefly discuss the evaluation of the matrices of **C** and **K**.

Evaluation of the mass and stiffness matrices

The basis functions B_1 and B_2 are shown in Fig. 2.23. The function B_1 is

$$\begin{aligned} B_1 &= 1 - \frac{1}{h}x_1, & 0 \le x_1 \le h,\\ B_1 &= 0, & x_1 > h. \end{aligned} \tag{2.55}$$

Its derivative is

$$\begin{aligned} \frac{dB_1}{dx_1} &= -\frac{1}{h}, & 0 \le x_1 \le h,\\ \frac{dB_1}{dx_1} &= 0, & x_1 > h. \end{aligned} \tag{2.56}$$

Using these expressions and Eq. (2.52), the coefficient C_{11} is

$$\begin{aligned} C_{11} &= \rho_0 \int_0^H B_1 B_1 \, dx_1 \\ &= \rho_0 \int_0^h \left(1 - \frac{1}{h}x_1\right)^2 \, dx_1 \\ &= \frac{\rho_0 h}{3}, \end{aligned} \tag{2.57}$$

---

[1] Solving two- and three-dimensional problems using finite elements involves dividing the domain of the solution into finite parts, or elements, and defining suitable basis functions to describe the solution within each element. If the problem is time dependent, the weak formulation of the governing differential equations leads to an equation identical to Eq. (2.54). If the problem is static, the first term in Eq. (2.54) does not appear.

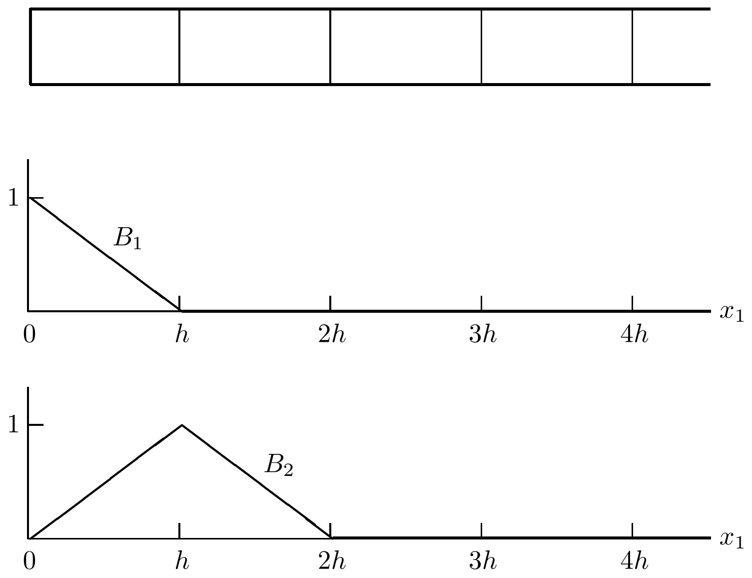


Figure 2.23: The basis functions B_1 and B_2.

and the coefficient K_{11} is

$$\begin{aligned} K_{11} &= (\lambda + 2\mu) \int_0^H \frac{dB_1}{dx_1} \frac{dB_1}{dx_1} \, dx_1 \\ &= (\lambda + 2\mu) \int_0^h \left(-\frac{1}{h} \right)^2 dx_1 \\ &= \frac{\lambda + 2\mu}{h}. \end{aligned} \tag{2.58}$$

The basis function B_2 is

$$\begin{aligned} B_2 &= \frac{1}{h} x_1, && 0 \leq x_1 \leq h, \\ B_2 &= 2 - \frac{1}{h} x_1, && h < x_1 \leq 2h, \\ B_2 &= 0, && 2h < x_1. \end{aligned} \tag{2.59}$$

Its derivative is

$$\begin{aligned} \frac{dB_2}{dx_1} &= \frac{1}{h}, && 0 \le x_1 \le h, \\ \frac{dB_2}{dx_1} &= -\frac{1}{h}, && h < x_1 \le 2h, \\ \frac{dB_2}{dx_1} &= 0, && 2h < x_1. \end{aligned} \tag{2.60}$$

Using these expressions, the coefficient C_{22} is

$$\begin{aligned} C_{22} &= \rho_0 \int_0^H B_2 B_2 \, dx_1 \\ &= \rho_0 \int_0^h \left(\frac{1}{h} x_1 \right)^2 dx_1 + \rho_0 \int_h^{2h} \left(2 - \frac{1}{h} x_1 \right)^2 dx_1 \\ &= \frac{2\rho_0 h}{3}, \end{aligned} \tag{2.61}$$

and the coefficient K_{22} is

$$\begin{aligned} K_{22} &= (\lambda + 2\mu) \int_0^H \frac{dB_2}{dx_1} \frac{dB_2}{dx_1} \, dx_1 \\ &= (\lambda + 2\mu) \left[\int_0^h \left(\frac{1}{h} \right)^2 dx_1 + \int_h^{2h} \left(-\frac{1}{h} \right)^2 dx_1 \right] \\ &= \frac{2(\lambda + 2\mu)}{h}. \end{aligned} \tag{2.62}$$

The coefficients $C_{12} = C_{21}$ and $K_{12} = K_{21}$ are

$$\begin{aligned} C_{12} &= \rho_0 \int_0^H B_1 B_2 \, dx_1 \\ &= \rho_0 \int_0^h \left(1 - \frac{1}{h} x_1 \right) \left(\frac{1}{h} x_1 \right) dx_1 \\ &= \frac{\rho_0 h}{6}, \end{aligned} \tag{2.63}$$

$$\begin{aligned} K_{12} &= (\lambda + 2\mu) \int_0^H \frac{dB_1}{dx_1} \frac{dB_2}{dx_1} \, dx_1 \\ &= (\lambda + 2\mu) \int_0^h \left(-\frac{1}{h} \right) \left(\frac{1}{h} \right) dx_1 \\ &= -\frac{\lambda + 2\mu}{h}. \end{aligned} \tag{2.64}$$

All of the coefficients C_{km} and K_{km} are determined by similar calculations. The task is greatly simplified by the repetitive nature of the basis functions.

For example, if $N = 4$,

$$C_{km} = \rho_0 h \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & \frac{1}{6} & \frac{2}{3} \end{bmatrix}, \tag{2.65}$$

$$K_{km} = \frac{\lambda + 2\mu}{h} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \tag{2.66}$$

$$f_k = -\bar{p}(t) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.67}$$

Numerical integration with respect to time

We now address the problem of integrating Eq. (2.54):

$$\mathbf{C} \frac{d^2}{dt^2} \mathbf{d} + \mathbf{K} \mathbf{d} = \mathbf{f}. \tag{2.68}$$

Let the velocities and accelerations of the material at the nodes be denoted by

$$\begin{aligned} \mathbf{v} &= \frac{d}{dt} \mathbf{d}, \\ \mathbf{a} &= \frac{d^2}{dt^2} \mathbf{d} = \frac{d}{dt} \mathbf{v}. \end{aligned} \tag{2.69}$$

Equation (2.68) can be solved for the accelerations in terms of the displacements:

$$\mathbf{a} = \mathbf{C}^{-1} \left(-\mathbf{K} \mathbf{d} + \mathbf{f} \right). \tag{2.70}$$

Let $\mathbf{v}(t)$ denote the nodal velocities at time t. If $\mathbf{v}(t)$ is known, the "trapezoidal" method of approximating the velocities at time $t + \Delta t$ is

$$\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + \frac{\Delta t}{2} \big[\mathbf{a}(t + \Delta t) + \mathbf{a}(t) \big]. \tag{2.71}$$

That is, the change in the velocity is calculated using the average acceleration during the interval of time Δt. Then the change in the nodal displacements can be determined in the same way using the average velocity:

$$\mathbf{d}(t + \Delta t) = \mathbf{d}(t) + \frac{\Delta t}{2} \big[\mathbf{v}(t + \Delta t) + \mathbf{v}(t) \big]. \tag{2.72}$$

However, Eqs. (2.71) and (2.72) cannot be solved sequentially, because **a** depends on **d**. From Eq. (2.70),

$$\begin{aligned} \mathbf{a}(t) &= \mathbf{C}^{-1}\big[-\mathbf{K}\mathbf{d}(t) + \mathbf{f}(t)\big], \\ \mathbf{a}(t+\Delta t) &= \mathbf{C}^{-1}\big[-\mathbf{K}\mathbf{d}(t+\Delta t) + \mathbf{f}(t+\Delta t)\big]. \end{aligned} \tag{2.73}$$

We substitute these expressions into Eq. (2.71):

$$\begin{aligned} \mathbf{v}(t+\Delta t) \;=\; & \mathbf{v}(t) + \frac{\Delta t}{2}\mathbf{C}^{-1}\Big\{-\mathbf{K}\big[\mathbf{d}(t+\Delta t) + \mathbf{d}(t)\big] \\ & +\mathbf{f}(t+\Delta t) + \mathbf{f}(t)\Big\}. \end{aligned} \tag{2.74}$$

Substituting this expression for $\mathbf{v}(t+\Delta t)$ into Eq. (2.72) and solving the resulting equation for $\mathbf{d}(t+\Delta t)$ yields

$$\begin{aligned} \mathbf{d}(t+\Delta t) \;=\; & \mathbf{R}^{-1}\Big\{\mathbf{S}\mathbf{d}(t) + \Delta t\mathbf{v}(t) \\ & +\Big(\frac{\Delta t}{2}\Big)^2\mathbf{C}^{-1}\big[\mathbf{f}(t+\Delta t) + \mathbf{f}(t)\big]\Big\}, \end{aligned} \tag{2.75}$$

where

$$\mathbf{R} = \mathbf{I} + \Big(\frac{\Delta t}{2}\Big)^2\mathbf{C}^{-1}\mathbf{K}, \quad \mathbf{S} = \mathbf{I} - \Big(\frac{\Delta t}{2}\Big)^2\mathbf{C}^{-1}\mathbf{K}, \tag{2.76}$$

and **I** is the identity transformation, $I_{km} = \delta_{km}$. Equation (2.75) determines $\mathbf{d}(t+\Delta t)$ using information known at time t. (The force vector **f** is a prescribed function of time.) Once $\mathbf{d}(t+\Delta t)$ has been determined, $\mathbf{v}(t+\Delta t)$ is given by Eq. (2.74). In this way the time evolution of the nodal displacements and velocities can be determined iteratively. The slope of the displacement in each element, and therefore the stress, can be calculated using the nodal displacements.

Example

Let the width of the layer be $H = 4$. Let the density of the material be $\rho = 1$ and let $\lambda + 2\mu = 1$, so that the wave velocity $\alpha = 1$. Suppose that at $t = 0$, the left side of the stationary layer is subjected to a pulse of compressive stress of unit duration:

$$T_{11}(0,t) = \begin{cases} -\sin \pi t, & 0 \le t \le 1, \\ 0, & 1 < t. \end{cases} \tag{2.77}$$

This pulse is shown in Fig. 2.24. The exact solution for the displacement field

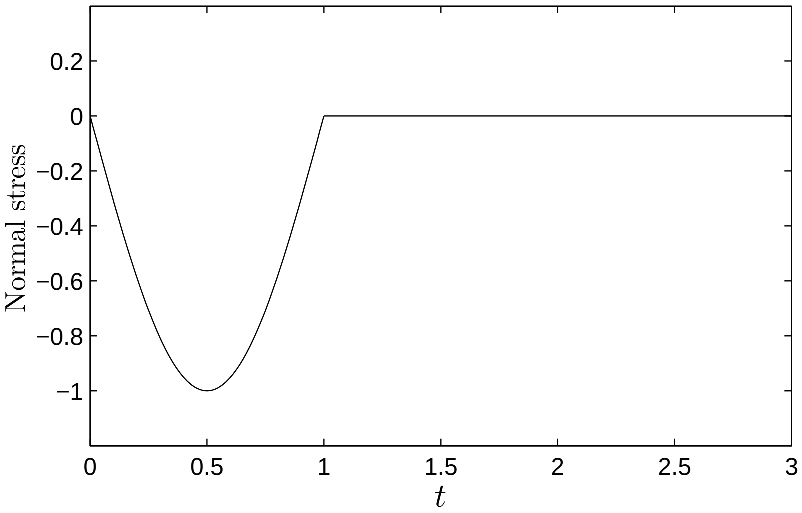


Figure 2.24: Normal stress applied to the right side of the layer.

due to the pulse boundary condition is

$$u_1(x_1,t) = \begin{cases} \dfrac{2\alpha}{\pi}, & x_1 < \alpha(t-1), \\ \dfrac{\alpha}{\pi}\left[1-\cos\pi\left(t-\dfrac{x_1}{\alpha}\right)\right], & \alpha(t-1) \le x_1 \le \alpha t, \\ 0, & \alpha t < x_1. \end{cases} \tag{2.78}$$

In Fig. 2.25, the nodal displacements obtained using $N = 80$ and $\Delta t = 0.01$ seconds are compared to the exact solution at $t = 2$ seconds. The circles are the nodal displacements and the line is the exact solution. In Fig. 2.26, the nodal velocities are compared to the exact solution at $t = 2$ seconds. The small oscillations near the leading edge of the wave are an artifact arising from the artificial discretization of the material into layers.

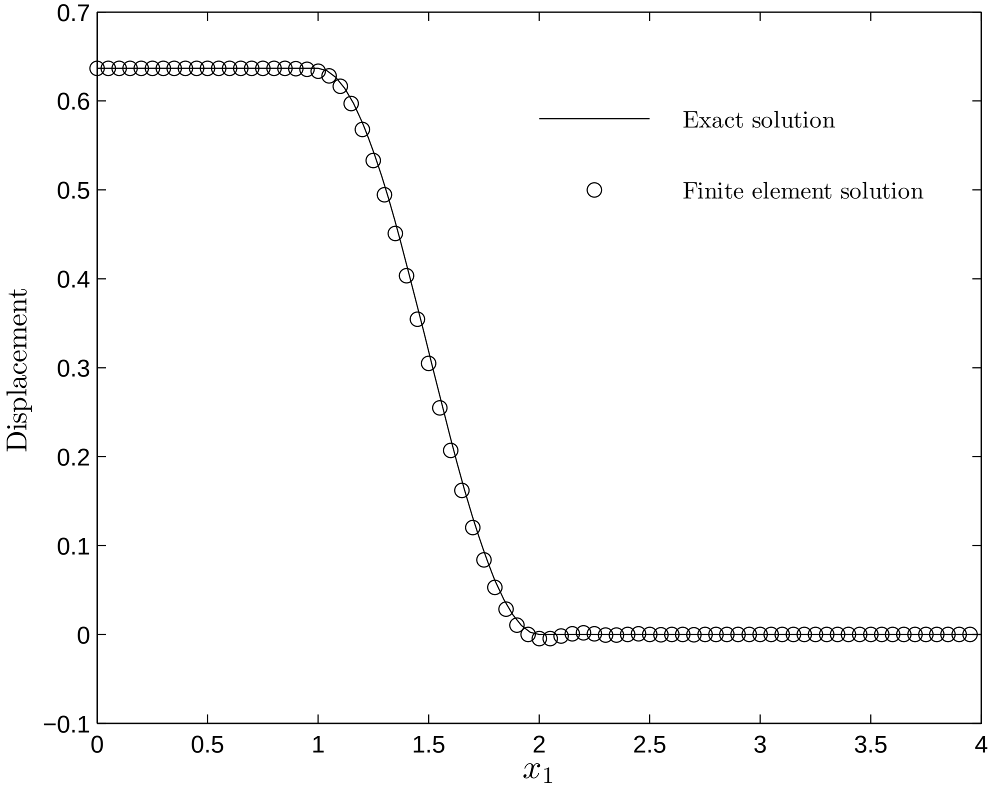


Figure 2.25: Displacement distribution at $t = 2$ seconds.

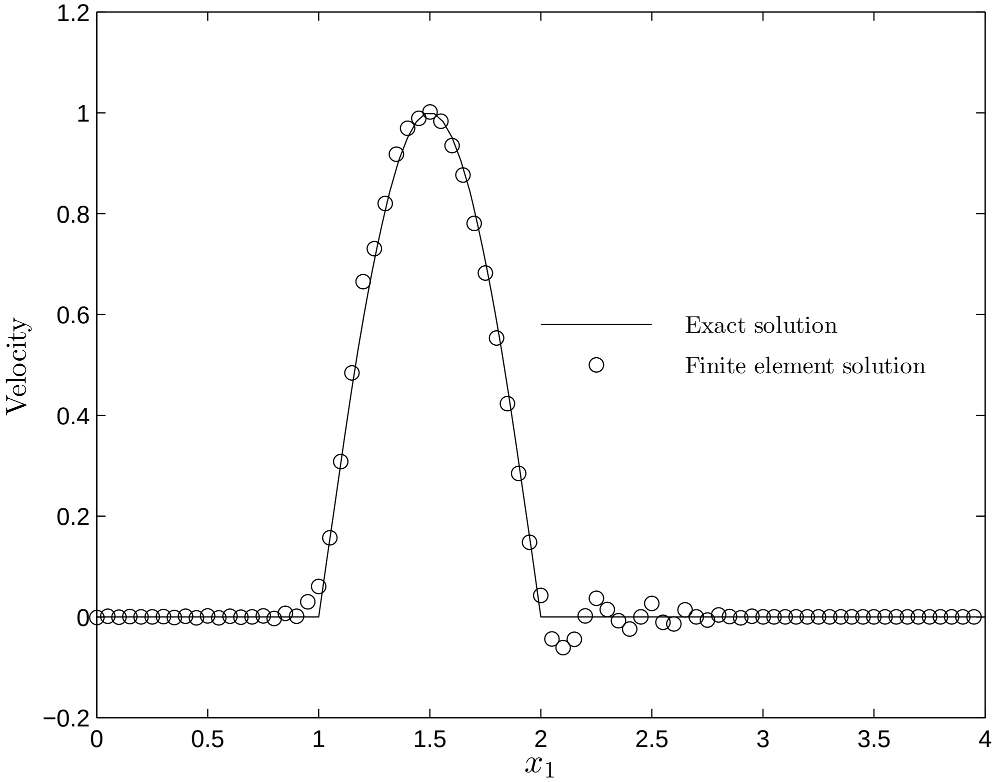


Figure 2.26: Velocity distribution at $t = 2$ seconds.

2.6 Method of Characteristics

Our objective is to define the characteristics of the one-dimensional wave equation and show how they can be used to obtain solutions. This approach is widely used in solving practical one-dimensional wave problems and also helps to explain the boundary and initial conditions that must be prescribed in such problems.

The one-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x_1^2}. \tag{2.79}$$

By introducing the notation

$$v = \frac{\partial u}{\partial t}, \quad \varepsilon = \frac{\partial u}{\partial x_1},$$

we can write Eq. (2.79) as a first-order equation:

$$\frac{\partial v}{\partial t} = \alpha^2 \frac{\partial \varepsilon}{\partial x_1}. \tag{2.80}$$

Because the variables v and ε are related by

$$\frac{\partial v}{\partial x_1} = \frac{\partial \varepsilon}{\partial t}, \tag{2.81}$$

we obtain a system of two first-order equations, Eqs. (2.80) and (2.81), in place of the second-order wave equation. When u represents the displacement of an elastic material, v is the velocity of the material and ε is the strain. When u represents the lateral displacement of a stretched string, v is the lateral velocity of the string and ε is the slope.

Let us assume that $v(x_1, t)$ and $\varepsilon(x_1, t)$ are solutions of Eqs. (2.80) and (2.81) in some region of the x_1-t plane. Our objective is to show that there are lines within that region along which the quantity $v - \alpha\varepsilon$ is constant. The change in the quantity $v - \alpha\varepsilon$ from the point x_1, t to the point $x_1 + dx_1, t + dt$ is

$$\begin{aligned} d(v - \alpha\varepsilon) &= \frac{\partial}{\partial t}(v - \alpha\varepsilon)\, dt + \frac{\partial}{\partial x_1}(v - \alpha\varepsilon)\, dx_1 \\ &= \left(\frac{\partial v}{\partial t} - \alpha \frac{\partial \varepsilon}{\partial t} \right) dt + \left(\frac{\partial v}{\partial x_1} - \alpha \frac{\partial \varepsilon}{\partial x_1} \right) dx_1. \end{aligned}$$

By using Eqs. (2.80) and (2.81), we can write this expression in the form

$$d(v - \alpha\varepsilon) = \left(\frac{\partial v}{\partial x_1} - \frac{1}{\alpha} \frac{\partial v}{\partial t} \right) (dx_1 - \alpha\, dt).$$

From this equation, $d(v - \alpha\varepsilon)$ is zero if

$$\frac{dx_1}{dt} = \alpha.$$

That is, the quantity $v - \alpha\varepsilon$ is constant along any straight line in the x_1-t plane having slope α. In the same way, it can be shown that the quantity $v + \alpha\varepsilon$ is constant along any straight line in the x_1-t plane having slope

$$\frac{dx_1}{dt} = -\alpha.$$

Straight lines in the x_1-t plane having slopes α and $-\alpha$ are called *characteristics* of the one-dimensional wave equation. When the x_1-t plane is represented as shown in Fig. 2.27, we refer to the two types as *right-running* and *left-running* characteristics. The quantity $v - \alpha\varepsilon$ is constant along a right-running characteristic, and the quantity $v + \alpha\varepsilon$ is constant along a left-running characteristic. In the following sections we show that these two conditions can be used to obtain solutions to Eqs. (2.80) and (2.81).

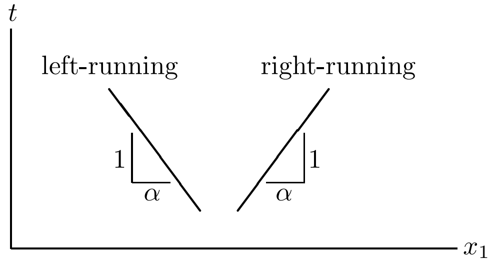


Figure 2.27: Left- and right-running characteristics in the x_1-t plane.

Initial-value problem

Suppose that at $t = 0$, the velocity and strain of an unbounded elastic material are prescribed:

$$v(x_1, 0) = \frac{\partial u_1}{\partial t}(x_1, 0) = p(x_1),$$
$$\varepsilon(x_1, 0) = \frac{\partial u_1}{\partial x_1}(x_1, 0) = q(x_1),$$

where $p(x_1)$ and $q(x_1)$ are given functions. We assume that the velocities and displacements of the material in the x_2 and x_3 directions are zero at $t = 0$. We can use characteristics to determine the velocity and strain of the material as functions of position and time.

Consider an arbitrary point x_1, t in the x_1-t plane. We can extend a right-running characteristic ab and a left-running characteristic ac from this point back to the x_1 axis (Fig. 2.28). They intersect the x_1 axis at $x_1 - \alpha t$ and at $x_1 + \alpha t$ respectively. Along the right-running characteristic ab, $v - \alpha\varepsilon$ is constant:

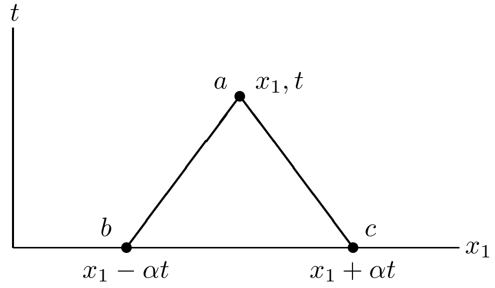


Figure 2.28: Characteristics for the initial-value problem.

$$v(x_1, t) - \alpha\varepsilon(x_1, t) = p(x_1 - \alpha t) - \alpha q(x_1 - \alpha t).$$

Along the left-running characteristic ac, $v + \alpha\varepsilon$ is constant:

$$v(x_1, t) + \alpha\varepsilon(x_1, t) = p(x_1 + \alpha t) + \alpha q(x_1 + \alpha t).$$

We can solve these two equations for $v(x_1, t)$ and $\varepsilon(x_1, t)$:

$$\begin{aligned} v(x_1, t) &= \frac{1}{2}[p(x_1 + \alpha t) + p(x_1 - \alpha t)] + \frac{\alpha}{2}[q(x_1 + \alpha t) - q(x_1 - \alpha t)], \\ \varepsilon(x_1, t) &= \frac{1}{2\alpha}[p(x_1 + \alpha t) - p(x_1 - \alpha t)] + \frac{1}{2}[q(x_1 + \alpha t) + q(x_1 - \alpha t)]. \end{aligned}$$

Thus we have determined the velocity and strain fields in the material in terms of the initial velocity and strain distributions.

Initial-boundary value problem

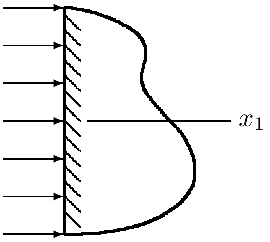
Suppose that a half space of elastic material is initially undisturbed, and at $t = 0$ we subject the boundary to the velocity boundary condition

$$v(0, t) = \frac{\partial u_1}{\partial t}(0, t) = p(t), \tag{2.82}$$

where $p(t)$ is a prescribed function of time that vanishes for $t < 0$ (Fig. 2.29).

Consider an arbitrary point x_1, t in the x_1-t plane. We can extend a left-running characteristic ab from this point back to the x_1 axis (Fig. 2.30). Because the material is undisturbed at $t = 0$, the velocity $v = 0$ and the strain $\varepsilon = 0$ at each point on the x_1 axis. Along the left-running characteristic ab, $v + \alpha\varepsilon$ is constant. Therefore

$$v(x_1, t) + \alpha\varepsilon(x_1, t) = 0. \tag{2.83}$$



$$v(0,t) = \frac{\partial u_1}{\partial t}(0,t) = p(t)$$

Figure 2.29: Half space subjected to a velocity boundary condition.

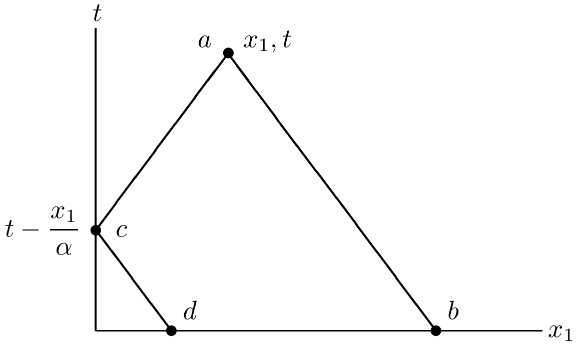


Figure 2.30: Characteristics for a velocity boundary condition.

We can also extend a right-running characteristic ac from the point x_1, t back to the t axis. As indicated in the figure, it intersects the t axis at $t - x_1/\alpha$. Along the right-running characteristic ac, $v - \alpha\varepsilon$ is constant. However, although we know the velocity v at point c because the velocity is prescribed as a function of time at $x_1 = 0$, we do not know the strain at point c. To determine it, we can extend a left-running characteristic from point c back to the x_1 axis. From the characteristic cd we obtain the relation

$$v\left(0, t - \frac{x_1}{\alpha}\right) + \alpha\varepsilon\left(0, t - \frac{x_1}{\alpha}\right) = 0. \tag{2.84}$$

Thus the value of the strain at point c is

$$\varepsilon\left(0, t - \frac{x_1}{\alpha}\right) = -\frac{1}{\alpha} v\left(0, t - \frac{x_1}{\alpha}\right) = -\frac{1}{\alpha} p\left(t - \frac{x_1}{\alpha}\right),$$

where we have used Eq. (2.82). Now that the strain at point c is known, the right-running characteristic ac gives the relation

$$v(x_1,t) - \alpha\varepsilon(x_1,t) \quad = \quad v\left(0, t - \frac{x_1}{\alpha}\right) - \alpha\varepsilon\left(0, t - \frac{x_1}{\alpha}\right)$$

$$= \quad 2p\left(t - \frac{x_1}{\alpha}\right).$$

We can solve this equation and Eq. (2.83) for $v(x_1, t)$ and $\varepsilon(x_1, t)$ in terms of the prescribed boundary condition:

$$\begin{aligned} v(x_1, t) &= p\left(t - \frac{x_1}{\alpha}\right), \\ \varepsilon(x_1, t) &= -\frac{1}{\alpha}p\left(t - \frac{x_1}{\alpha}\right). \end{aligned}$$

These results show that the solution has a remarkably simple structure: the velocity v and the strain ε are constant along right-running characteristics.

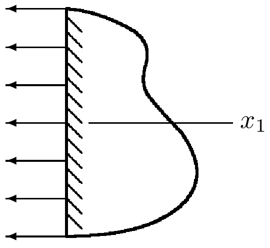
On page 95, we considered initial-boundary value problems in a half space and assumed that either the displacement or the stress was prescribed at the boundary. Our discussion in this section explains why both the displacement and the stress are not prescribed in such problems. Prescribing the displacement implies that the velocity is prescribed, and prescribing the stress is equivalent to prescribing the strain. Equation (2.84), which we obtained from the characteristic *cd*, shows that the values of the velocity and the strain at the boundary are not independent, but are related through the initial condition. As a result, it is possible to prescribe either the velocity or the strain at the boundary, but not both.

Exercises

Exercise 2.14 By using Eqs. (2.80) and (2.81), show that the quantity $v + \alpha\varepsilon$ is constant along any straight line in the x-t plane having slope

$$\frac{dx}{dt} = -\alpha.$$

Exercise 2.15



$T_{11}(0, t) = p(t)$

Suppose that a half space of elastic material is initially undisturbed, and at $t = 0$ the boundary is subjected to the stress boundary condition

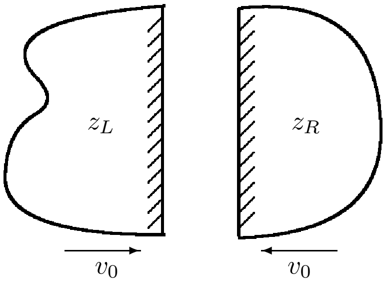
$$T_{11}(0,t) = p(t),$$

where $p(t)$ is a prescribed function of time that vanishes for $t < 0$. Use the method of characteristics to determine the velocity and strain of the material at an arbitrary point x_1, t in the x_1-t plane.

Answer:

$$\begin{aligned} v(x_1,t) &= -\frac{\alpha}{\lambda+2\mu} p\left(t - \frac{x_1}{\alpha}\right), \\ \varepsilon(x_1,t) &= \frac{1}{\lambda+2\mu} p\left(t - \frac{x_1}{\alpha}\right). \end{aligned}$$

Exercise 2.16



Half spaces of materials with acoustic impedances $z_L = \rho_L \alpha_L$ and $z_R = \rho_R \alpha_R$ approach each other with equal velocities v_0. Use characteristics to show that the velocity of their interface after the collision is $v_0(z_L - z_R)/(z_L + z_R)$.

Exercise 2.17 Consider the first-order partial differential equation

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = 0,$$

where α is a constant. Show that its solution $u(x,t)$ is constant along characteristics with slope $dx/dt = \alpha$.

2.7 Waves in Layered Media

In the previous section we used the method of characteristics to analyze one-dimensional waves in a plate, or layer, of elastic material. The same approach works for waves in several bonded layers (Fig. 2.31), although it is tedious because of the number of equations involved. In this section we describe two procedures that efficiently analyze waves in a particular class of layered elastic media.

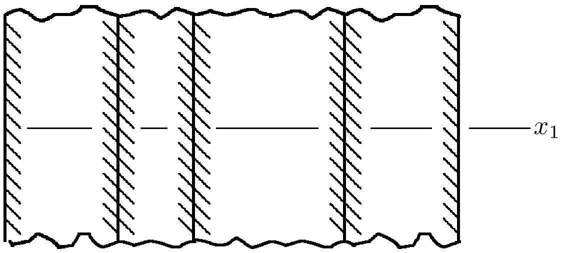


Figure 2.31: Bonded layers of elastic materials.

Along right-running characteristics within each layer,

$$v - \alpha\varepsilon = \text{constant}, \tag{2.85}$$

and along left-running characteristics,

$$v + \alpha\varepsilon = \text{constant}. \tag{2.86}$$

The product of the acoustic impedance $z = \rho_0\alpha$ and the term $\alpha\varepsilon$ is

$$z\alpha\varepsilon = \rho_0\alpha^2\frac{\partial u_1}{\partial x_1} = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1} = T_{11}.$$

In the following development we denote the component of normal stress T_{11} by σ. Multiplying Eqs. (2.85) and (2.86) by z yields the relations

$$zv - \sigma = \text{constant} \tag{2.87}$$

along right-running characteristics and

$$zv + \sigma = \text{constant} \tag{2.88}$$

along left-running characteristics. This transformation of the equations is convenient because, while the strain ε is not continuous across the interfaces between layers, both the velocity v and normal stress σ are.

Define new independent variables n and j by

$$t = \Delta j, \tag{2.89}$$

$$dx_1 = \alpha\Delta n, \tag{2.90}$$

where Δ is a time interval. Consider a finite number of bonded layers with the left boundary of the first layer at $n = 0$ and the right boundary of the last layer at $n = L$, where L is an integer. We assume that the time interval Δ can be chosen so that interfaces between layers occur only at integer values of n.

First algorithm

As a consequence of Eqs. (2.89) and (2.90), left-running and right-running characteristics join points defined by integer values of j and n (Fig. 2.32). The methods we describe evaluate the dependent variables $v(n, j)$ and $\sigma(n, j)$ only at these points. However, that does not mean that solutions can be obtained only at boundaries of layers. By dividing an individual layer into several *sublayers,* solutions can be determined at points within a layer.

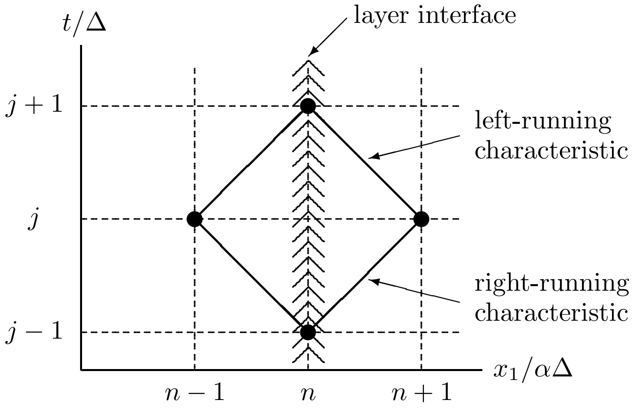


Figure 2.32: Points corresponding to integer values of j and n.

For a given position n, we denote the acoustic impedances to the left and right by $z(n)^-$ and $z(n)^+$, respectively. Notice that $z(n)^- = z(n-1)^+$. We also define

$$z(n) = \tfrac{1}{2}[z(n)^+ + z(n)^-]. \tag{2.91}$$

With Eqs. (2.87) and (2.88), we can establish relationships between the values of v and σ at the points shown in Fig. 2.32. From the two characteristics at the top of the “diamond,” we obtain

$$z(n)^{-}v(n,j+1)-\sigma(n,j+1)=z(n)^{-}v(n-1,j)-\sigma(n-1,j) \tag{2.92}$$

and

$$z(n)^{+}v(n,j+1)+\sigma(n,j+1)=z(n)^{+}v(n+1,j)+\sigma(n+1,j). \tag{2.93}$$

Addition of these two equations yields

$$\begin{aligned} 2z(n)v(n,j+1) \quad = \quad & z(n)^{+}v(n+1,j)+z(n)^{-}v(n-1,j) \\ & +\sigma(n+1,j)-\sigma(n-1,j). \end{aligned} \tag{2.94}$$

We can use Eqs. (2.92) and (2.94) to analyze initial-boundary value problems in the layered medium. Initial conditions $\sigma(n,0)$ and $v(n,0)$ must be specified. Then Eq. (2.94) determines $v(n,1)$ and Eq. (2.92) determines $\sigma(n,1)$. These calculations can be carried out in any order for all values of n except $n=0$ and $n=L$. At the left boundary, $n=0$, Eq. (2.93) must be used, and either $v(0,1)$ or $\sigma(0,1)$ must be specified. Similarly, at the right boundary, $n=L$, Eq. (2.92) must be used, and either $v(L,1)$ or $\sigma(L,1)$ must be specified. This process can be repeated for successive values of j until a prescribed final time is reached.

Each application of Eqs. (2.92) and (2.94) requires nine arithmetic operations. In the next section we describe an alternative algorithm that determines only the velocity, but is simpler and more efficient, both for computing simple examples by hand and for solving problems involving many layers by computer.

Second algorithm

In this method we employ four relations: Eqs. (2.92) and (2.93), together with the two characteristic relations obtained from the bottom of the diamond in Fig. 2.32,

$$z(n)^{-}v(n,j-1)+\sigma(n,j-1)=z(n)^{-}v(n-1,j)+\sigma(n-1,j) \tag{2.95}$$

and

$$z(n)^{+}v(n,j-1)-\sigma(n,j-1)=z(n)^{+}v(n+1,j)-\sigma(n+1,j). \tag{2.96}$$

Addition of these four relations yields the new algorithm:

$$z(n)[v(n,j+1)+v(n,j-1)]=z(n)^{+}v(n+1,j)+z(n)^{-}v(n-1,j). \tag{2.97}$$

This equation determines $v(n, j+1)$ with only four arithmetic operations. When the acoustic impedance is continuous at n, $z(n) = z(n)^+ = z(n)^-$, it further simplifies to

$$v(n, j+1) + v(n, j-1) = v(n+1, j) + v(n-1, j). \tag{2.98}$$

The sequence of calculations using Eq. (2.97) is roughly the same as for the first algorithm, but because σ does not appear explicitly, the initial and boundary conditions are handled somewhat differently. The initial conditions $\sigma(n, 0)$ and $v(n, 0)$ must be specified as before. Then Eq. (2.94) determines $v(n, 1)$, and from that point onward, Eq. (2.97) determines v. When σ is specified at the left boundary, Eqs. (2.93) and (2.96) are evaluated for $n = 0$ and summed to obtain

$$v(0, j+1) + v(0, j-1) = 2v(1, j) - \frac{1}{z(0)^+}[\sigma(0, j+1) - \sigma(0, j-1)], \tag{2.99}$$

and when σ is specified at the right boundary,

$$v(L, j+1) + v(L, j-1) = 2v(L-1, j) + \frac{1}{z(L)^-}[\sigma(L, j+1) - \sigma(L, j-1)]. \tag{2.100}$$

These two algorithms have an interesting property. The values of σ and v at points where $j+n$ is even are determined by the boundary and initial conditions at points where $j+n$ is even, and the values of σ and v at points where $j+n$ is odd are determined by the boundary and initial conditions at points where $j+n$ is odd. This is most apparent in Eq. (2.98). We can compute either the even or odd solution independently, requiring one-half the number of computations.

Examples

1. A single layer Consider a layer of an elastic material that is initially undisturbed, and suppose that for $t > 0$ the left boundary is subjected to a unit step in velocity, $v(0, t) = 1$. Let the right boundary be fixed. We divide the layer into 5 sublayers to determine the solution, so that in terms of n and j, the boundary conditions are $v(0, j) = 1$ and $v(5, j) = 0$.

Using Eq. (2.98) to obtain the odd solution, we obtain the results in Fig. 2.33. The numbers in the figure are the values of the velocity corresponding to the indicated values of n and j. The three zero values of the solution for $j = 0$ come from the initial conditions. The next row of values, at $j = 1$, is determined by the left boundary condition $v(0, 1) = 1$ and Eq. (2.94), which reduces to

$$2v(n, 1) = v(n+1, 0) + v(n-1, 0).$$

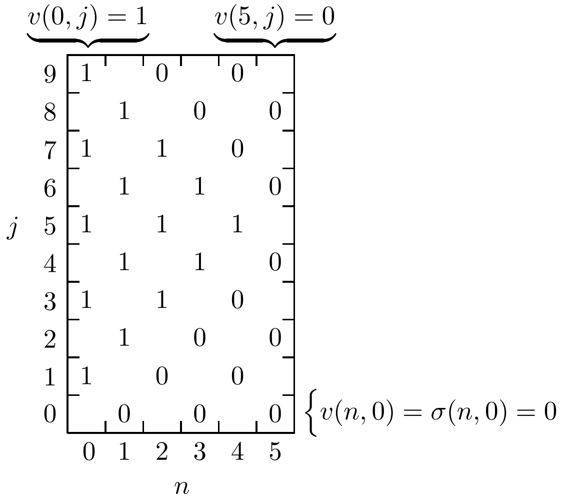


Figure 2.33: A layer with a specified velocity at the left boundary and fixed at the right boundary.

(Because $\sigma(n, 0) = 0$ and $z(n) = z(n)^+ = z(n)^-$, the solution does not depend on the acoustic impedance.)

The next row of values is obtained using the right boundary condition and Eq. (2.98),

$$v(n, 2) = v(n + 1, 1) + v(n - 1, 1) - v(n, 0).$$

The remaining rows are determined by repeated applications of Eq. (2.98). A hand calculation of this part of the solution is simplified by visualizing each solution point as the top of the diamond in Fig. 2.32 and noticing from Eq. (2.98) that the sum of values at the top and bottom of the diamond equals the sum of the values on the left and right sides.

Figure 2.33 shows the wave initiated at the left boundary at $j = 0$ propagating at velocity α, advancing one integer value of n for each integer increase in j. Behind the wave front, the velocity equals the constant velocity at the left boundary. When the wave reaches the fixed right boundary, a reflected wave propagates back toward the left boundary. The velocity behind the reflected wave is zero.

If the right boundary of the plate is free instead of fixed, the only difference in the computation is that we use Eq. (2.100),

$$v(5, j + 1) = 2v(4, j) - v(5, j - 1),$$

to calculate the values of $v(5, j+1)$. In this case we obtain the results in Fig. 2.34. The velocity behind the reflected wave is twice the velocity imposed at the left boundary.

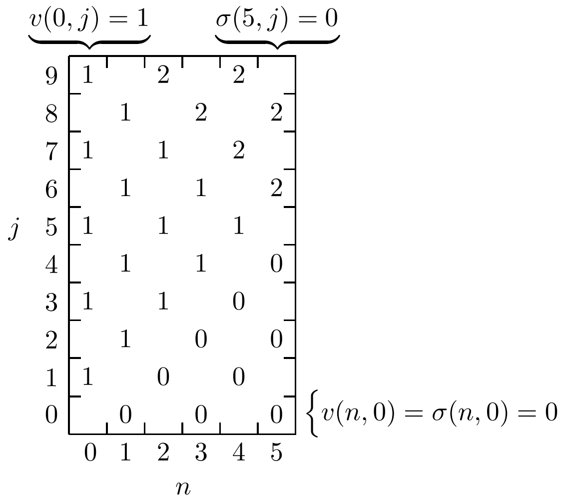


Figure 2.34: A layer with a specified velocity at the left boundary and free at the right boundary.

2. Two layers Let two bonded layers have acoustic impedances $z = 1$ in the left layer and $z = 2$ in the right one, and suppose that the left layer is subjected to a step in velocity $v(0,t) = 3$ for $t > 0$. We divide each layer into 5 sublayers. The solution is given by Eq. (2.98) for all values of n except the interface between layers at $n = 5$. At the interface, $z(5)^+ \neq z(5)^-$, so we must use Eq. (2.97),

$$v(5, j+1) = \tfrac{4}{3}v(6,j) + \tfrac{2}{3}v(4,j) - v(5, j-1).$$

The results (Fig. 2.35) show that when the wave initiated at the left boundary reaches the interface, a reflected wave propagates back toward the boundary and a transmitted wave propagates into the right layer. The velocity behind both the reflected and transmitted waves is $v = 2$.

3. Unbounded medium Here we solve an initial-value problem in an infinite elastic medium by dividing a portion of it into sublayers. At the left boundary of the first layer, $n = 0$, and the right boundary of the last layer, $n = L$, we

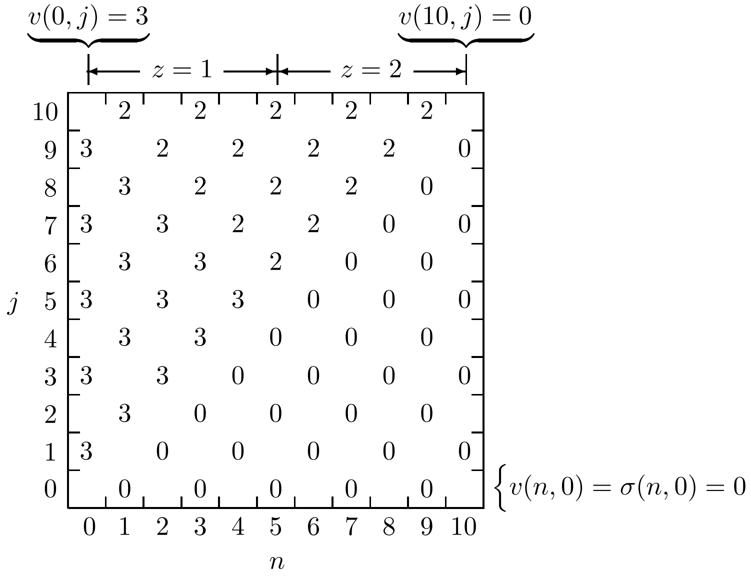


Figure 2.35: Two layers with a specified velocity at the left boundary.

impose conditions that prevent reflections of waves and thus simulate an infinite medium. The D'Alembert solution for a right-traveling wave is

$$u_1 = f(\xi),$$

where $\xi = x_1 - \alpha t$. The velocity and normal stress are

$$v = -\alpha \frac{df}{d\xi}, \qquad \sigma = (\lambda + 2\mu)\frac{df}{d\xi},$$

so in a right-traveling wave, v and σ are constant along characteristics $\xi =$ constant. Equations (2.89) and (2.90) imply that within a sublayer, ξ is constant when $n - j$ is constant. Because

$$n - (j+1) = (n-1) - j,$$

reflected waves are prevented at the right boundary, $n = L$, if we impose the conditions

$$\begin{aligned} v(L, j+1) &= v(L-1, j), \\ \sigma(L, j+1) &= \sigma(L-1, j). \end{aligned}$$

By similar arguments, reflected waves are prevented at the left boundary, $n = 0$, if we impose the conditions

$$v(0, j+1) = v(1, j), \tag{2.101}$$

$$\sigma(0, j+1) = \sigma(1, j). \tag{2.102}$$

We assume that at $t = 0$ the strain is zero and the velocity is zero except in a finite region in which $v(x, 0) = 2$ (Fig. 2.36). The solution is given by Eqs. (2.94) and (2.98) together with the conditions imposed at the boundaries. The results show waves propagating to the left and right within which the velocity of the material equals one-half of the velocity imposed at $t = 0$. Behind these waves, the medium is stationary. Notice that the conditions imposed at the boundaries prevent reflected waves.

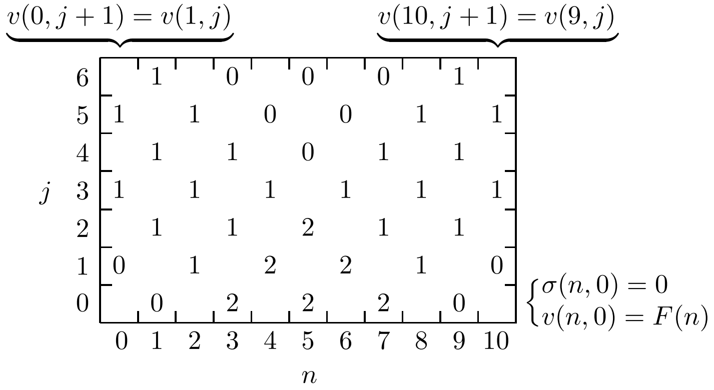


Figure 2.36: An unbounded medium subjected to an initial velocity.

4. Periodically layered medium Our first three examples illustrate wave behavior using cases simple enough to compute by hand, but the real utility of these methods becomes evident when they are applied to layered media with large numbers of layers. As an example, we consider a medium consisting of alternating layers of two elastic materials. One material has acoustic impedance $z = 1$ and consists of layers of thickness $10\alpha\Delta$, and the other material has acoustic impedance $z = 200$ and consists of layers of thickness $\alpha\Delta$.

Suppose that the left boundary of the medium is subjected to a unit step in velocity, $v(0, t) = 1$, for $t > 0$. We evaluate the odd solution from Eq. (2.97) by using the following program:

```
• Initialize storage:
• z is a vector with $1101$ elements initialized to $1$. v is a matrix with
  $(1101, 1501)$ elements initialized to $0$. Subscripts start at $0$.
• for ($n = 11$ to $1100$ by increments of $11$) $z(n) = 200$.
```

```
• Using Eq. (2.91), compute $\frac{z(n)^+}{z(n)}$ and $\frac{z(n)^-}{z(n)}$:
• for ($n = 1$ to 1099 by increments of 1) begin
    – $zp(n) = 2 * z(n+1)/(z(n) + z(n+1))$
    – $zm(n) = 2 * z(n)/(z(n) + z(n+1))$
• end for
• Specify left boundary condition:
• for ($j = 1$ to 1499 by increments of 2) $v(0, j) = 1$.
• Start time loop to compute solution:
• for ($j = 1$ to 1499 by increments of 1) begin
    – if $j$ is even $ns = 2$ else $ns = 1$
    – Using Eq. (2.97), compute odd solution—exclude right boundary:
    – for ($n = ns$ to 1099 by increments of 2) $v(n, j+1) =$
          $zp(n) * v(n+1, j) + zm(n) * v(n-1, j) - v(n, j-1)$
• end for
```

The computer's memory must be large enough to hold the values of v for all values of n and j simultaneously, including the values of the even solution that remain as zeros. To simplify the program, we used only Eq. (2.97), even though Eq. (2.98) is more efficient at interfaces between sublayers where $zp(n) = zm(n) = 1$.

The computed velocity at the center of the thirty-first layer (n=170) is shown as a function of j (nondimensional time) in Fig. 2.37. As a result of the multiple reflections and transmissions at the layer interfaces, the velocity has an oscillatory history. This oscillatory profile is seen in sound propagation through ducts, elastic waves in channels and bars, and ocean tidal surges, or *bores,* into rivers and channels, and is called an *undulating bore.* This behavior results from a phenomenon known as *dispersion,* which we discuss in the next chapter.

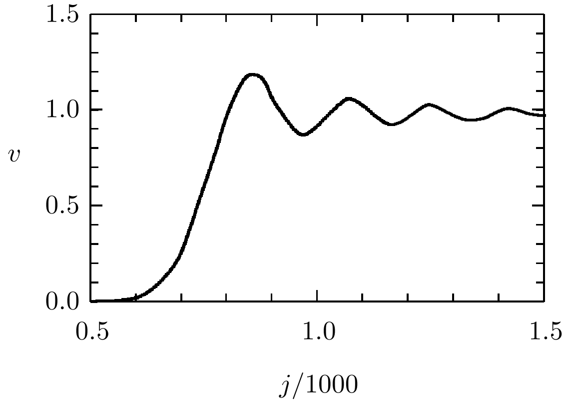


Figure 2.37: Velocity at a point in a periodically layered medium resulting from a unit step in velocity at the boundary.

Exercises

Exercise 2.18 Use D'Alembert solutions to verify the values shown in Fig. 2.34.

Exercise 2.19
(a) Use Eq. (2.98) to verify the values shown in Fig. 2.34.
(b) Extend the calculation shown in Fig. 2.34 to $j = 15$.

Exercise 2.20 Extend the calculations used to obtain Fig. 2.34 to $j = 15$, assuming that at $t = 0$ the left boundary is subjected to a unit step in stress, $\sigma(0, t) = 1$.

Exercise 2.21 Repeat the calculations used to obtain Fig. 2.33, replacing the unit step in velocity at the left boundary by the pulse

$$v(0, j) = \begin{cases} 1 & j = 0, \\ 0 & j > 0. \end{cases}$$

Exercise 2.22 Calculate the results shown in Fig. 2.37.

Chapter 3

Steady-State Waves

Steady-state waves are waves in which the dependent variables are harmonic, or oscillatory functions of time. Waves that are not steady-state are said to be *transient.* Most of the waves we encounter, both natural and man-made, are transient. We study steady-state waves for the insights they provide and because, at least conceptually, transient wave solutions can be obtained by superimposing steady-state solutions. Furthermore, with modern *ultrasonic* devices it is possible to create steady-state waves.

We begin by discussing one-dimensional steady-state waves, then derive representations of steady-state compressional and shear waves in two dimensions. Using these representations, we obtain solutions for the reflection of plane waves at a free boundary, Rayleigh waves, waves in a layer of elastic material, and waves in layered media.

3.1 Steady-State One-Dimensional Waves

We will first show that the one-dimensional wave equation admits a solution that has a harmonic dependence on time. We then introduce the terminology of steady-state waves and consider some one-dimensional problems having harmonic boundary conditions.

Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}. \tag{3.1}$$

Let us seek a solution of this equation that is a harmonic, or oscillatory function of time:

$$u = f(x) \sin \omega t, \tag{3.2}$$

where ω is a constant and $f(x)$ is a function that must be determined. Substituting this expression into Eq. (3.1) shows that Eq. (3.1) is satisfied if the function $f(x)$ is a solution of the equation

$$\frac{d^2 f(x)}{dx^2} + \frac{\omega^2}{\alpha^2} f(x) = 0.$$

The general solution of this ordinary differential equation can be expressed in the form

$$f(x) = 2A \sin \frac{\omega}{\alpha} x + 2B \cos \frac{\omega}{\alpha} x,$$

where A and B are arbitrary constants. Substituting this expression into Eq. (3.2), we obtain a solution of the one-dimensional wave equation:

$$u = 2A \sin \frac{\omega}{\alpha} x \sin \omega t + 2B \cos \frac{\omega}{\alpha} x \sin \omega t.$$

We can write this equation in the form

$$\begin{aligned} u \quad = \quad & A \left[\cos \omega \left(\frac{x}{\alpha} - t \right) - \cos \omega \left(\frac{x}{\alpha} + t \right) \right] \\ & -B \left[\sin \omega \left(\frac{x}{\alpha} - t \right) - \sin \omega \left(\frac{x}{\alpha} + t \right) \right]. \end{aligned}$$

Observe that we have obtained D'Alembert solutions in terms of sines and cosines. Solutions of this form are called *steady-state solutions*. With x held fixed, the dependent variable undergoes a steady harmonic oscillation in time.

Let us consider a wave represented by the steady-state solution

$$u = A \cos \omega \left(\frac{x}{\alpha} - t \right) = A \cos(kx - \omega t). \tag{3.3}$$

The constant

$$k = \frac{\omega}{\alpha} \tag{3.4}$$

is called the *wave number* of the steady-state solution. The absolute value of the constant A is called the *amplitude* of the wave, and the speed α is called the *phase velocity* of the wave.

Figure 3.1.a shows a graph of u as a function of x with the time t held fixed. The distance λ required for the solution to undergo a complete oscillation is called the *wavelength* of the wave. From Eq. (3.3) we see that with t held fixed the solution undergoes one complete oscillation when kx changes by 2π. Therefore $k\lambda = 2\pi$, so the wave number is related to the wavelength by

$$k = \frac{2\pi}{\lambda}. \tag{3.5}$$

Figure 3.1.b shows a graph of u as a function of t with the position x held fixed. The time T required for the solution to undergo one complete oscillation

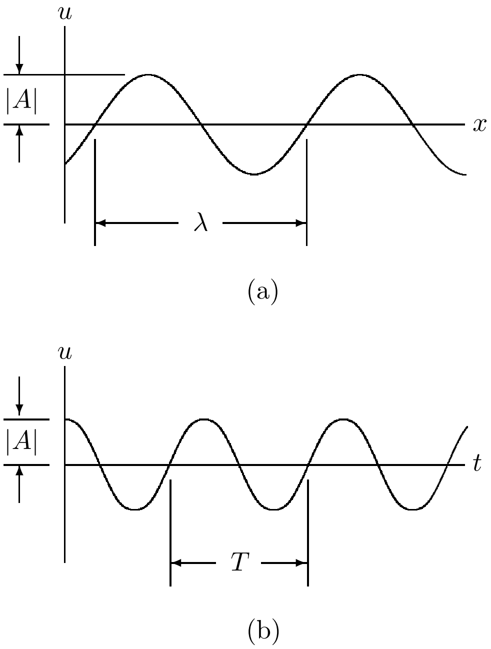


Figure 3.1: The dependent variable u: (a) as a function of x; (b) as a function of t.

is called the *period* of the wave. From Eq. (3.3), we see that with x held fixed the solution undergoes one complete oscillation when ωt changes by 2π. Therefore $\omega T = 2\pi$, so the constant ω is related to the period by

$$\omega = \frac{2\pi}{T}. \tag{3.6}$$

The inverse of the period $f = 1/T$ is the number of oscillations per unit time at a fixed position x. It is called the *frequency* of the wave. From Eq. (3.6), we see that

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{3.7}$$

When the position x is held fixed, there is a one-to-one correspondence between the value of u and the vertical position of a point moving in a circular path of radius $|A|$ with constant angular velocity ω. Figure 3.2 shows this correspondence for the position $x = 0$. The period T is the time required for the point to move once around the circular path. The frequency f is the number of revolutions of the point around the circular path per unit time, and $\omega = 2\pi f$ is the number of radians around the circular path per unit time. Thus f and ω

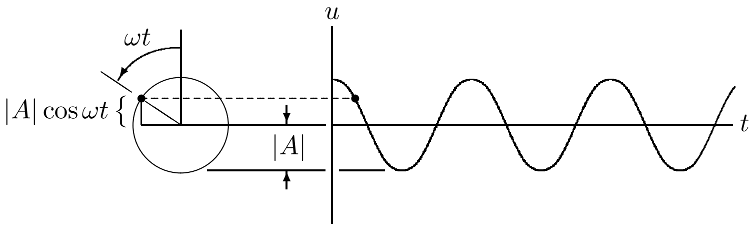


Figure 3.2: Correspondence between the dependent variable u and the motion of a point in a circular path.

are both measures of the frequency. The term f is measured in oscillations or *cycles* per second, which are called Hertz (Hz). The term ω is measured in radians per second, and is sometimes called the *circular frequency* to distinguish it from f. From Eqs. (3.4), (3.5), and (3.7), we obtain a simple relation between the phase velocity, the wavelength, and the frequency:

$$\alpha = \lambda f. \tag{3.8}$$

It is often convenient to use the complex exponential function, Eq. (A.2), to express steady-state waves in the form

$$u = Ae^{i(kx - \omega t)} = A[\cos(kx - \omega t) + i \sin(kx - \omega t)],$$

where the constant A may be complex. The magnitude of A is the amplitude of the wave. We call A the *complex amplitude* of the wave.

For example, suppose that a plate of elastic material of thickness L is subjected to the steady-state displacement boundary condition

$$u_1(0, t) = Ue^{-i\omega t}$$

at the left boundary, where U is a constant, and the plate is bonded to a fixed support at the right boundary (Fig. 3.3). Let us determine the resulting steady-state motion of the material.

The steady-state boundary condition gives rise to a steady-state wave propagating in the positive x_1 direction. Due to reflection at the boundary, there is also a steady-state wave propagating in the negative x_1 direction. Therefore we assume a steady-state solution of the form

$$u_1 = Te^{i(kx_1 - \omega t)} + Re^{i(-kx_1 - \omega t)}, \tag{3.9}$$

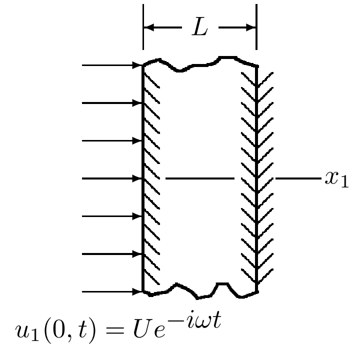


Figure 3.3: Plate with a steady-state displacement boundary condition.

where T is the complex amplitude of the forward-propagating *transmitted* wave and R is the complex amplitude of the rearward-propagating *reflected* wave. Substituting this expression into the boundary condition at the left boundary, we obtain the equation

$$T + R = U.$$

Requiring the displacement to be zero at the right boundary yields the equation

$$Te^{ikL} + Re^{-ikL} = 0.$$

The solution of these two equations for the values of the complex amplitudes are

$$T = \frac{e^{-ikL}U}{e^{-ikL} - e^{ikL}}, \quad R = \frac{-e^{ikL}U}{e^{-ikL} - e^{ikL}}.$$

By using these results, we can express Eq. (3.9) in the form

$$u_1 = -\frac{U\sin[k(x_1 - L)]}{\sin kL}e^{-i\omega t}.$$

This solution exhibits the interesting phenomenon of *resonance*. It predicts that the amplitude of the displacement is infinite when $\sin kL = 0$; that is, when $kL = n\pi$, where n is an integer. From Eq. (3.5), we see that this occurs when the wavelength satisfies the relation $n\lambda/2 = L$. That is, the width of the plate equals an integral number of half-wavelengths (Fig. 3.4).

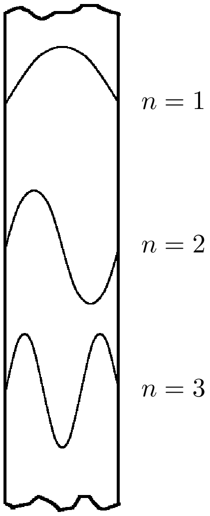


Figure 3.4: Resonance occurs when the width of the plate is equal to an integral number of half-wavelengths.

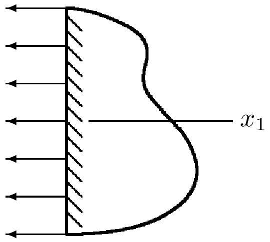
Exercises

Exercise 3.1 Show that

$$u = Ae^{i(kx - \omega t)}$$

is a solution of Eq. (3.1) if $k = \omega/\alpha$.

Exercise 3.2



$T_{11}(0,t) = T_0 e^{-i\omega t}$

A half space of elastic material is subjected to the stress boundary condition

$$T_{11}(0,t) = T_0 e^{-i\omega t},$$

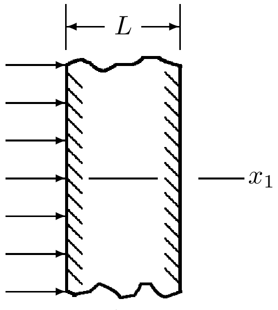
where T_0 is a constant. Determine the resulting steady-state displacement field in the material.

Answer:

$$u_1 = \frac{T_0}{ik(\lambda + 2\mu)} e^{i(kx_1 - \omega t)},$$

where $k = \omega/\alpha$.

Exercise 3.3



$u_1(0,t) = Ue^{-i\omega t}$

A plate of elastic material of thickness L is subjected to the steady-state displacement boundary condition

$$u_1(0,t) = Ue^{-i\omega t}$$

at the left boundary. The plate is free at the right boundary. Determine the resulting steady-state displacement field in the material.

Discussion—Assume a steady-state solution of the form

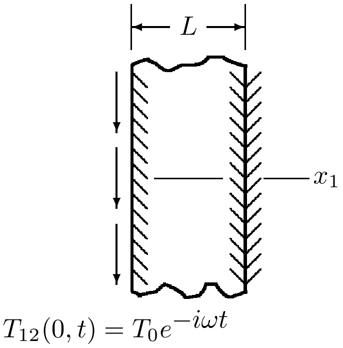
$$u_1 = Te^{i(kx_1 - \omega t)} + Re^{i(-kx_1 - \omega t)},$$

where T and R are constants and $k = \omega/\alpha$.

Answer:

$$u_1 = \frac{U\cos[k(x_1 - L)]}{\cos kL} e^{-i\omega t}.$$

Exercise 3.4



A plate of elastic material of thickness L is subjected to the steady-state shear stress boundary condition

$$T_{12}(0, t) = T_0 e^{-i\omega t}$$

at the left boundary. The plate is bonded to a rigid material at the right boundary. Determine the resulting steady-state displacement field in the material.

Answer:

$$u_1 = \frac{T_0 \sin[k(x_1 - L)]}{k\mu \cos kL} e^{-i\omega t},$$

where $k = \omega/\beta$.

3.2 Steady-State Two-Dimensional Waves

We will begin by obtaining "building-block" solutions that will be used in the following sections to analyze several important problems in elastic wave propagation. Consider the elastic half space and coordinate system shown in Fig. 3.5. We assume that the motion of the material is described by the displacement field

$$\begin{aligned} u_1 &= u_1(x_1, x_3, t), \\ u_2 &= 0, \\ u_3 &= u_3(x_1, x_3, t). \end{aligned} \tag{3.10}$$

Thus the motion is two-dimensional: it does not depend on the coordinate normal to the plane in Fig. 3.5, and the component of the displacement normal to the plane is zero. This type of motion is called *plane strain*. Using the Helmholtz decomposition

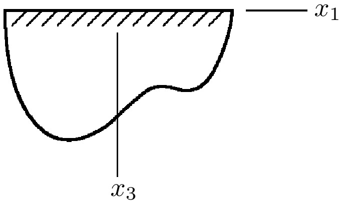


Figure 3.5: An elastic half space.

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi},$$

we can express the displacement components in terms of two scalar potentials:

$$\begin{aligned} u_1 &= \frac{\partial\phi}{\partial x_1} - \frac{\partial\psi_2}{\partial x_3}, \\ u_3 &= \frac{\partial\phi}{\partial x_3} + \frac{\partial\psi_2}{\partial x_1}, \end{aligned} \tag{3.11}$$

where $\phi = \phi(x_1, x_3, t)$ and $\psi_2 = \psi_2(x_1, x_3, t)$. The scalar ψ_2 is the x_2 component of the vector potential $\boldsymbol{\psi}$. To simplify the notation, we will drop the subscript 2 and write the scalar ψ_2 as ψ.

The potentials ϕ and ψ are governed by the wave equations

$$\frac{\partial^2\phi}{\partial t^2} = \alpha^2 \left(\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_3^2} \right) \tag{3.12}$$

and

$$\frac{\partial^2\psi}{\partial t^2} = \beta^2 \left(\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_3^2} \right), \tag{3.13}$$

where the compressional and shear wave speeds α and β are defined in terms of the Lamé constants λ and μ and the density ρ_0 of the material:

$$\alpha = \left(\frac{\lambda + 2\mu}{\rho_0}\right)^{1/2}, \quad \beta = \left(\frac{\mu}{\rho_0}\right)^{1/2}.$$

Let us assume that the potential ϕ is given by the steady-state equation

$$\phi = f(x_3)e^{i(k_1x_1 - \omega t)}, \tag{3.14}$$

where the wave number k_1 and the frequency ω are prescribed and the function $f(x_3)$ must be determined. Thus we assume that the solution for the potential ϕ has the form of a steady-state wave propagating in the positive x_1 direction, but we make no assumption about the behavior of the solution in the x_3 direction. Substituting this expression into Eq. (3.12), we find that the function $f(x_3)$ must satisfy the ordinary differential equation

$$\frac{d^2f(x_3)}{dx_3^2} + \left(\frac{\omega^2}{\alpha^2} - k_1^2\right) f(x_3) = 0. \tag{3.15}$$

The solutions of this equation result in solutions for ϕ that have very different characters depending on whether $k_1^2 < \omega^2/\alpha^2$ or $k_1^2 > \omega^2/\alpha^2$. We discuss these cases separately.

Plane waves propagating in the x_1-x_3 plane

If $k_1^2 < \omega^2/\alpha^2$, it is convenient to write the solution of Eq. (3.15) in the form

$$f(x_3) = Ae^{ik_3x_3} + Be^{-ik_3x_3},$$

where A and B are constants and

$$k_3 = \left(\frac{\omega^2}{\alpha^2} - k_1^2\right)^{1/2}.$$

Substituting this form of the solution for $f(x_3)$ into Eq. (3.14), we obtain the solution for the potential ϕ in the form

$$\phi = Ae^{i(k_1x_1 + k_3x_3 - \omega t)} + Be^{i(k_1x_1 - k_3x_3 - \omega t)}. \tag{3.16}$$

Consider the first term of this solution:

$$\phi = Ae^{i(k_1x_1 + k_3x_3 - \omega t)}. \tag{3.17}$$

This function describes a *plane wave.* That is, at any time t the value of ϕ is constant on planes defined by the equation

$$k_1x_1 + k_3x_3 = \text{ constant.}$$

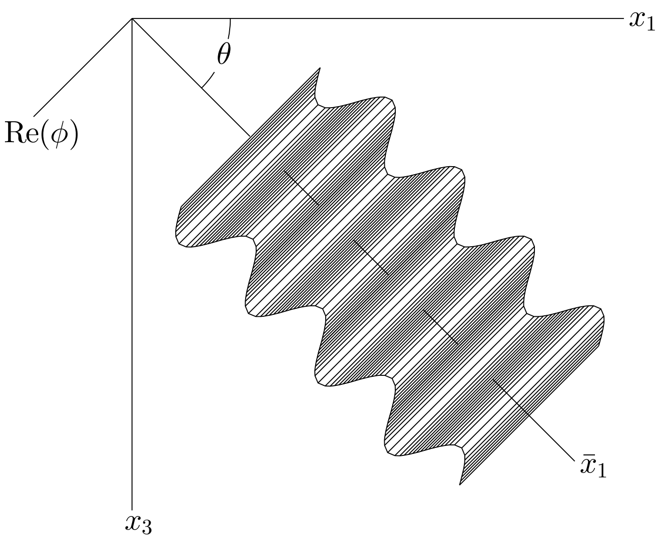


Figure 3.6: Real part of ϕ.

Figure 3.6 shows a plot of the real part of ϕ as a function of x_1 and x_3. The value is constant along lines $x_3 = -(k_1/k_3)x_1 + \text{constant}$. The wave propagates in the *propagation direction* θ with wave speed α.

Figure 3.7 shows straight lines along two succeeding "peaks" of the wave. The distance λ is the wavelength of the plane wave. It is related to the wave number k of the wave by $\lambda = 2\pi/k$. From Fig. 3.7 we see that the wavelength λ_1 of the wave in the x_1 direction is related to λ by

$$\lambda = \lambda_1 \cos\theta,$$

and the wavelength λ_3 of the wave in the x_3 direction is related to λ by

$$\lambda = \lambda_3 \sin\theta.$$

Using these expressions, we can write the wave numbers k_1 and k_3 in Eq. (3.17) in terms of the wave number k of the wave and the propagation direction θ:

$$\begin{aligned} k_1 &= \frac{2\pi}{\lambda_1} = \frac{2\pi}{\lambda}\cos\theta = k\cos\theta, \\ k_3 &= \frac{2\pi}{\lambda_3} = \frac{2\pi}{\lambda}\sin\theta = k\sin\theta. \end{aligned}$$

With these results, we can express Eq. (3.17) in terms of the propagation direction θ:

$$\phi = Ae^{i(kx_1\cos\theta + kx_3\sin\theta - \omega t)}, \tag{3.18}$$

where the wave number $k = \omega/\alpha$.

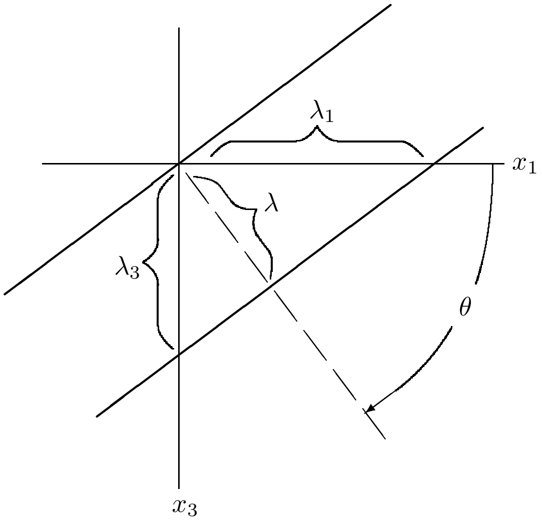


Figure 3.7: Lines along two succeeding peaks of the plane wave.

We can further explain the form of Eq. (3.18) by deriving it in another way. Let us express ϕ as a one-dimensional steady-state wave propagating in the $\bar{x}_1$ direction of a coordinate system $\bar{x}_1, \bar{x}_2, \bar{x}_3$ (Fig. 3.8):

$$\phi = Ae^{i(k\bar{x}_1 - \omega t)}. \tag{3.19}$$

If we orient a coordinate system x_1, x_2, x_3 as shown in the figure, with the x_1 axis at an angle θ relative to the $\bar{x}_1$ axis, the position of a point on the $\bar{x}_1$ axis is given in terms of the coordinates of the point in the x_1, x_2, x_3 coordinate system by

$$\bar{x}_1 = x_1 \cos\theta + x_3 \sin\theta.$$

Substituting this coordinate transformation into Eq. (3.19), we obtain Eq. (3.18).

From Eqs. (3.11), the components of the displacement field of the steady-state compressional wave, Eq. (3.18), are

$$u_1 = \frac{\partial\phi}{\partial x_1} = ikA\cos\theta e^{i(kx_1\cos\theta + kx_3\sin\theta - \omega t)},$$

$$u_3 = \frac{\partial\phi}{\partial x_3} = ikA\sin\theta e^{i(kx_1\cos\theta + kx_3\sin\theta - \omega t)}.$$

From these expressions we can see that the displacements of the points of the material are parallel to the propagation direction of the compressional wave. The displacement parallel to the propagation direction is

$$u_C \quad = \quad u_1\cos\theta + u_3\sin\theta$$

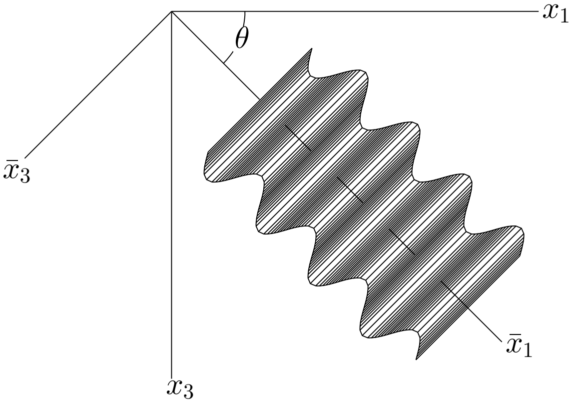


Figure 3.8: One-dimensional plane wave propagating in the direction of the $\bar{x}_1$ axis.

$$= \quad ikAe^{i(kx_1 \cos\theta + kx_3 \sin\theta - \omega t)}.$$

The amplitude of the displacement of the compressional wave is $|u_C| = k|A|$.

Equation (3.18) is a convenient expression for a plane compressional wave with propagation direction θ relative to the x_1 axis. We can obtain an expression for a plane shear wave with propagation direction θ relative to the x_1 axis by assuming a solution for the potential ψ of the same form:

$$\psi = Ce^{i(k_S x_1 \cos\theta + k_S x_3 \sin\theta - \omega t)}, \tag{3.20}$$

where the wave number $k_S = \omega/\beta$. From Eqs. (3.11), the components of the displacement field of this steady-state shear wave are

$$\begin{aligned} u_1 &= -\frac{\partial\psi}{\partial x_3} = -ik_S C \sin\theta e^{i(k_S x_1 \cos\theta + k_S x_3 \sin\theta - \omega t)}, \\ u_3 &= \quad \frac{\partial\psi}{\partial x_1} = \quad ik_S C \cos\theta e^{i(k_S x_1 \cos\theta + k_S x_3 \sin\theta - \omega t)}. \end{aligned}$$

From these expressions we can see that the displacements of the points of the material are transverse to the propagation direction of the shear wave. The displacement transverse to the propagation direction is

$$\begin{aligned} u_S &= -u_1 \sin\theta + u_3 \cos\theta \\ &= ik_S Ce^{i(k_S x_1 \cos\theta + k_S x_3 \sin\theta - \omega t)}. \end{aligned}$$

The amplitude of the displacement of the shear wave is $|u_S| = k_S|C|$.

Plane waves propagating in the x_1 direction and attenuating in the x_3 direction

If $k_1^2 > \omega^2/\alpha^2$, it is convenient to write the solution of Eq. (3.15) in the form

$$f(x_3) = Ae^{-hx_3} + Be^{hx_3},$$

where A and B are constants and

$$h = \left(k_1^2 - \frac{\omega^2}{\alpha^2}\right)^{1/2}.$$

Substituting this form of the solution for $f(x_3)$ into Eq. (3.14), we obtain the solution for the potential ϕ in the form

$$\phi = Ae^{-hx_3}e^{i(k_1x_1 - \omega t)} + Be^{hx_3}e^{i(k_1x_1 - \omega t)}. \tag{3.21}$$

These solutions represent waves that propagate in the x_1 direction and whose amplitudes attenuate or grow exponentially in the x_3 direction. If this expression is to represent a solution for ϕ in the half space shown in Fig. 3.5, the constant B must be zero since the second term grows without bound in the positive x_3 direction. The first term

$$\phi = Ae^{-hx_3}e^{i(k_1x_1 - \omega t)} \tag{3.22}$$

represents a steady-state wave propagating in the x_1 direction whose amplitude decays exponentially in the positive x_3 direction. The real part of this solution is plotted as a function of x_1 and x_3 in Fig. 3.9.

Equation (3.22) describes a compressional wave. We can obtain an expression for a shear wave that propagates in the x_1 direction and whose amplitude decays exponentially in the positive x_3 direction by assuming a solution for the potential ψ of the same form:

$$\psi = Ce^{-h_S x_3}e^{i(k_1x_1 - \omega t)}, \tag{3.23}$$

where

$$h_S = \left(k_1^2 - \frac{\omega^2}{\beta^2}\right)^{1/2}.$$

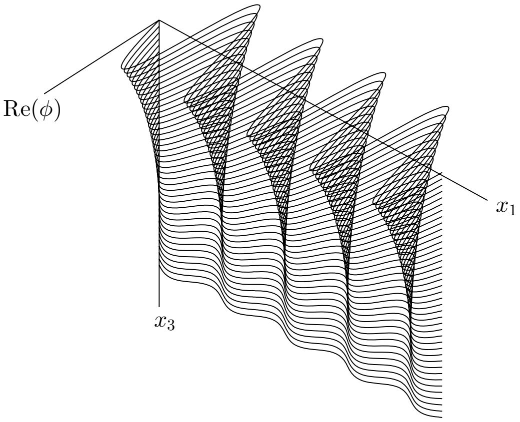


Figure 3.9: A wave propagating in the x_1 direction whose amplitude decays exponentially in the positive x_3 direction.

Exercise

Exercise 3.5 A compressional wave is described by the potential

$$\phi = Ae^{i(kx_1 \cos\theta + kx_3 \sin\theta - \omega t)}.$$

(a) The Lamé constants of the material are λ and μ. Determine the stress component T_{33} as a function of position and time.
(b) The density of the material in the reference state is ρ_0. Determine the density ρ of the material as a function of position and time.

Answer: (a) $T_{33} = -k^2\phi(\lambda + 2\mu \sin^2\theta)$, (b) $\rho = \rho_0(1 + k^2\phi)$.

3.3 Reflection at a Plane Boundary

Here we determine the solutions for the reflected waves that result when a steady-state plane compressional or shear wave is incident on the free boundary of a half space of elastic material.

Incident compressional wave

Consider the elastic half space shown in Fig. 3.10, and let us assume that a plane compressional wave with known frequency and known complex amplitude propagates toward the free boundary of the half space in the direction shown. We can express the compressional wave with a potential of the form of Eq. (3.18):

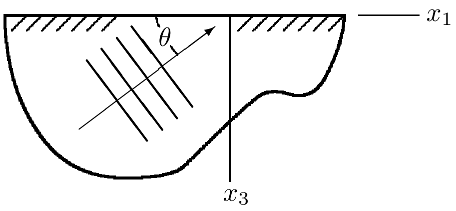


Figure 3.10: Elastic half space with a plane compressional wave incident on the boundary.

$$\phi = Ie^{i(kx_1 \cos\theta - kx_3 \sin\theta - \omega t)}, \tag{3.24}$$

where I is the complex amplitude and the wave number $k = \omega/\alpha$. We have chosen the signs in the exponential term so that the wave propagates in the positive x_1 direction and the negative x_3 direction. Our objective is to determine the wave reflected from the boundary.

The boundary condition at the free boundary of the half space is that the traction is zero (see Section 1.4):

$$[t_k]_{x_3=0} = [T_{mk}n_m]_{x_3=0} = 0.$$

The unit vector normal to the boundary has components $n_1 = 0$, $n_2 = 0$, $n_3 = -1$, so from this condition we obtain the three stress boundary conditions

$$[T_{13}]_{x_3=0} = 0, \quad [T_{23}]_{x_3=0} = 0, \quad [T_{33}]_{x_3=0} = 0.$$

That is, the normal stress and the two components of shear stress on the boundary are zero. For the two-dimensional motion we are considering, the stress component T_{23} is identically zero. The other two boundary conditions give us two conditions that the displacement field must satisfy:

$$\begin{aligned} [T_{13}]_{x_3=0} &= \left[\mu\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)\right]_{x_3=0} = 0, \\ [T_{33}]_{x_3=0} &= \left[\lambda\frac{\partial u_1}{\partial x_1}+(\lambda+2\mu)\frac{\partial u_3}{\partial x_3}\right]_{x_3=0} = 0. \end{aligned} \tag{3.25}$$

It is easy to show that the boundary conditions cannot be satisfied if we assume that the incident compressional wave causes only a reflected compressional wave. The boundary conditions can be satisfied only if we assume that there is also a reflected shear wave. Therefore we assume that the compressional potential ϕ and the shear potential ψ are of the forms

$$\begin{aligned} \phi &= Ie^{i(kx_1\cos\theta - kx_3\sin\theta - \omega t)} \\ &\quad + Pe^{i(kx_1\cos\theta_P + kx_3\sin\theta_P - \omega t)}, \\ \psi &= Se^{i(k_Sx_1\cos\theta_S + k_Sx_3\sin\theta_S - \omega t)}, \end{aligned} \tag{3.26}$$

where P and S are the complex amplitudes of the reflected compressional and shear waves. The propagation direction of the reflected compressional wave is θ_P and the propagation direction of the reflected shear wave is θ_S (Fig. 3.11). The wave numbers $k = \omega/\alpha$ and $k_S = \omega/\beta$. We have chosen the signs so that the reflected waves propagate in the positive x_1 direction and the positive x_3 direction.

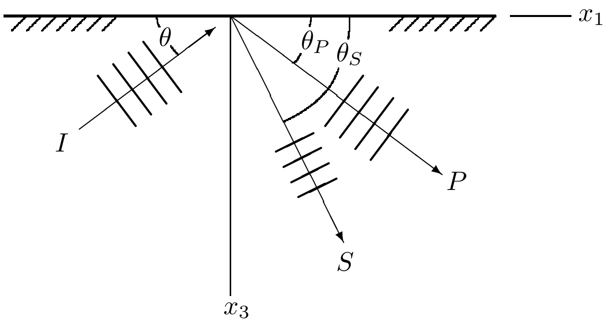


Figure 3.11: The reflected compressional and shear waves.

Substituting Eqs. (3.26) into Eqs. (3.11), we obtain the displacement field

$$\begin{aligned}
u_1 &= (u_1)_I + (u_1)_P + (u_1)_S \\
&= ik\cos\theta I e^{i(kx_1\cos\theta - kx_3\sin\theta - \omega t)} \\
&\quad + ik\cos\theta_P P e^{i(kx_1\cos\theta_P + kx_3\sin\theta_P - \omega t)} \\
&\quad - ik_S\sin\theta_S S e^{i(k_S x_1\cos\theta_S + k_S x_3\sin\theta_S - \omega t)}, \\
u_3 &= (u_3)_I + (u_3)_P + (u_3)_S \\
&= -ik\sin\theta I e^{i(kx_1\cos\theta - kx_3\sin\theta - \omega t)} \\
&\quad + ik\sin\theta_P P e^{i(kx_1\cos\theta_P + kx_3\sin\theta_P - \omega t)} \\
&\quad + ik_S\cos\theta_S S e^{i(k_S x_1\cos\theta_S + k_S x_3\sin\theta_S - \omega t)},
\end{aligned} \tag{3.27}$$

where the subscripts I, P, and S refer to the incident, compressional, and shear waves. Substituting these expressions into the boundary conditions, we obtain the two equations

$$\begin{aligned}
&-[\lambda\cos^2\theta + (\lambda+2\mu)\sin^2\theta]k^2 I e^{i(kx_1\cos\theta - \omega t)} \\
&-[\lambda\cos^2\theta_P + (\lambda+2\mu)\sin^2\theta_P]k^2 P e^{i(kx_1\cos\theta_P - \omega t)} \\
&-2\mu k_S^2\sin\theta_S\cos\theta_S S e^{i(k_S x_1\cos\theta_S - \omega t)} = 0, \\
&\ 2k^2\sin\theta\cos\theta I e^{i(kx_1\cos\theta - \omega t)} \\
&-2k^2\sin\theta_P\cos\theta_P P e^{i(kx_1\cos\theta_P - \omega t)} \\
&+k_S^2(\sin^2\theta_S - \cos^2\theta_S) S e^{i(k_S x_1\cos\theta_S - \omega t)} = 0.
\end{aligned} \tag{3.28}$$

The exponential terms in these equations are linearly independent functions of x_1, and therefore the equations have only trivial solutions for the complex amplitudes P and S unless the coefficients of x_1 are equal:

$$k\cos\theta = k\cos\theta_P = k_S\cos\theta_S. \tag{3.29}$$

From these relations we see that the propagation direction of the reflected compressional wave $\theta_P = \theta$. We also obtain an equation for the propagation direction of the reflected shear wave:

$$\begin{aligned}
\cos\theta_S &= \frac{k}{k_S}\cos\theta \\
&= \frac{\beta}{\alpha}\cos\theta.
\end{aligned} \tag{3.30}$$

Equation (3.29) has an important interpretation. Let $\lambda = 2\pi/k$ be the wavelength of the incident and reflected compressional waves and let $\lambda_S = 2\pi/k_S$ be the wavelength of the reflected shear wave. We can write Eq. (3.29) as

$$\frac{\lambda}{\cos\theta} = \frac{\lambda_S}{\cos\theta_S}. \tag{3.31}$$

Figure 3.12 shows straight lines representing successive crests of the incident compressional wave and the reflected shear wave. From this figure we see that Eq. (3.31) implies that the wavelengths of the waves along the boundary are equal:

$$\lambda_B = \frac{\lambda}{\cos\theta} = \frac{\lambda_S}{\cos\theta_S}.$$

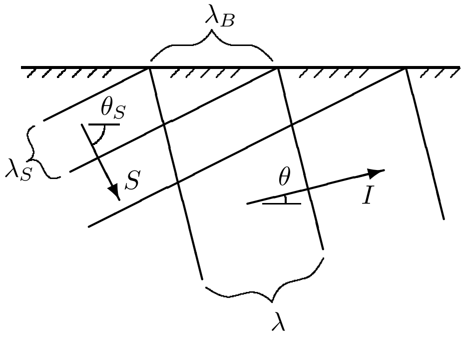


Figure 3.12: Successive crests of the incident compressional wave and the reflected shear wave.

The Poisson's ratio ν of an elastic material is related to its Lamé constants by (see Exercise 1.29)

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

From this relation it is easy to show that

$$\frac{\beta^2}{\alpha^2} = \frac{\mu}{\lambda + 2\mu} = \frac{1 - 2\nu}{2(1-\nu)}. \tag{3.32}$$

With this result, we can express Eq. (3.30), which relates the propagation direction θ_S of the shear wave to the angle θ, in terms of the Poisson's ratio. Figure 3.13 shows a plot of θ_S as a function of θ for $\nu = 0.3$.

Because of Eqs. (3.29), the exponential terms cancel from Eqs. (3.28) and we can write them in the forms

$$\begin{aligned}(P/I) + \left[\frac{2\sin\theta_S\cos\theta_S}{1 - (2\beta^2/\alpha^2)\cos^2\theta}\right](S/I) &= -1, \\ (P/I) - \left[\frac{1 - 2\cos^2\theta_S}{(2\beta^2/\alpha^2)\sin\theta\cos\theta}\right](S/I) &= 1.\end{aligned}$$

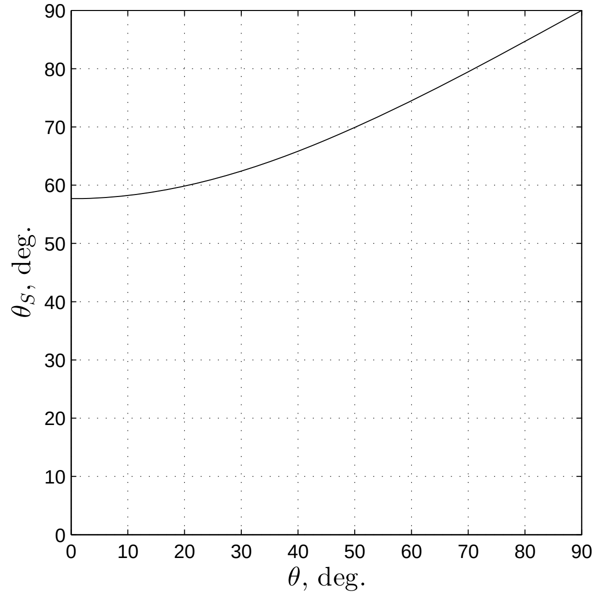


Figure 3.13: The propagation direction of the reflected shear wave for Poisson's ratio $\nu = 0.3$.

We can solve these two equations for the ratios P/I and S/I. Due to Eqs. (3.30) and (3.32), the coefficients of these equations depend only on θ and the Poisson's ratio ν.

From Eq. (3.27), the amplitudes of the displacement of the incident, reflected compressional, and reflected shear waves are

$$\begin{aligned} |u_I| &= |(u_1)_I \cos\theta - (u_3)_I \sin\theta| = k|I|, \\ |u_P| &= |(u_1)_P \cos\theta + (u_3)_P \sin\theta| = k|P|, \\ |u_S| &= |(u_1)_S \sin\theta_S - (u_3)_S \cos\theta_S| = k_S|S|. \end{aligned}$$

Figure 3.14 shows plots of the amplitude ratios $|u_P/u_I| = |P/I|$ and $|u_S/u_I| = |k_S S/(kI)|$ as functions of θ for $\nu = 0.3$. The reflected shear wave vanishes at $\theta = 0$ and at $\theta = 90°$. The amplitude of the shear wave is a maximum at approximately $\theta = 40°$, where its amplitude is actually larger than the amplitude of the incident compressional wave.

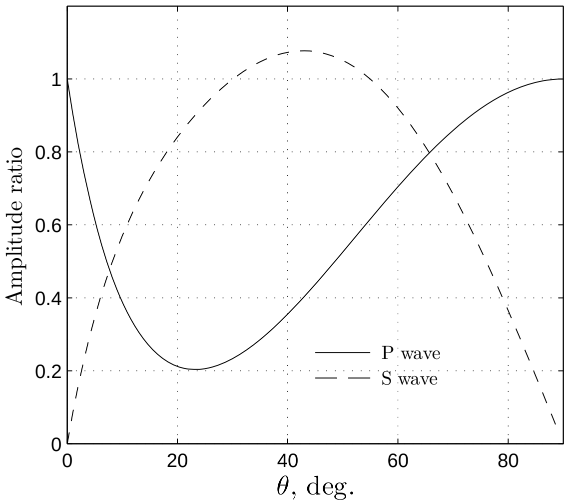


Figure 3.14: Amplitude ratios of the reflected compressional and shear waves for Poisson's ratio $\nu = 0.3$.

Incident shear wave

We now assume that a plane shear wave with known frequency and known complex amplitude is incident on the free boundary of an elastic half space. We begin by assuming that it causes a reflected shear wave with propagation direction θ_S and a reflected compressional wave with propagation direction θ_P, as shown in Fig. 3.15. (We show subsequently that this is true only for a particular range of values of the propagation direction θ of the incident wave.)

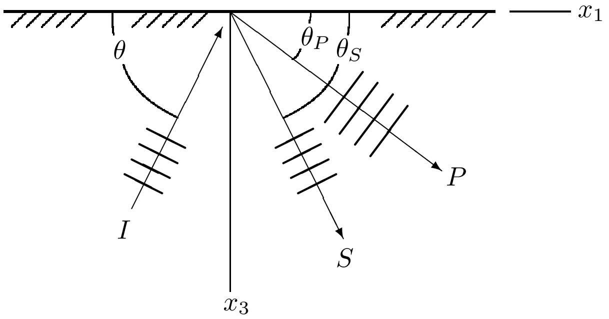


Figure 3.15: Elastic half space with a plane shear wave incident on the boundary.

We write the compressional potential ϕ and the shear potential ψ in the forms

$$\begin{aligned} \phi &= P e^{i(k_P x_1 \cos\theta_P + k_P x_3 \sin\theta_P - \omega t)}, \\ \psi &= I e^{i(k x_1 \cos\theta - k x_3 \sin\theta - \omega t)} \\ &\quad + S e^{i(k x_1 \cos\theta_S + k x_3 \sin\theta_S - \omega t)}, \end{aligned} \tag{3.33}$$

where I, P, and S are the complex amplitudes of the incident wave, the reflected compressional wave and the reflected shear wave. The wave numbers $k = \omega/\beta$ and $k_P = \omega/\alpha$. From these potentials we obtain the displacement field

$$\begin{aligned} u_1 &= (u_1)_P + (u_1)_I + (u_1)_S \\ &= ik_P \cos\theta_P P e^{i(k_P x_1 \cos\theta_P + k_P x_3 \sin\theta_P - \omega t)} \\ &\quad + ik \sin\theta I e^{i(k x_1 \cos\theta - k x_3 \sin\theta - \omega t)} \\ &\quad - ik \sin\theta_S S e^{i(k x_1 \cos\theta_S + k x_3 \sin\theta_S - \omega t)}, \\ u_3 &= (u_3)_P + (u_3)_I + (u_3)_S \\ &= ik_P \sin\theta_P P e^{i(k_P x_1 \cos\theta_P + k_P x_3 \sin\theta_P - \omega t)} \\ &\quad + ik \cos\theta I e^{i(k x_1 \cos\theta - k x_3 \sin\theta - \omega t)} \\ &\quad + ik \cos\theta_S S e^{i(k x_1 \cos\theta_S + k x_3 \sin\theta_S - \omega t)}, \end{aligned} \tag{3.34}$$

where the subscripts I, P, and S refer to the incident, compressional, and shear waves. Substituting these expressions into the boundary conditions, we obtain

$$\begin{array}{l}
-[\lambda\cos^2\theta_P + (\lambda+2\mu)\sin^2\theta_P]k_P^2 P e^{i(k_P x_1\cos\theta_P - \omega t)} \\
+2\mu k^2 \sin\theta\cos\theta I e^{i(k x_1\cos\theta - \omega t)} \\
-2\mu k^2 \sin\theta_S\cos\theta_S S e^{i(k x_1\cos\theta_S - \omega t)} = 0, \\
-2k_P^2\sin\theta_P\cos\theta_P P e^{i(k_P x_1\cos\theta_P - \omega t)} \\
+k^2(\sin^2\theta - \cos^2\theta) I e^{i(k x_1\cos\theta - \omega t)} \\
+k^2(\sin^2\theta_S - \cos^2\theta_S) S e^{i(k x_1\cos\theta_S - \omega t)} = 0.
\end{array} \tag{3.35}$$

These equations have only trivial solutions for the complex amplitudes P and S unless the coefficients of x_1 are equal:

$$k\cos\theta = k\cos\theta_S = k_P\cos\theta_P. \tag{3.36}$$

Thus $\theta_S = \theta$. The angle between the direction of propagation of the reflected shear wave and the boundary is equal to the angle between the direction of propagation of the incident shear wave and the boundary. The equation for the angle between the direction of propagation of the reflected compressional wave and the boundary is

$$\begin{aligned}
\cos\theta_P &= \frac{k}{k_P}\cos\theta \\
&= \frac{\alpha}{\beta}\cos\theta.
\end{aligned} \tag{3.37}$$

Let $\lambda = 2\pi/k$ be the wavelength of the incident and reflected shear waves and let $\lambda_P = 2\pi/k_P$ be the wavelength of the reflected shear wave. Equation (3.36) states that

$$\frac{\lambda}{\cos\theta} = \frac{\lambda_P}{\cos\theta_P}. \tag{3.38}$$

Figure 3.16 shows straight lines representing successive crests of the incident shear wave and the reflected compressional wave. The wavelengths of the waves along the boundary are equal:

$$\lambda_B = \frac{\lambda}{\cos\theta} = \frac{\lambda_P}{\cos\theta_P}.$$

So far, our analysis of the reflection of a shear wave has been similar to the reflection of a compressional wave. However, at this stage a new and interesting phenomenon arises. Recall that the wave speed α of a compressional wave is larger than the wave speed β of a shear wave. For angles θ such that

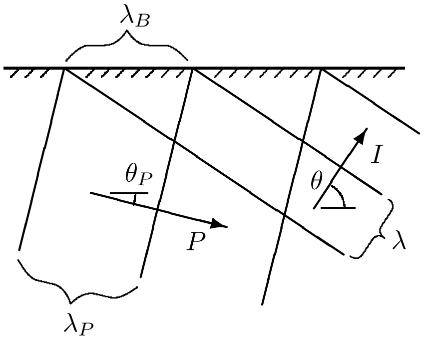


Figure 3.16: Successive crests of the incident shear wave and the reflected compressional wave.

$(\alpha/\beta)\cos\theta > 1$, we cannot solve Eq. (3.37) for the direction of propagation of the reflected compressional wave. This condition can be written

$$\frac{\alpha}{\beta}\cos\theta = \frac{\lambda_P}{\left(\dfrac{\lambda}{\cos\theta}\right)} > 1.$$

We can explain what happens with Fig. 3.16. When the propagation direction θ of the incident shear wave decreases to a certain value, the wavelength $\lambda_B = \lambda/\cos\theta$ of the shear wave along the boundary becomes equal to the wavelength λ_P of the reflected compressional wave. This value of θ is called the *critical angle* θ_C. When θ decreases below this value, the reflected compressional wave cannot have the same wavelength along the boundary as the incident shear wave, and Eq. (3.38) cannot be satisfied. Figure 3.17 shows the propagation direction θ_P as a function of θ for $\nu = 0.3$. In this case the critical angle $\theta_C = 57.7°$.

When the angle θ is greater than the critical angle, the analysis of the reflection of a shear wave is similar to our analysis of the reflection of a compressional wave. When the angle θ is less than the critical angle, the solution is obtained by assuming that the compressional wave is one that propagates in the x_1 direction and attenuates in the x_3 direction. (See our discussion of this type of wave on page 123.) We present these cases separately.

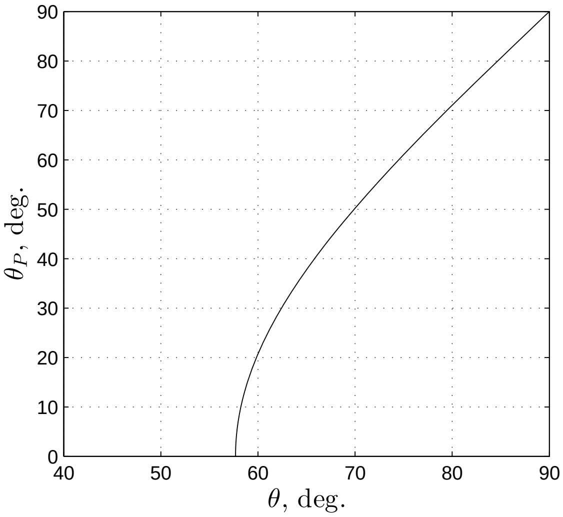


Figure 3.17: Propagation direction of the reflected compressional wave for Poisson's ratio $\nu = 0.3$.

Angle θ greater than the critical angle θ_C

Because of Eqs. (3.36), we can write Eqs. (3.35) in the forms

$$\begin{aligned}(S/I) \quad &- \quad \left[\frac{2(\beta^2/\alpha^2)\sin\theta_P\cos\theta_P}{\sin^2\theta - \cos^2\theta}\right](P/I) = -1,\\ (S/I) \quad &+ \quad \left[\frac{1 - 2(\beta^2/\alpha^2)\cos^2\theta_P}{2\sin\theta\cos\theta}\right](P/I) = \quad 1.\end{aligned} \tag{3.39}$$

We can solve these two equations for the ratios S/I and P/I. As a result of Eqs. (3.32) and (3.37), the coefficients of these equations depend only on θ and the Poisson's ratio ν.

From Eq. (3.34), the amplitudes of the displacement of the incident, reflected shear, and reflected compressional waves are

$$\begin{aligned}|u_I| &= \quad |(u_1)_I\sin\theta + (u_3)_I\cos\theta| \quad = \quad k|I|,\\ |u_S| &= \quad |(u_1)_S\sin\theta - (u_3)_S\cos\theta| \quad = \quad k|S|,\\ |u_P| &= |(u_1)_P\cos\theta_P + (u_3)_P\sin\theta_P| = k_P|P|.\end{aligned}$$

With these relations and the solutions of Eqs. (3.39), we can determine the amplitude ratios $|u_S/u_I| = |S/I|$ and $|u_P/u_I| = |k_P P/(kI)|$ as functions of θ for angles greater than the critical angle.

Angle θ less than the critical angle θ_C

In this case the boundary conditions cannot be satisfied by assuming that there is a reflected compressional wave. Instead, we assume that the compressional wave propagates in the x_1 direction and attenuates in the x_3 direction. To do so, we write the compressional potential ϕ in the form of Eq. (3.22):

$$\begin{aligned} \phi &= Pe^{-hx_3}e^{i(k_P x_1 - \omega t)}, \\ \psi &= Ie^{i(kx_1 \cos\theta - kx_3 \sin\theta - \omega t)} \\ &\quad +Se^{i(kx_1 \cos\theta_S + kx_3 \sin\theta_S - \omega t)}, \end{aligned} \tag{3.40}$$

where

$$h = \left(k_P^2 - \frac{\omega^2}{\alpha^2}\right)^{1/2}.$$

The wave number $k = \omega/\beta$. From these potentials we obtain the displacement field

$$\begin{aligned} u_1 &= (u_1)_P + (u_1)_I + (u_1)_S \\ &= ik_P Pe^{-hx_3}e^{i(k_P x_1 - \omega t)} \\ &\quad +ik\sin\theta Ie^{i(kx_1 \cos\theta - kx_3 \sin\theta - \omega t)} \\ &\quad -ik\sin\theta_S Se^{i(kx_1 \cos\theta_S + kx_3 \sin\theta_S - \omega t)}, \\ u_3 &= (u_3)_P + (u_3)_I + (u_3)_S \\ &= -hPe^{-hx_3}e^{i(k_P x_1 - \omega t)} \\ &\quad +ik\cos\theta Ie^{i(kx_1 \cos\theta - kx_3 \sin\theta - \omega t)} \\ &\quad +ik\cos\theta_S Se^{i(kx_1 \cos\theta_S + kx_3 \sin\theta_S - \omega t)}. \end{aligned} \tag{3.41}$$

where the subscripts I, P, and S refer to the incident, compressional, and shear waves. Substituting these expressions into the boundary conditions, Eqs. (3.25), we obtain

$$\begin{aligned} & [-\lambda k_P^2 + (\lambda + 2\mu)h^2]Pe^{i(k_P x_1 - \omega t)} \\ & +2\mu k^2 \sin\theta\cos\theta Ie^{i(kx_1 \cos\theta - \omega t)} \\ & -2\mu k^2 \sin\theta_S \cos\theta_S Se^{i(kx_1 \cos\theta_S - \omega t)} = 0, \\ & -2ik_P hPe^{i(k_P x_1 - \omega t)} \\ & +k^2(\sin^2\theta - \cos^2\theta)Ie^{i(kx_1 \cos\theta - \omega t)} \\ & +k^2(\sin^2\theta_S - \cos^2\theta_S)Se^{i(kx_1 \cos\theta_S - \omega t)} = 0. \end{aligned} \tag{3.42}$$

These equations have only trivial solutions for the complex amplitudes P and S unless

$$k\cos\theta = k_P. \tag{3.43}$$

By using this result, we can write Eqs. (3.42) in the forms

$$\begin{aligned}(S/I) - \left[\frac{2i\cos\theta(\cos^2\theta - \beta^2/\alpha^2)^{1/2}}{\sin^2\theta - \cos^2\theta}\right](P/I) &= -1,\\ (S/I) + \left[\frac{1-2\cos^2\theta}{2\sin\theta\cos\theta}\right](P/I) &= 1.\end{aligned} \tag{3.44}$$

From Eqs. (3.41), the amplitudes of the displacement of the incident and reflected shear waves are

$$\begin{aligned}|u_I| &= |(u_1)_I\sin\theta + (u_3)_I\cos\theta| = k|I|,\\ |u_S| &= |(u_1)_S\sin\theta - (u_3)_S\cos\theta| = k|S|,\end{aligned}$$

and the amplitude of the horizontal motion of the compressional wave at $x_3 = 0$ is

$$|(u_1)_P| = k_P|P| = k\cos\theta|P|.$$

With these relations and the solutions of Eqs. (3.44), we can determine the amplitude ratios $|u_S/u_I| = |S/I|$ and $|(u_1)_P/u_I| = \cos\theta|P/I|$ as functions of θ for angles less than the critical angle.

Figure 3.18 shows the amplitude ratios of the shear and compressional waves as functions of θ for $\nu = 0.3$. (For angles below the critical angle θ_C, we plot the amplitude ratio of the horizontal motion of the compressional wave at $x_3 = 0$.) Below the critical angle, the amplitude of the reflected shear wave is equal to the amplitude of the incident shear wave.

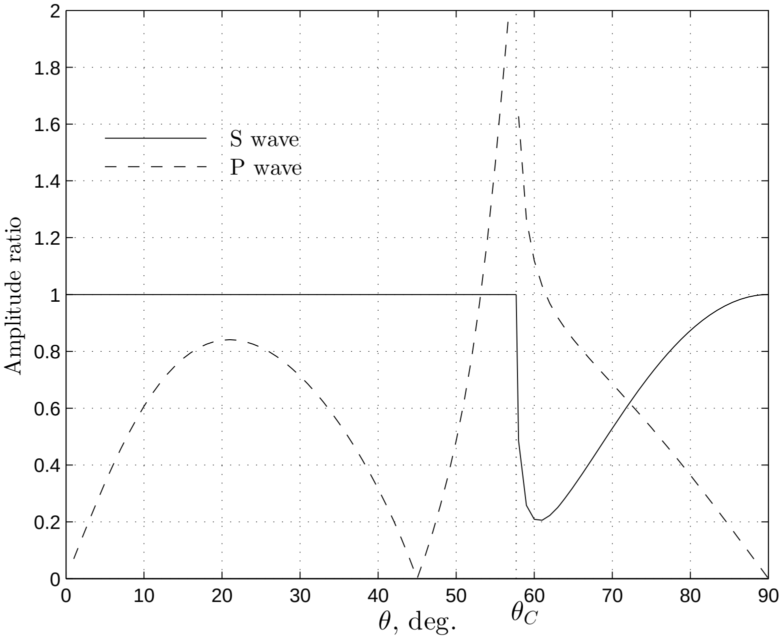


Figure 3.18: Amplitude ratios of the reflected shear and compressional waves for Poisson's ratio $\nu = 0.3$.

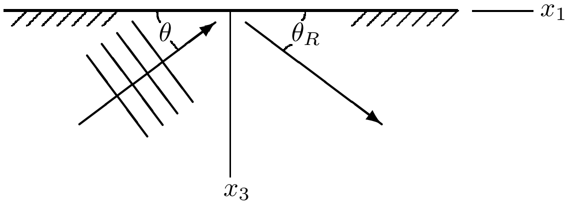
Exercise

Exercise 3.6 The motion of an elastic material is described by the displacement field

$$\begin{aligned} u_1 &= 0, \\ u_2 &= u_2(x_1, x_3, t), \\ u_3 &= 0. \end{aligned}$$

(a) Show that u_2 satisfies the wave equation

$$\frac{\partial^2 u_2}{\partial t^2} = \beta^2 \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_3^2} \right).$$



For the half-space shown, suppose that the displacement field consists of the incident and reflected waves

$$\begin{aligned} u_2 \quad = \quad & Ie^{i(kx_1 \cos\theta - kx_3 \sin\theta - \omega t)} \\ & + Re^{i(kx_1 \cos\theta_R + kx_3 \sin\theta_R - \omega t)}. \end{aligned}$$

(b) Show that $k = \omega/\beta$.

(c) Show that $\theta_R = \theta$ and $R = I$.

Discussion—The waves described in this exercise are *horizontally polarized shear waves.* They are plane waves that propagate in the x_1-x_3 plane, but the motion of the material particles is perpendicular to the x_1-x_3 plane.

3.4 Rayleigh Waves

Consider the elastic half space and coordinate system shown in Fig. 3.19. In Section 3.2, we obtained two types of solution for two-dimensional steady-state waves. One type consisted of plane waves propagating in an arbitrary direction in the x_1-x_3 plane, and the second type consisted of waves that propagate in the x_1 direction and whose amplitudes attenuate exponentially in the x_3 direction. Let us determine whether a wave of the second type can exist in an elastic half space. That is, can a wave that propagates in the x_1 direction and whose amplitude attenuates exponentially in the x_3 direction satisfy the boundary conditions at the free surface?

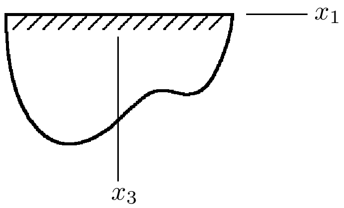


Figure 3.19: An elastic half space.

We assume solutions for the compressional and shear potentials ϕ and ψ of the forms given by Eqs. (3.22) and (3.23):

$$\begin{aligned} \phi &= Ae^{-hx_3}e^{i(k_1x_1 - \omega t)}, \\ \psi &= Ce^{-h_Sx_3}e^{i(k_1x_1 - \omega t)}, \end{aligned} \tag{3.45}$$

where A and C are constants and

$$h = \left(k_1^2 - \frac{\omega^2}{\alpha^2}\right)^{1/2}, \quad h_S = \left(k_1^2 - \frac{\omega^2}{\beta^2}\right)^{1/2}. \tag{3.46}$$

From Eqs. (3.45) we obtain the displacement field

$$\begin{aligned} u_1 &= (ik_1Ae^{-hx_3} + h_SCe^{-h_Sx_3})e^{i(k_1x_1 - \omega t)}, \\ u_3 &= (-hAe^{-hx_3} + ik_1Ce^{-h_Sx_3})e^{i(k_1x_1 - \omega t)}. \end{aligned} \tag{3.47}$$

The boundary conditions at the free surface are

$$[T_{13}]_{x_3=0} = 0, \quad [T_{33}]_{x_3=0} = 0.$$

From Eq. (3.25), the expressions for the boundary conditions in terms of displacements are

$$\left[\mu\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)\right]_{x_3=0}=0, \qquad \left[\lambda\frac{\partial u_1}{\partial x_1}+(\lambda+2\mu)\frac{\partial u_3}{\partial x_3}\right]_{x_3=0}=0.$$

Substituting Eqs. (3.47) into these boundary conditions, we obtain a system of homogeneous equations for the constants A and C that we write as

$$\begin{bmatrix} 2ik_1h & 2k_1^2-\omega^2/\beta^2 \\ 2k_1^2-\omega^2/\beta^2 & -2ik_1h_S \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix}=0. \tag{3.48}$$

This system has a nontrivial solution for A and C only if the determinant of the coefficients is equal to zero:

$$4k_1^2hh_S-\left(2k_1^2-\frac{\omega^2}{\beta^2}\right)^2=0. \tag{3.49}$$

We denote the phase velocity of the wave in the x_1 direction by

$$c_R=\frac{\omega}{k_1}.$$

By using Eqs. (3.46), we can write Eq. (3.49) in terms of c_R in the form

$$\left(2-\frac{c_R^2}{\beta^2}\right)^2-4\left(1-\frac{\beta^2}{\alpha^2}\frac{c_R^2}{\beta^2}\right)^{1/2}\left(1-\frac{c_R^2}{\beta^2}\right)^{1/2}=0. \tag{3.50}$$

This expression is called the *Rayleigh characteristic equation*. It has one root which yields a wave that propagates in the x_1 direction and whose amplitude attenuates exponentially in the x_3 direction. This type of wave is called a *surface wave,* because it has significant amplitude only near the surface. The particular example we have derived is called a *Rayleigh wave*.

From Eq. (3.32), we see that the solution of Eq. (3.50) for c_R/β depends only on the value of the Poisson's ratio ν of the material. Figure 3.20 shows a plot of the value of c_R/β as a function of Poisson's ratio. The Rayleigh wave phase velocity is slightly less than the shear wave velocity.

From Eq. (3.48), we can write the constant C in terms of the constant A as

$$C=\frac{2ik_1h}{\dfrac{\omega^2}{\beta^2}-2k_1^2}A.$$

Substituting this result into Eqs. (3.47), we obtain expressions for the displacement components in terms of A. The real parts of the resulting equations can

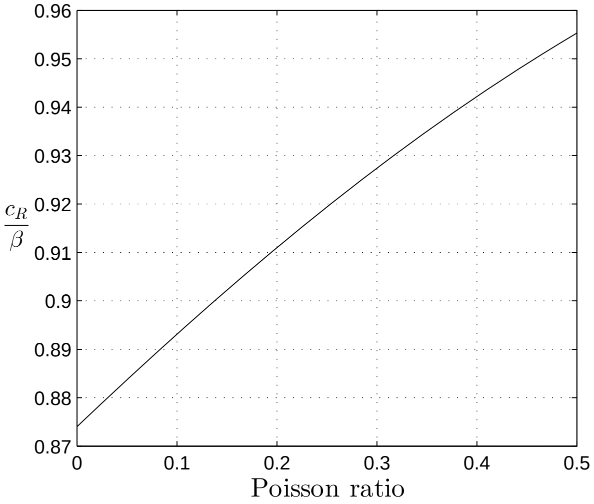


Figure 3.20: Rayleigh wave phase velocity as a function of Poisson's ratio.

be written in the forms

$$\begin{aligned}
\text{Real}\left(\frac{u_1}{k_1 A}\right) &= -\Bigg\{ e^{-[1-(\beta/\alpha)^2(c_R/\beta)^2]^{1/2}k_1x_3} \\
&\quad + \frac{2\left(1-\dfrac{\beta^2}{\alpha^2}\dfrac{c_R^2}{\beta^2}\right)^{1/2}\left(1-\dfrac{c_R^2}{\beta^2}\right)^{1/2}}{\dfrac{c_R^2}{\beta^2}-2} e^{-\left(1-c_R^2/\beta^2\right)^{1/2}k_1x_3} \Bigg\} \sin(k_1x_1-\omega t), \\
\text{Real}\left(\frac{u_3}{k_1 A}\right) &= -\Bigg\{ \left(1-\frac{\beta^2}{\alpha^2}\frac{c_R^2}{\beta^2}\right)^{1/2} e^{-[1-(\beta/\alpha)^2(c_R/\beta)^2]^{1/2}k_1x_3} \\
&\quad + \frac{2\left(1-\dfrac{\beta^2}{\alpha^2}\dfrac{c_R^2}{\beta^2}\right)^{1/2}}{\dfrac{c_R^2}{\beta^2}-2} e^{-\left(1-c_R^2/\beta^2\right)^{1/2}k_1x_3} \Bigg\} \cos(k_1x_1-\omega t).
\end{aligned}$$

These equations determine the displacement field in terms of the arbitrary constant A. The wave number $k_1 = \omega/c_R$ depends on the frequency. For a given Poisson's ratio ν, we can hold the position fixed and determine the displacement components of a point as functions of time. Figure 3.21 shows the resulting trajectories of points of the material at depths $k_1x_3 = 0$, 2, 4, and 6 for a Rayleigh wave propagating in the positive x_1 direction. The points of the material move in elliptical paths. The points near the surface move in the counterclockwise direction. At approximately $k_1x_3 = 1$, the horizontal motion vanishes, and below that depth the points move in the clockwise direction.

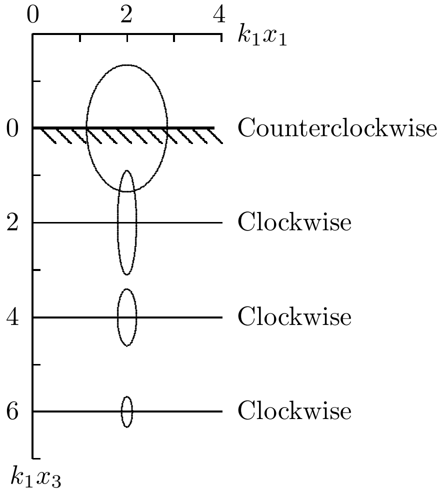


Figure 3.21: Particle trajectories at the surface and at several depths for Poisson's ratio $\nu = 0.3$.

3.5 Steady-State Waves in a Layer

We want to analyze the propagation of waves along a layer of elastic material. So that some of the concepts can be more easily understood, we first discuss the simpler case of acoustic waves in a channel.

Acoustic waves in a channel

The theory of acoustic or sound waves can be obtained from the equations of linear elastic wave propagation by setting the shear modulus $\mu = 0$. (See page 44.) Shear waves do not exist, and the displacement field can be expressed in terms of the potential ϕ:

$$\mathbf{u} = \nabla\phi.$$

Suppose that an acoustic medium occupies the space between two unbounded, rigid plates, as shown in Fig. 3.22, and consider two-dimensional motion of the medium described by the displacement field

$$\begin{aligned} u_1 &= u_1(x_1, x_3, t), \\ u_2 &= 0, \\ u_3 &= u_3(x_1, x_3, t). \end{aligned}$$

In terms of ϕ, the displacement field is

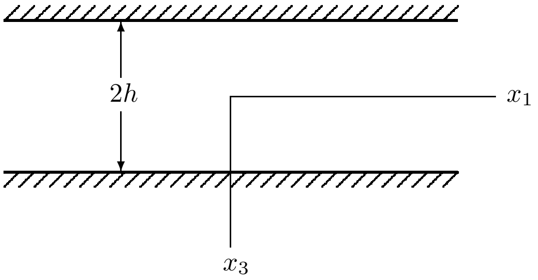


Figure 3.22: Acoustic medium between two unbounded rigid plates.

$$u_1 = \frac{\partial\phi}{\partial x_1}, \qquad u_3 = \frac{\partial\phi}{\partial x_3}.$$

We assume a solution for ϕ that describes a steady-state wave propagating along the channel:

$$\phi = f(x_3)e^{i(k_1x_1 - \omega t)},$$

where the function $f(x_3)$ that describes the distribution of ϕ across the channel must be determined. This is the same form of solution we discussed in Section 3.2. Substituting it into the wave equation

$$\frac{\partial^2\phi}{\partial t^2} = \alpha^2\left(\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_3^2}\right),$$

we find that the function $f(x_3)$ must satisfy the equation

$$\frac{d^2 f(x_3)}{dx_3^2} + \left(\frac{\omega^2}{\alpha^2} - k_1^2\right) f(x_3) = 0.$$

In this case we express the solution of this equation in the form

$$f(x_3) = A\sin k_3x_3 + B\cos k_3x_3,$$

where

$$k_3 = \left(\frac{\omega^2}{\alpha^2} - k_1^2\right)^{1/2}.$$

The resulting solution for ϕ is

$$\phi = (A \sin k_3 x_3 + B \cos k_3 x_3) e^{i(k_1 x_1 - \omega t)}.$$

From this expression we can determine the displacement component u_1:

$$u_1 = \frac{\partial \phi}{\partial x_1} = (ik_1 A \sin k_3 x_3 + ik_1 B \cos k_3 x_3) e^{i(k_1 x_1 - \omega t)}.$$

The term containing A describes an odd, or *antisymmetric* distribution of u_1 with respect to the centerline of the channel (Fig. 3.23.a). The term containing B describes an even, or *symmetric* distribution of u_1 with respect to the centerline of the channel (Fig. 3.23.b).

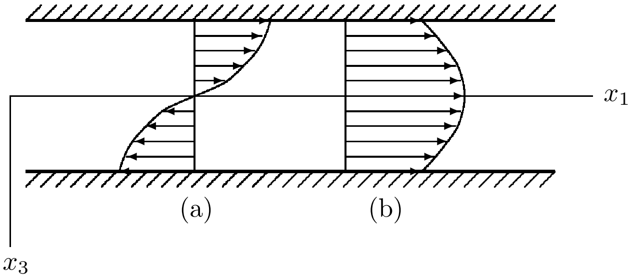


Figure 3.23: (a) Antisymmetric and (b) symmetric distributions of u_1.

Consider the part of the solution for ϕ that results in a symmetric distribution of u_1:

$$\phi = B \cos k_3 x_3 \, e^{i(k_1 x_1 - \omega t)}. \tag{3.51}$$

The boundary condition at the rigid walls of the channel is that the normal component of the displacement must equal zero:

$$[u_3]_{x_3 = \pm h} = 0. \tag{3.52}$$

The solution for u_3 is

$$u_3 = \frac{\partial \phi}{\partial x_3} = -B k_3 \sin k_3 x_3 \, e^{i(k_1 x_1 - \omega t)}.$$

Substituting this expression into the boundary condition, we find that it is satisfied if $\sin k_3 h = 0$. This equation requires that

$$k_3 h = \left(\frac{\omega^2}{\alpha^2} - k_1^2\right)^{1/2} h = n\pi,$$

where n is any integer. We can solve this equation for the wave number k_1 as a function of the frequency:

$$k_1 = \left(\frac{\omega^2}{\alpha^2} - \frac{n^2 \pi^2}{h^2} \right)^{1/2}. \tag{3.53}$$

With this result, we can also solve for the phase velocity of the wave in the x_1 direction $c_1 = \omega / k_1$ as a function of the frequency:

$$\frac{c_1}{\alpha} = \frac{1}{\left[1 - \left(\frac{\alpha n \pi}{\omega h} \right)^2 \right]^{1/2}}. \tag{3.54}$$

For a given frequency ω and integer n, we can substitute Eq. (3.53) into Eq. (3.51) to determine the potential ϕ. Thus we have obtained an infinite number of solutions that satisfy the boundary condition. These solutions are called *propagation modes*, or simply *modes*. The solution for $n = 0$ is called the first mode, the solution for $n = 1$ is called the second mode, and so forth.

The solutions of Eq. (3.54) for the phase velocities of the modes are shown as functions of frequency in Fig. 3.24. The phase velocities of all the modes except the first one depend on the frequency. When the phase velocity of a wave varies with the frequency, the wave is said to be *dispersive*, or to exhibit *dispersion*.

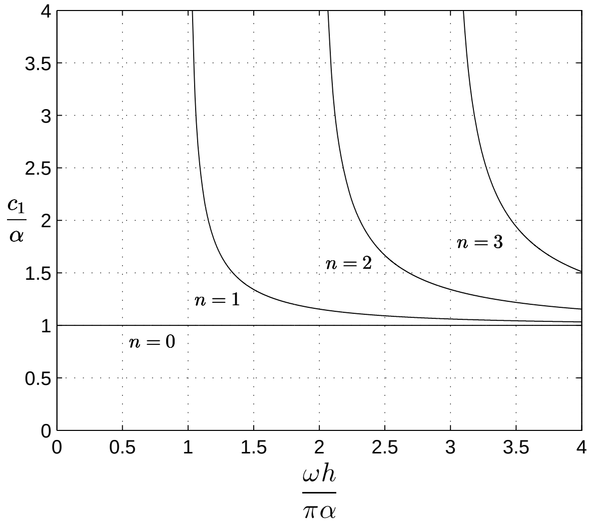


Figure 3.24: Phase velocities of the modes as functions of frequency.

For this reason, Eqs. (3.53) and (3.54) are called *dispersion relations.* In this example all of the modes except the first one are dispersive.

The phase velocities of all of the modes except the first one approach infinity when the frequency approaches the value $\omega = \alpha n\pi/h$. From Eq. (3.53), we see that when the frequency approaches this value the wave number k_1 approaches zero (which means that the wavelength of the wave in the x_1 direction approaches infinity), and below this frequency the wave number k_1 is imaginary. When k_1 is imaginary Eq. (3.51) does not describe a propagating wave, but oscillates in time and attenuates exponentially in the x_1 direction. The frequencies below which the wave number is imaginary are called *cutoff frequencies.* A mode does not propagate when the frequency is below its cutoff frequency.

Figure 3.25 shows the distributions of u_1 as a function of x_3 for the first three modes. The first mode is one-dimensional motion: the distribution of u_1 is uniform and $u_3 = 0$. This mode propagates at the wave speed α for any frequency.

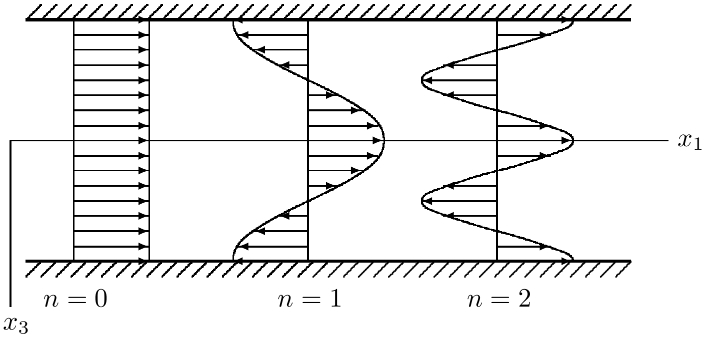


Figure 3.25: Distributions of u_1 for the first three modes.

Waves in an elastic layer

We now turn to the propagation of two-dimensional steady-state waves along a plate of elastic material with free surfaces (Fig. 3.26). The analysis is very similar to our treatment of acoustic waves in a channel in the previous section.

We consider two-dimensional motion described by the displacement field

$$\begin{aligned} u_1 &= u_1(x_1, x_3, t), \\ u_2 &= 0, \\ u_3 &= u_3(x_1, x_3, t). \end{aligned}$$

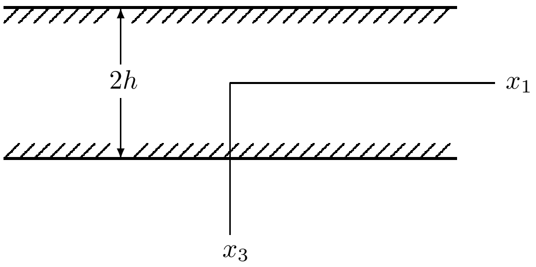


Figure 3.26: A plate of elastic material with free surfaces.

In terms of the potentials ϕ and ψ, the displacement field is

$$u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3},$$
$$u_3 = \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi}{\partial x_1}.$$

We assume solutions for ϕ and ψ that describe a steady-state wave propagating along the plate:

$$\phi = f(x_3)e^{i(k_1 x_1 - \omega t)}, \qquad \psi = g(x_3)e^{i(k_1 x_1 - \omega t)}.$$

Substituting these expressions into Eqs. (3.12) and (3.13), we find that the solutions for the functions $f(x_3)$ and $g(x_3)$ can be written in the forms

$$f(x_3) = A \sin k_P x_3 + B \cos k_P x_3,$$
$$g(x_3) = C \sin k_S x_3 + D \cos k_S x_3,$$

where

$$k_P = \left(\frac{\omega^2}{\alpha^2} - k_1^2\right)^{1/2}, \qquad k_S = \left(\frac{\omega^2}{\beta^2} - k_1^2\right)^{1/2}.$$

The resulting solutions for ϕ and ψ are

$$\phi = (A \sin k_P x_3 + B \cos k_P x_3)e^{i(k_1 x_1 - \omega t)},$$
$$\psi = (C \sin k_S x_3 + D \cos k_S x_3)e^{i(k_1 x_1 - \omega t)}.$$

Let us use these solutions to determine the displacement component u_1:

$$u_1 = [ik_1(A \sin k_P x_3 + B \cos k_P x_3) - k_S(C \cos k_S x_3 - D \sin k_S x_3)]e^{i(k_1 x_1 - \omega t)}.$$

The terms containing A and D describe antisymmetric distributions of u_1 with respect to the centerline of the channel (Fig. 3.23.a). The terms containing B and C describe symmetric distributions of u_1 with respect to the centerline of the channel (Fig. 3.23.b). Consider the parts of the solutions for ϕ and ψ that result in a symmetric distribution of u_1:

$$\phi = B\cos k_P x_3\, e^{i(k_1 x_1 - \omega t)}, \qquad \psi = C\sin k_S x_3\, e^{i(k_1 x_1 - \omega t)}. \tag{3.55}$$

The resulting expressions for u_1 and u_3 are

$$\begin{aligned} u_1 &= (ik_1 B\cos k_P x_3 - k_S C\cos k_S x_3)e^{i(k_1 x_1 - \omega t)}, \\ u_3 &= (-k_P B\sin k_P x_3 + ik_1 C\sin k_S x_3)e^{i(k_1 x_1 - \omega t)}. \end{aligned} \tag{3.56}$$

The boundary conditions at the free surfaces of the plate are

$$\begin{aligned} [T_{13}]_{x_3=\pm h} &= \left[\mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)\right]_{x_3=\pm h} = 0, \\ [T_{33}]_{x_3=\pm h} &= \left[\lambda\frac{\partial u_1}{\partial x_1} + (\lambda+2\mu)\frac{\partial u_3}{\partial x_3}\right]_{x_3=\pm h} = 0. \end{aligned} \tag{3.57}$$

Substituting Eqs. (3.56) into these boundary conditions, we obtain two homogeneous equations in terms of B and C:

$$\begin{bmatrix} [2\mu k_1^2 - (\lambda+2\mu)\omega^2/\alpha^2]\cos k_P h & 2i\mu k_1 k_S\cos k_S h \\ -2ik_1 k_P\sin k_P h & (\omega^2/\beta^2 - 2k_1^2)\sin k_S h \end{bmatrix} \begin{bmatrix} B \\ C \end{bmatrix} = 0.$$

This system has a nontrivial solution for B and C only if the determinant of the coefficients is equal to zero:

$$\begin{aligned} &[2\mu k_1^2 - (\lambda+2\mu)\omega^2/\alpha^2](\omega^2/\beta^2 - 2k_1^2)\cos k_P h\sin k_S h \\ &\qquad -4\mu k_1^2 k_P k_S\sin k_P h\cos k_S h = 0. \end{aligned}$$

This is the *dispersion relation* for symmetric longitudinal waves in a layer. It determines the wave number k_1 as a function of the frequency. If we express it in terms of the phase velocity in the x_1 direction $c_1 = \omega/k_1$, it can be written in the form

$$\left(2 - \frac{c_1^2}{\beta^2}\right)^2 \frac{\tan\left[\left(1 - \frac{\beta^2}{c_1^2}\right)^{1/2}\frac{\omega h}{\beta}\right]}{\tan\left[\left(\frac{\beta^2}{\alpha^2} - \frac{\beta^2}{c_1^2}\right)^{1/2}\frac{\omega h}{\beta}\right]} + 4\left(\frac{c_1^2}{\beta^2}\frac{\beta^2}{\alpha^2} - 1\right)^{1/2}\left(\frac{c_1^2}{\beta^2} - 1\right)^{1/2} = 0. \tag{3.58}$$

For a given frequency ω, this equation can be solved numerically for the phase velocity c_1. The equation is transcendental, and has an infinite number of roots. Each root yields a propagation mode that satisfies the boundary conditions at the free surfaces of the plate. These solutions are called the *Rayleigh-Lamb modes.*

The solution for c_1/β as a function of $\omega h/\beta$ depends only on the parameter β/α. From Eq. (3.32), we see that this parameter depends only on the Poisson's ratio ν of the material. Figure 3.27 shows a plot c_1/β as a function of $\omega h/\beta$ for the first and second propagation modes for $\nu = 0.3$. This figure is qualitatively similar to Fig. 3.24 for acoustic waves in a channel. However, in the acoustic problem the first mode was not dispersive. In the case of waves along an elastic plate, even the first mode is dispersive. At high frequency, the phase velocities of the modes approach the Rayleigh-wave phase velocity c_R (Section 3.4).

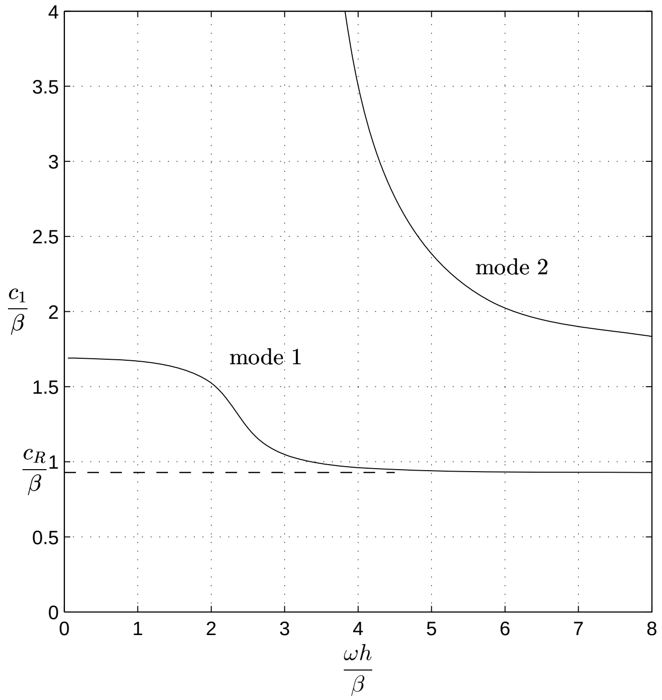


Figure 3.27: Phase velocities of the first two Rayleigh-Lamb modes as functions of frequency.

Elementary theory of waves in a layer

It is instructive to compare the results of the previous section with a simple approximate analysis of longitudinal waves in a layer. Consider a layer of linear elastic material with arbitrary dimensions in the x_1 and x_2 directions that is unbounded in the x_3 direction. If the stationary layer is subjected to a uniform normal stress T_{11} (Fig. 3.28), the relation between T_{11} and the resulting longitudinal strain E_{11} is

$$T_{11} = E_p E_{11} = E_p \frac{\partial u_1}{\partial x_1}, \tag{3.59}$$

where

$$E_p = \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \tag{3.60}$$

is called the *plate modulus.*

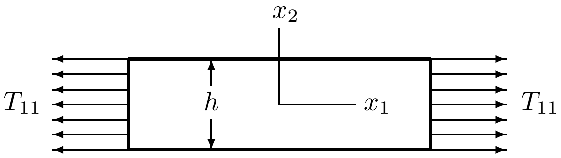


Figure 3.28: Subjecting a stationary layer of finite length to a uniform normal stress.

Now we assume that the layer undergoes motion in the x_1 direction and neglect variations in the displacement u_1 and the normal stress T_{11} in the x_2 direction. From the free-body diagram of an element of the plate of width dx_1 and unit depth in the x_3 direction (Fig. 3.29), we obtain the equation of motion

$$(\rho_0 h \, dx_1) \frac{\partial^2 u_1}{\partial t^2} = -T_{11} h + \left(T_{11} + \frac{\partial T_{11}}{\partial x_1} \, dx_1 \right) h.$$

Substituting Eq. (3.59) into this equation, we obtain the one-dimensional wave equation

$$\frac{\partial^2 u_1}{\partial t^2} = c_p^2 \frac{\partial^2 u_1}{\partial x_1^2}, \tag{3.61}$$

where the *plate velocity* c_p is

$$c_p = \sqrt{\frac{E_p}{\rho_0}}. \tag{3.62}$$

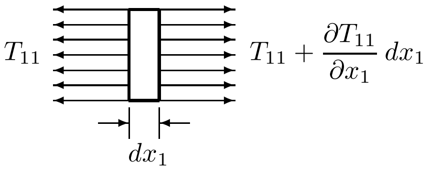


Figure 3.29: Free-body diagram of an element of the layer of width dx_1.

Equation (3.61), which predicts a single, nondispersive propagation mode with phase velocity c_p, provides an approximate model for longitudinal waves in a layer valid for very low frequencies. (Notice that we obtained Eq. (3.59) by applying a uniform normal stress to a *stationary* layer.) In fact, the exact theory we discussed in the previous section reduces to this elementary model in the limit as $\omega \to 0$, and the phase velocity of the first mode of the exact theory (Fig. 3.27) approaches c_p. (For Poisson's ratio $\nu = 0.3$, the ratio of the plate velocity to the shear wave velocity is $c_p/\beta = 1.69$.)

Exercises

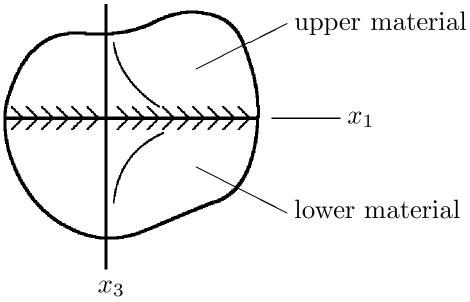
EXERCISE 3.7 Show that in terms of Poisson's ratio ν, the ratio of the plate velocity to the shear wave velocity is

$$\frac{c_p}{\beta} = \left(\frac{2}{1 - \nu} \right)^{1/2}.$$

EXERCISE 3.8 Show that in the limit as $\omega \to \infty$, Eq. (3.58) for the velocities of the Rayleigh-Lamb modes becomes identical to Eq. (3.50) for the velocity of a Rayleigh wave.

EXERCISE 3.9 Show that in the limit as $\omega \to 0$, the solution of Eq. (3.58) for the velocities of the Rayleigh-Lamb modes is the plate velocity c_p.

EXERCISE 3.10



A wave analogous to a Rayleigh wave can exist at a plane, bonded interface between two different elastic materials. This wave, called a Stoneley wave, attenuates exponentially with distance away from the interface in each material. Derive the characteristic equation for a Stoneley wave.

3.6 Steady-State Waves in Layered Media

Layered media occur in nature, and are also manufactured by bonding layers of different materials together to obtain *composite* materials with desired mechanical properties. A wave propagating through a layered elastic medium behaves quite differently from a wave in a homogeneous medium, due to the effects of the interfaces between the layers. In this section, we consider steady-state waves in a medium consisting of alternating plane layers of two elastic materials a and b (Fig. 3.30). Two adjacent layers form what is called a *unit cell* of the material.

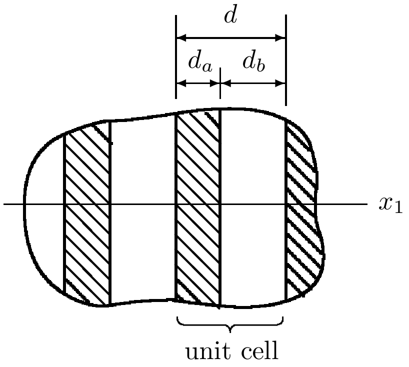


Figure 3.30: Medium of alternating layers of elastic materials a and b.

Suppose that a plane compressional steady-state wave propagates in the x_1 direction perpendicular to the interfaces between layers. Let the subscript ξ denote either a or b. The displacement within each layer of a unit cell is governed by the one-dimensional wave equation:

$$\frac{\partial^2 u_\xi}{\partial t^2} = \alpha_\xi^2 \frac{\partial^2 u_\xi}{\partial x_1^2}.$$

We express the solution within each layer as a sum of forward and rearward propagating waves,

$$u_\xi = A_\xi e^{i(k_\xi x_1 - \omega t)} + B_\xi e^{i(-k_\xi x_1 - \omega t)}, \tag{3.63}$$

where the wave number k_ξ is

$$k_\xi = \frac{\omega}{\alpha_\xi}. \tag{3.64}$$

In terms of the displacement, the normal stress T_{11} within each layer is

$$T_\xi = \rho_\xi \alpha_\xi^2 \frac{\partial u_\xi}{\partial x_1} = \frac{z_\xi^2}{\rho_\xi} \frac{\partial u_\xi}{\partial x_1}, \tag{3.65}$$

where the acoustic impedance $z_\xi = \rho_\xi \alpha_\xi$.

Let $x_1 = 0$ be the interface between the layers of the unit cell (Fig. 3.31). The displacements and normal stresses in the two materials must be equal at the interface:

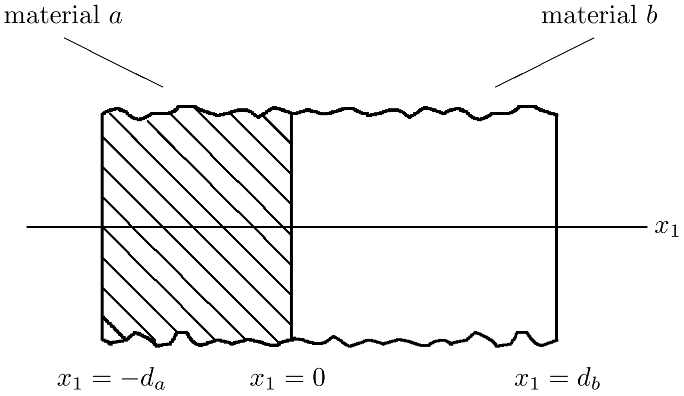


Figure 3.31: A unit cell.

$$u_a(0,t) = u_b(0,t), \tag{3.66}$$

$$T_a(0,t) = T_b(0,t). \tag{3.67}$$

Because the solutions given by Eq. (3.63) contain four constants, A_a, B_a, A_b, and B_b, two more conditions are needed. We obtain them from the periodicity of the problem with a result called the *Floquet theorem.* We recast Eqs. (3.63) and (3.65) into steady-state wave expressions having the same wave number k for both materials,

$$u_\xi = \mathcal{U}_\xi(x_1) e^{i(kx_1 - \omega t)}, \tag{3.68}$$

$$T_\xi = \mathcal{T}_\xi(x_1) e^{i(kx_1 - \omega t)}, \tag{3.69}$$

where we define

$$\mathcal{U}_\xi(x_1) = A_\xi e^{iK_\xi^- x_1} + B_\xi e^{-iK_\xi^+ x_1},$$
$$\mathcal{T}_\xi(x_1) = iz_\xi \omega [A_\xi e^{iK_\xi^- x_1} - B_\xi e^{-iK_\xi^+ x_1}],$$

and

$$K_\xi^\pm = k_\xi \pm k.$$

Because the functions $\mathcal{U}_\xi(x_1)$ and $\mathcal{T}_\xi(x_1)$ are continuous across the layer interfaces and do not depend on time, the Floquet theorem states that they must be periodic functions of x_1 with period equal to the length $d = d_a + d_b$ of the unit cell:

$$\begin{aligned} \mathcal{U}_a(-d_a) &= \mathcal{U}_b(d_b), \quad (3.70) \\ \mathcal{T}_a(-d_a) &= \mathcal{T}_b(d_b). \quad (3.71) \end{aligned}$$

We can write these two conditions, together with Eqs. (3.66) and (3.67), in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ z_a & -z_a & z_b & -z_b \\ e^{-iK_a^- d_a} & e^{iK_a^+ d_a} & e^{iK_b^- d_b} & e^{-iK_b^+ d_b} \\ z_a e^{-iK_a^- d_a} & -z_a e^{iK_a^+ d_a} & z_b e^{iK_b^- d_b} & -z_b e^{-iK_b^+ d_b} \end{bmatrix} \begin{bmatrix} A_a \\ B_a \\ -A_b \\ -B_b \end{bmatrix} = [0]. \quad (3.72)$$

Equating the determinant of the coefficients to zero, we obtain

$$\begin{aligned} \cos kd &= \cos\left(\frac{\omega d_a}{\alpha_a}\right)\cos\left(\frac{\omega d_b}{\alpha_b}\right) \\ &\quad - \tfrac{1}{2}\left(\frac{z_a}{z_b} + \frac{z_b}{z_a}\right)\sin\left(\frac{\omega d_a}{\alpha_a}\right)\sin\left(\frac{\omega d_b}{\alpha_b}\right). \end{aligned} \quad (3.73)$$

This is the dispersion relation for the steady-state wave. For a given frequency ω, we can solve it for the wave number k and then determine the phase velocity $c_1 = \omega/k$.

Special cases

When $z_a = z_b$, Eq. (3.73) yields a simple expression for the phase velocity c_1 in terms of the compressional wave speeds of the two materials:

$$\frac{d}{c_1} = \frac{d_a}{\alpha_a} + \frac{d_b}{\alpha_b}. \quad (3.74)$$

If, in addition, $\alpha_a = \alpha_b = \alpha$, the two materials are identical and $c_1 = \alpha$.

In the limit $\omega \to 0$, the solution for the phase velocity is

$$c_1 = \frac{d}{\Delta_d}, \quad (3.75)$$

where

$$\Delta_d^2 = \left(\frac{d_a}{\alpha_a}\right)^2 + \left(\frac{z_a}{z_b} + \frac{z_b}{z_a}\right)\frac{d_a d_b}{\alpha_a \alpha_b} + \left(\frac{d_b}{\alpha_b}\right)^2. \quad (3.76)$$

Equation 3.76 gives the approximate value of the phase velocity if the wavelength is large in comparison to the width of the unit cell. In the special case when $\alpha_a = \alpha_b = \alpha$ but the densities of the two materials are unequal, it yields the curious result that the phase velocity of the wave is smaller than the compressional wave speed in the two materials, $c_1 < \alpha$.

Pass and stop bands

Equation (3.73) yields real solutions for k only for values of frequency within distinct *bands.* Between these bands, they are complex. Writing k in terms of its real and imaginary parts as $k = k_R + ik_I$ and substituting it into Eqs. (3.68) and (3.69), we see that the expressions for the displacement and stress fields contain the term $e^{-k_I x_1}$. This means that within the bands of frequency in which k is complex, the amplitudes of the displacement and stress decrease, or *attenuate* exponentially with x_1. For this reason, these bands of frequency are called *stop bands,* and the bands within which k is real are called *pass bands.*

Figure 3.32 shows the real solutions for k for a layered material in which the ratio of the acoustic impedances is $z_b/z_a = 5$ and the ratio of *transit times* across the layers is $d_a\alpha_b/d_b\alpha_a = 10$. The shaded areas are the stop bands. The frequency range of the first pass band is $0 \leq \omega\Delta_d/2\pi \leq 0.42$. Its upper limit occurs at $kd/2\pi = 0.5$, which corresponds to a wavelength twice the thickness of the unit cell.

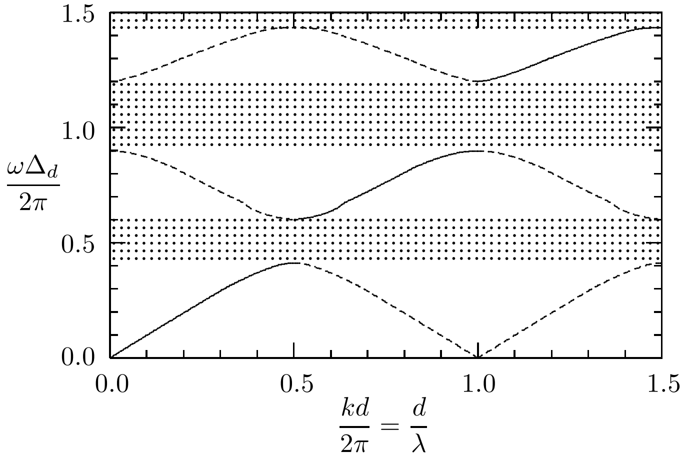


Figure 3.32: Pass and stop bands for a layered medium.

Although an infinite number of solutions for k are obtained from Eq. (3.73) for a given value of frequency, these roots do *not* represent different propagation modes. They result in identical solutions for the displacement field, so there is a single propagation mode. To emphasize this point, we plot only one “branch” of each pass band as a solid line and show the other branches as dashed lines. As the ratio of the acoustic impedances of the two materials approaches one, the stop bands narrow, and the branches we show as solid lines approach a single straight line, the dispersion relation of the resulting homogeneous material.

When superimposed waves having different frequencies propagate through a layered medium, the waves having frequencies within stop bands are attenuated, altering the character of the superimposed wave. We examine this phenomenon in the next chapter.

Exercises

Exercise 3.11 Show that when $z_a = z_b$, Eq. (3.73) yields the expression given in Eq. (3.74) for the phase velocity $c_1 = \omega/k$ in a layered material.

Exercise 3.12 Show that in the limit $\omega \to 0$, the solution of Eq. (3.73) for the phase velocity of a steady-state wave in a layered material is given by Eq. (3.75).

Exercise 3.13 For the layered material discussed in Section 3.6, the fraction of the volume of the material occupied by material a is $\phi = d_a/(d_a + d_b) = d_a/d$. Derive an equation for the low-frequency limit of the phase velocity as a function of ϕ. Using the properties of tungsten for material a and aluminum for material b (see Table B.2 in Appendix B), plot your equation for values of ϕ from zero to one.

Chapter 4

Transient Waves

Analytical and numerical solutions to transient wave problems in elastic materials can be obtained by integral transform methods, particularly Fourier and Laplace transforms. We discuss these transforms in the next two sections. We then define the discrete Fourier transform and demonstrate the use of the numerical algorithm called the fast Fourier transform (FFT). In the concluding section, we introduce the most important technique for transient waves in elastic wave propagation, the Cagniard-de Hoop method, by determining the response of an elastic half space to an impulsive line load. (This chapter involves integration in the complex plane. The necessary material from complex analysis is summarized in Appendix A).

4.1 Laplace Transform

The Laplace transform of a function $f(t)$ is defined by

$$f^L(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

where s is a parameter called the *transform variable.* The Laplace transform $f^L(s)$ can be *inverted* to recover the function $f(t)$ through the equation

$$f(t) = \frac{1}{2\pi i}\int_{C_\infty} f^L(s)e^{st}\,ds, \tag{4.1}$$

where C_∞ denotes a contour integral evaluated along an unbounded straight line parallel to the imaginary axis of the complex s-plane (Fig. 4.1). The distance D must be chosen so that all singularities of $f^L(s)$ are to the left of C_∞.

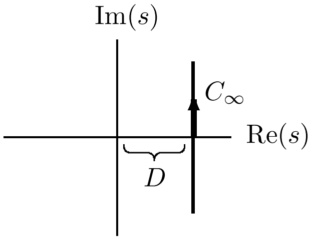


Figure 4.1: Contour for inversion of the Laplace transform.

The Laplace transform is an example of an *integral transform.* Applying an integral transform to a partial differential equation reduces its number of independent variables. In particular, applying the Laplace transform to the one-dimensional wave equation reduces it from a partial differential equation in two independent variables to an ordinary differential equation. Solving the ordinary equation yields the Laplace transform of the solution, which must be inverted to obtain the solution.

To demonstrate this process, we consider a half space of elastic material that is initially undisturbed and is subjected to the displacement boundary condition

$$u_1(0, t) = H(t)te^{-bt}, \tag{4.2}$$

where $H(t)$ is the step function and b is a positive real number (Fig. 4.2). The

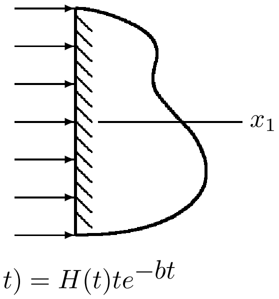


Figure 4.2: Half space with a displacement boundary condition.

motion is governed by the one-dimensional wave equation

$$\frac{\partial^2 u_1}{\partial t^2} = \alpha^2 \frac{\partial^2 u_1}{\partial x_1^2}. \tag{4.3}$$

To take the Laplace transform of this equation with respect to time, we multiply it by e^{-st} and integrate with respect to time from 0 to ∞:

$$\int_0^\infty \frac{\partial^2 u_1}{\partial t^2} e^{-st}\, dt = \int_0^\infty \alpha^2 \frac{\partial^2 u_1}{\partial x_1^2} e^{-st}\, dt. \tag{4.4}$$

Integrating the term on the left side by parts, we obtain the expression

$$\int_0^\infty e^{-st} \frac{\partial^2 u_1}{\partial t^2}\, dt = e^{-st} \frac{\partial u_1}{\partial t}\bigg|_0^\infty + s\int_0^\infty e^{-st} \frac{\partial u_1}{\partial t}\, dt. \tag{4.5}$$

If we define the step function so that $H(t) = 0$ for $t \leq 0$, the velocity $\partial u_1/\partial t$ is zero at $t = 0$. Because the inversion integral for the Laplace transform is evaluated along a contour for which $\mathrm{Re}(s) > 0$ (Fig. 4.1), $e^{-st} \to 0$ as $t \to \infty$. Evaluating the integral on the right side of Eq. (4.5) by parts, we obtain the result

$$\int_0^\infty e^{-st} \frac{\partial^2 u_1}{\partial t^2}\, dt = se^{-st} u_1\bigg|_0^\infty + s^2\int_0^\infty e^{-st} u_1\, dt.$$

We see that the left side of Eq. (4.4) assumes the form

$$\begin{aligned} \int_0^\infty \frac{\partial^2 u_1}{\partial t^2} e^{-st}\, dt &= s^2\int_0^\infty u_1 e^{-st}\, dt \\ &= s^2 u_1^L. \end{aligned}$$

By using this expression, we can write Eq. (4.4) as

$$\frac{d^2 u_1^L}{dx_1^2} - \frac{s^2}{\alpha^2} u_1^L = 0. \tag{4.6}$$

The Laplace transform of the boundary condition, Eq. (4.2), with respect to time is

$$u_1^L(0, s) = \int_0^\infty te^{-(b+s)t}\, dt = \frac{1}{(b+s)^2}. \tag{4.7}$$

We can write the solution of Eq. (4.6) in the form

$$u_1^L = Ae^{sx_1/\alpha} + Be^{-sx_1/\alpha}. \tag{4.8}$$

Substituting this result into Eq. (4.1), we can see that the first term represents a wave propagating in the negative x_1 direction. Thus we conclude that $A = 0$. From Eqs. (4.8) and (4.7), we find that

$$B = \frac{1}{(b+s)^2},$$

so that the solution for the displacement is

$$u_1 = \frac{1}{2\pi i}\int_{C_\infty} \frac{1}{(b+s)^2} e^{s(t - x_1/\alpha)}\, ds. \tag{4.9}$$

The Laplace transform is usually inverted by using one of the closed contours shown in Fig. 4.3. We denote the straight vertical part of the contour by C_1. The straight horizontal parts of the contour in Fig. 4.3.a are denoted by C_2 and C_4, and the semicircular part of the contours is denoted by C_3.

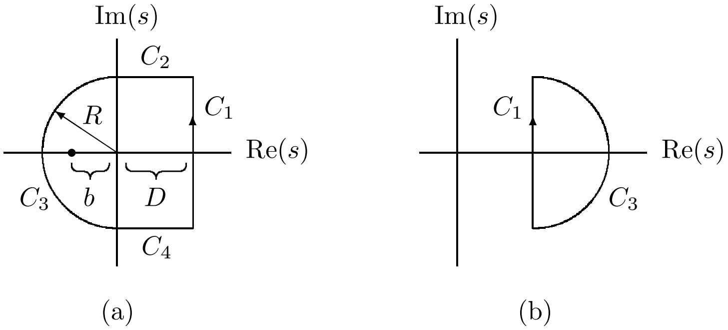


Figure 4.3: Closed contours in the complex s-plane.

If we denote the integrand in Eq. (4.9) by

$$g(s) = \frac{1}{2\pi i(b+s)^2} e^{s(t - x_1/\alpha)}, \tag{4.10}$$

the integral we must evaluate to determine the displacement is

$$u_1 = \lim_{R\to\infty} \int_{C_1} g(s)\, ds. \tag{4.11}$$

The integral along the closed contour in Fig. 4.3.a can be expressed in terms of the residues of the integrand within the contour:

$$\sum_{k=1}^{4} \int_{C_k} g(s)\, ds = 2\pi i \sum \text{Residues}.$$

The integrand, Eq. (4.10), has a second-order pole at $s = -b$. From Eq. (A.13), the residue of the second-order pole is

$$\lim_{s\to -b} \frac{d}{ds}[(s+b)^2 g(s)] = \frac{1}{2\pi i}\left(t - \frac{x_1}{\alpha}\right) e^{-b(t - x_1/\alpha)}.$$

We see that the limit of the integral over the closed contour as $R \to \infty$ is

$$\lim_{R\to\infty} \sum_{k=1}^{4} \int_{C_k} g(s)\, ds = \left(t - \frac{x_1}{\alpha}\right) e^{-b(t - x_1/\alpha)}. \tag{4.12}$$

We can show that the integrals along C_2 and C_4 approach zero as $R \to \infty$. On C_2, $s = x + iR$, where $x \leq D$. Therefore, on C_2

$$\left| e^{s(t - x_1/\alpha)} \right| \leq e^{D|t - x_1/\alpha|}.$$

On C_2, $|b + s| \geq (R^2 + b^2)^{1/2}$, so $\left|(b+s)^2\right| \geq (R^2 + b^2)$. As a consequence,

$$\left| \int_{C_2} g(s)\, ds \right| \leq \frac{e^{D|t - x_1/\alpha|}}{2\pi(R^2 + b^2)} \int_{C_2} |ds| = \frac{De^{D|t - x_1/\alpha|}}{2\pi(R^2 + b^2)}.$$

From this expression we see that the integral along C_2 vanishes as $R \to \infty$. The same argument can be used to show that the integral along C_4 vanishes as $R \to \infty$.

To evaluate the integral along the semicircular contour C_3, we use a result called *Jordan's lemma:*

> Let C_A denote a circular arc of radius R in the complex s-plane, and consider the integral
>
> $$I = \int_{C_A} h(s) e^{as}\, ds,$$
>
> where $h(s)$ is analytic on C_A and a is a complex constant. Jordan's lemma states that if the maximum value of the magnitude of $h(s)$ on C_A approaches zero as $R \to \infty$ and $\text{Re}(as) \leq 0$ on C_S, then $I \to 0$ as $R \to \infty$.

From Eq. (4.10), we see that on C_3, $\text{Re}[s(t - x_1/\alpha)] \leq 0$ when $t > x_1/\alpha$. Also, on C_3

$$\left| \frac{1}{2\pi i (b+s)^2} \right| \leq \frac{1}{(R-b)^2},$$

which approaches zero as $R \to \infty$. Therefore Jordan's lemma states that

$$\lim_{R \to \infty} \int_{C_3} g(s)\, ds = 0 \quad \text{when} \quad t > \frac{x_1}{\alpha}.$$

From Eqs. (4.11) and (4.12), we obtain the solution for the displacement:

$$u_1 = \left(t - \frac{x_1}{\alpha} \right) e^{-b(t - x_1/\alpha)} \quad \text{when} \quad t > \frac{x_1}{\alpha}.$$

When $t < x_1/\alpha$, we can evaluate the displacement by using the closed contour shown in Fig. 4.3.b. In this case the contour does not contain the pole, so the integral over the closed contour is zero. Jordan's lemma states that the integral over C_3' vanishes as $R \to \infty$ when $t < x_1/\alpha$, so the displacement is

$$u_1 = \lim_{R \to \infty} \int_{C_1} g(s)\, ds = 0 \quad \text{when} \quad t < \frac{x_1}{\alpha}.$$

The time $t = x_1/\alpha$ is the time at which the wave arrives at the position x_1.

4.2 Fourier Transform

The Fourier transform of a function $f(t)$ is defined by

$$f^F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt, \tag{4.13}$$

where ω is the transform variable. The Fourier transform $f^F(\omega)$ can be inverted to recover the function $f(t)$ through the expression

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^F(\omega)e^{i\omega t}\, d\omega. \tag{4.14}$$

These two equations are called the Fourier integral theorem.

Example

For comparison with the Laplace transform, we apply the Fourier transform to the example discussed in the previous section: an undisturbed half space is subjected to the displacement boundary condition

$$u_1(0,t) = H(t)te^{-bt}, \tag{4.15}$$

where $H(t)$ is the step function and b is a positive real number (Fig. 4.2).

The motion is governed by the one-dimensional wave equation

$$\frac{\partial^2 u_1}{\partial t^2} = \alpha^2 \frac{\partial^2 u_1}{\partial x_1^2}. \tag{4.16}$$

To take the Fourier transform of this equation with respect to time, we multiply it by $e^{-i\omega t}$ and integrate with respect to time from $-\infty$ to ∞:

$$\int_{-\infty}^{\infty} \frac{\partial^2 u_1}{\partial t^2} e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} \alpha^2 \frac{\partial^2 u_1}{\partial x_1^2} e^{-i\omega t}\, dt. \tag{4.17}$$

Integrating the term on the left side by parts, we obtain the expression

$$\int_{-\infty}^{\infty} e^{-i\omega t} \frac{\partial^2 u_1}{\partial t^2}\, dt = e^{-i\omega t} \frac{\partial u_1}{\partial t}\bigg|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} e^{-i\omega t} \frac{\partial u_1}{\partial t}\, dt. \tag{4.18}$$

Because the material is initially undisturbed, the velocity $\partial u_1/\partial t$ is zero at $t = -\infty$. We assume that the velocity is zero at $t = \infty$. Evaluating the integral on the right side of Eq. (4.18) by parts, we obtain the result

$$\int_{-\infty}^{\infty} e^{-i\omega t} \frac{\partial u_1}{\partial t}\, dt = e^{-i\omega t} i\omega u_1 \bigg|_{-\infty}^{\infty} - \omega^2 \int_{-\infty}^{\infty} e^{-i\omega t} u_1\, dt.$$

The displacement is zero at $t = -\infty$ and we assume it is zero at $t = \infty$. By doing so, we obtain the left side of Eq. (4.17) in the form

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\partial^2 u_1}{\partial t^2} e^{-i\omega t}\, dt &= -\omega^2 \int_{-\infty}^{\infty} u_1 e^{-i\omega t}\, dt \\ &= -\omega^2 u_1^F.\end{aligned}$$

By using this expression, we can write Eq. (4.17) as

$$\frac{d^2 u_1^F}{dx_1^2} + \frac{\omega^2}{\alpha^2} u_1^F = 0. \tag{4.19}$$

By using the Fourier transform, we have reduced the partial differential equation, Eq. (4.16), to an ordinary equation. We also take the Fourier transform of Eq. (4.15) with respect to time:

$$\int_{-\infty}^{\infty} u_1(0, t) e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} H(t) t e^{-bt} e^{-i\omega t}\, dt,$$

which we can write as

$$u_1^F(0, \omega) = \int_0^{\infty} t e^{-(b + i\omega)t}\, dt.$$

Evaluating the integral on the right yields the Fourier transformed boundary condition:

$$u_1^F(0, \omega) = \frac{1}{(b + i\omega)^2}. \tag{4.20}$$

We now proceed to obtain the solution for u_1^F. We write the solution of Eq. (4.19) in the form

$$u_1^F = A e^{i\omega x_1/\alpha} + B e^{-i\omega x_1/\alpha}. \tag{4.21}$$

Substituting this result into the inversion integral, Eq. (4.14), we obtain an expression for the displacement:

$$u_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} A e^{i\omega(x_1/\alpha + t)}\, d\omega + \frac{1}{2\pi} \int_{-\infty}^{\infty} B e^{i\omega(-x_1/\alpha + t)}\, d\omega. \tag{4.22}$$

We see that the first term represents a wave propagating in the negative x_1 direction and the second term represents a wave propagating in the positive x_1 direction. Because the boundary condition can give rise only to a wave propagating in the positive x_1 direction, we conclude that $A = 0$. From Eqs. (4.21) and (4.20), we find that

$$u_1^F = \frac{1}{(b + i\omega)^2} e^{-i\omega x_1/\alpha}. \tag{4.23}$$

Therefore Eq. (4.22) for the displacement assumes the form

$$u_1 = \int_{-\infty}^{\infty} \frac{1}{2\pi(b + i\omega)^2} e^{i\omega(t - x_1/\alpha)}\, d\omega. \tag{4.24}$$

This expression for the displacement is an integral with respect to a real variable ω. However, to evaluate the integral it is convenient to let ω be a complex variable. By doing so, we can interpret Eq. (4.24) as a contour integral along the real axis from $\mathrm{Re}(\omega) = -\infty$ to $\mathrm{Re}(\omega) = +\infty$ (Fig. 4.4.a).

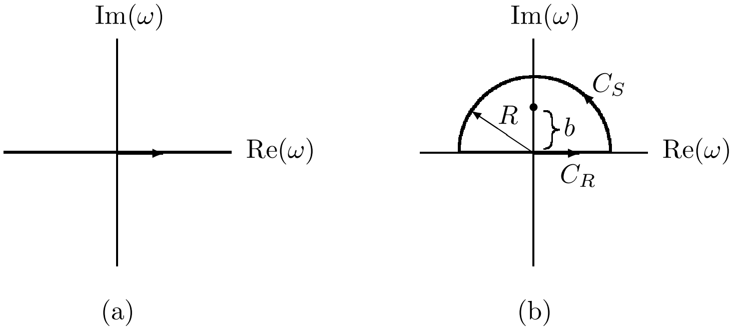


Figure 4.4: (a) Path along the real axis in the complex ω-plane. (b) A closed contour.

Consider the closed contour shown in Fig. 4.4.b, where C_R is the part of the contour along the real axis and C_S is the semicircular part of the contour. If we denote the integrand in Eq. (4.24) by

$$g(\omega) = \frac{1}{2\pi(b + i\omega)^2} e^{i\omega\,(t - x_1/\alpha)}, \tag{4.25}$$

the integral we must evaluate to determine the displacement is

$$u_1 = \lim_{R\to\infty} \int_{C_R} g(\omega)\, d\omega. \tag{4.26}$$

The integral along the closed contour can be expressed in terms of the residues of the integrand within the contour:

$$\int_{C_R} g(\omega)\, d\omega + \int_{C_S} g(\omega)\, d\omega = 2\pi i \sum \text{Residues}.$$

The integrand, Eq. (4.25), has a second-order pole at $\omega = bi$. From Eq. (A.13), the residue of the second-order pole is

$$\lim_{\omega\to bi} \frac{d}{d\omega}[(\omega - bi)^2 g(\omega)] = \frac{1}{2\pi i}\left(t - \frac{x_1}{\alpha}\right) e^{-b(t - x_1/\alpha)}.$$

Therefore we find that the integral along the closed contour is

$$\int_{C_R} g(\omega)\,d\omega + \int_{C_S} g(\omega)\,d\omega = \left(t - \frac{x_1}{\alpha}\right) e^{-b(t - x_1/\alpha)}.$$

This result is independent of R. Therefore, the limit of the integral over the closed contour as $R \to \infty$ is

$$\lim_{R\to\infty} \left[\int_{C_R} g(\omega)\,d\omega + \int_{C_S} g(\omega)\,d\omega\right] = \left(t - \frac{x_1}{\alpha}\right) e^{-b(t - x_1/\alpha)}. \qquad (4.27)$$

By using Jordan's lemma (page 163), it is easy to show that

$$\lim_{R\to\infty} \int_{C_S} g(\omega)\,d\omega = 0 \quad \text{when} \quad t > \frac{x_1}{\alpha}.$$

Therefore, from Eqs. (4.26) and (4.27), we obtain the solution for the displacement:

$$u_1 = \left(t - \frac{x_1}{\alpha}\right) e^{-b(t - x_1/\alpha)} \quad \text{when} \quad t > \frac{x_1}{\alpha}. \qquad (4.28)$$

Notice that $t = x_1/\alpha$ is the time at which the wave arrives at the position x_1.

Fourier superposition and impulse response

If we know the solution of a problem with a steady-state boundary condition, we can (in principle) use the Fourier transform to determine the solution of the same problem with a transient boundary condition. This powerful and commonly used procedure is called *Fourier superposition.* Suppose that a problem has a steady-state boundary condition represented by the expression $h(\mathbf{x})e^{i\omega t}$, where $h(\mathbf{x})$ may depend on position but not time, and let the resulting steady-state solution of the problem be represented by $g(\mathbf{x}, \omega)e^{i\omega t}$. Our objective is to determine the solution for a transient boundary condition $h(\mathbf{x})f(t)$, where $f(t)$ is some prescribed function of time.

Let $f^F(\omega)$ be the Fourier transform of $f(t)$. For a given value of ω, if we multiply the steady-state boundary condition by $(1/2\pi)f^F(\omega)$, obtaining $(1/2\pi)f^F(\omega)h(\mathbf{x})e^{i\omega t}$, the resulting steady-state solution is $(1/2\pi)f^F(\omega)g(\mathbf{x}, \omega)e^{i\omega t}$. (Notice that this is true only if the governing equations are linear.) Integrating the boundary condition with respect to frequency from $-\infty$ to ∞, we obtain the transient boundary condition,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f^F(\omega)h(\mathbf{x})e^{i\omega t}\,d\omega = h(\mathbf{x})f(t),$$

and (again because the governing equations are linear), the solution is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f^F(\omega)g(\mathbf{x}, \omega)e^{i\omega t}\,d\omega. \qquad (4.29)$$

This is Fourier superposition; by integrating, we have superimposed steady-state solutions to obtain the solution of the transient problem. Notice that the product $f^F(\omega)g(\mathbf{x},\omega)$ is the Fourier transform of the transient solution.

Now suppose that $f(t)$ is a delta function, $f(t) = \delta(t)$. This transient boundary condition is called an *impulse,* and the resulting solution is the *impulse response.* The Fourier transform of $f(t)$ is

$$f^F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t}\, dt = 1,$$

so that the Fourier transform of the transient solution is

$$f^F(\omega)g(\mathbf{x},\omega) = g(\mathbf{x},\omega).$$

The Fourier transform of the transient solution resulting from an impulse boundary condition equals the complex amplitude of the steady-state solution. The function $g(\mathbf{x},\omega)$ is the impulse response in the *frequency domain* and its inverse is the impulse response in the *time domain.* This result has important implications from both computational and experimental points of view: by applying an impulse boundary condition, the steady state solution can be obtained *as a function of frequency.*

As an example, we apply Fourier superposition to the problem discussed in the previous section. We considered a half space subjected to a transient displacement boundary condition $u_1(0,t) = H(t)te^{-bt}$ (Fig. 4.5.a). This boundary condition is of the form $h(\mathbf{x})f(t)$ with $h(\mathbf{x}) = 1$ and $f(t) = H(t)te^{-bt}$. We must first solve the problem for the steady-state boundary condition $u_1(0,t) =$

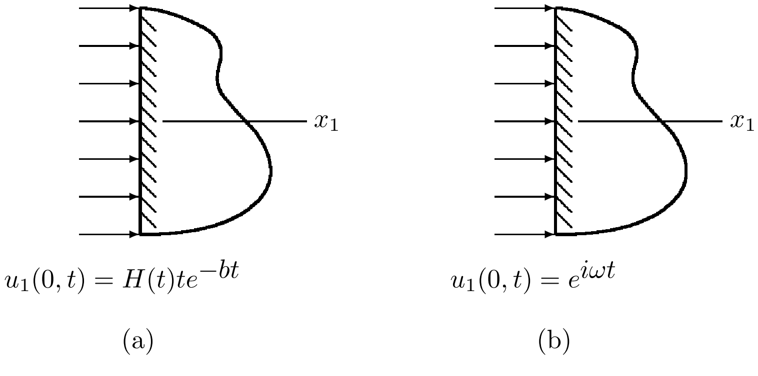


Figure 4.5: (a) Half space subjected to a transient boundary condition. (b) The corresponding steady-state problem.

$h(\mathbf{x})e^{i\omega t} = e^{i\omega t}$ (Fig. 4.5.b). The result is

$$u_1(x_1, t) = e^{i\omega(t - x_1/\alpha)} = e^{-x_1/\alpha}e^{i\omega t},$$

so the steady-state solution is of the form $g(\mathbf{x}, \omega)e^{i\omega t}$ with $g(\mathbf{x}, \omega) = e^{-x_1/\alpha}$.

The next step is to determine the Fourier transform of $f(t)$:

$$\begin{aligned} f^F(\omega) &= \int_{-\infty}^{\infty} H(t)te^{-bt}e^{-i\omega t}\,dt \\ &= \frac{1}{(b + i\omega)^2}. \end{aligned}$$

Substituting our expressions for $g(\mathbf{x}, \omega)$ and $f^F(\omega)$ into Eq. (4.29), we obtain the transient solution

$$u_1 = \int_{-\infty}^{\infty} \frac{1}{2\pi(b + i\omega)^2} e^{i\omega\,(t - x_1/\alpha)}\,d\omega, \tag{4.30}$$

which is identical to Eq. (4.24). Thus Fourier superposition gives the same result obtained by applying the Fourier transform to the transient problem. The task of inverting the transform, evaluating the integral in Eq. (4.30), remains. As we saw in the previous section, analytical evaluation of such integrals is difficult even in simple cases. We discuss efficient computational methods for evaluating them in the next section.

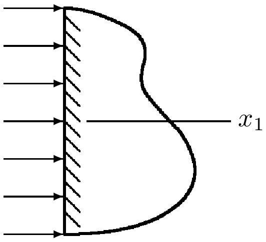
Notice that if the half-space in this example is subjected to an impulse boundary condition, $u_1(0, t) = \delta(t)$, the Fourier transform of the transient solution is

$$f^F(\omega)g(\mathbf{x}, \omega) = (1)\left(e^{-x_1/\alpha}\right) = e^{-x_1/\alpha},$$

which is the complex amplitude of the steady-state solution.

Exercises

Exercise 4.1



Suppose that an elastic half space is initially undisturbed and is subjected to the velocity boundary condition

$$\frac{\partial u_1}{\partial t}(0,t) = H(t)e^{-bt},$$

where $H(t)$ is the step function and b is a positive real number.
(a) By using a Laplace transform with respect to time, show that the solution for the Laplace transform of the displacement is

$$u_1^L = \frac{e^{-sx_1/\alpha}}{s(s+b)}.$$

(b) By inverting the Laplace transform obtained in Part (a), show that the solution for the displacement is

$$u_1 = \frac{1}{b}\left[1 - e^{-b(t - x_1/\alpha)}\right] \qquad \text{when} \quad t > \frac{x_1}{\alpha},$$

$$u_1 = 0 \qquad \text{when} \quad t < \frac{x_1}{\alpha}.$$

Exercise 4.2 Use Jordan's lemma to show that

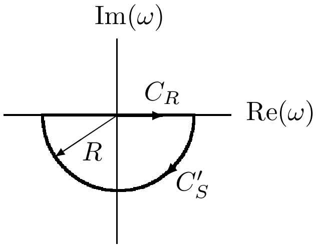
$$\lim_{R\to\infty}\int_{C_S}\frac{1}{2\pi(b+i\omega)^2}e^{i\omega(t - x_1/\alpha)}\,d\omega = 0 \quad \text{when} \quad t > \frac{x_1}{\alpha},$$

where C_S is the semicircular contour shown in Fig. 4.4.b.

Exercise 4.3 Show that when $t < x_1/\alpha$, the solution of Eq. (4.26) is

$$u_1 = \lim_{R\to\infty}\int_{C_R} g(\omega)\,d\omega = 0.$$

Discussion—Evaluate the integral by using a closed contour with the semicircular contour in the lower half of the complex ω-plane:



Exercise 4.4 Suppose that an unbounded elastic material is subjected to the initial displacement and velocity fields

$$\begin{aligned} u_1(x_1,0) &= p(x_1), \\ \frac{\partial u_1}{\partial t}(x_1,0) &= \quad 0, \end{aligned}$$

where $p(x_1)$ is a prescribed function. By taking a Fourier transform with respect to x_1 with the transform variable denoted by k, show that the solution for the Fourier transform of the displacement is

$$u_1^F = \tfrac{1}{2}p^F\left(e^{-i\alpha kt} + e^{i\alpha kt}\right),$$

where p^F is the Fourier transform of the function $p(x_1)$.

Discussion—To determine the solution for the displacement, it is not necessary to invert the Fourier transform. By substituting the expression for u_1^F into the inversion integral, you obtain

$$\begin{aligned} u_1 \quad = \quad & \frac{1}{2\pi}\int_{-\infty}^{\infty} \tfrac{1}{2}p^F e^{ik(x_1 - \alpha t)}\, dk \\ & +\frac{1}{2\pi}\int_{-\infty}^{\infty} \tfrac{1}{2}p^F e^{ik(x_1 + \alpha t)}\, dk. \end{aligned}$$

You can see by inspection of this expression that

$$u_1 = \tfrac{1}{2}p(x_1 - \alpha t) + \tfrac{1}{2}p(x_1 + \alpha t).$$

4.3 Discrete Fourier Transform

The exponential form of the Fourier series of a function of time $f(t)$ over the interval $0 \le t \le T$ is

$$f(t) = \sum_{n=-\infty}^{\infty} A_n e^{i\omega_n t}, \tag{4.31}$$

where the discrete frequencies ω_n are

$$\omega_n = 2\pi n/T. \tag{4.32}$$

The terms in this series are called the *frequency components* of $f(t)$. This representation is periodic over the interval T:

$$f(t+T) = f(t).$$

By multiplying Eq. (4.31) by $e^{-i\omega_m t}$, integrating from $t = 0$ to $t = T$, and using the *orthogonality condition*

$$\int_0^T e^{i\omega_n t} e^{-i\omega_m t} dt = \left\{ \begin{array}{ll} 0 & m \neq n, \\ T & m = n, \end{array} \right. \tag{4.33}$$

we obtain an expression for the coefficients A_m in terms of $f(t)$:

$$A_m = \frac{1}{T} \int_0^T f(t) e^{-i\omega_m t} dt. \tag{4.34}$$

In general, these coefficients are complex. A graph of their magnitudes as a function of the frequencies ω_m is called the *frequency spectrum* of $f(t)$.

We divide the time interval T into an even number N of intervals of length Δ, so that $T = N\Delta$. Let the value of $f(t)$ at some time within the nth interval be denoted by f_n, $n = 0, 1, 2, \ldots, N-1$. In terms of these *samples* of the function $f(t)$, we can approximate Eq. (4.34) by

$$A_m = f_m^{DF}, \tag{4.35}$$

where

$$f_m^{DF} = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-2\pi i m n/N}. \tag{4.36}$$

The array f_m^{DF}, $m = 0, 1, 2, \ldots, N-1$, is called the *discrete Fourier transform* of the discrete function f_n. It converts the N *time-domain* samples f_n into N *frequency-domain* samples f_n^{DF}. Evaluation of this equation for $m = n + N$ shows that the f_n^{DF} are periodic over the interval N:

$$f_{n+N}^{DF} = f_n^{DF}. \tag{4.37}$$

Comparison of Eq. (4.34) to Eq. (4.13) shows that the Fourier transform of the sampled function $f(t)$ is related to the discrete Fourier transform by

$$f^F(\omega_n) = T f_n^{DF}. \tag{4.38}$$

A function $f(t)$ that has a finite Fourier series representation over a time interval $0 \leq t \leq T$ is said to be *band-limited* on that interval. That is, such a function has a maximum frequency in its Fourier series. The truncated Fourier series

$$f(t) = \sum_{n=-N/2}^{N/2-1} A_n e^{i\omega_n t}, \tag{4.39}$$

where

$$\omega_n = 2\pi n / N\Delta, \tag{4.40}$$

is an exact representation of a band-limited function $f(t)$ in the interval $0 \leq t \leq T$ provided the frequency $\omega_{N/2} = 2\pi/(2\Delta)$ is greater than the maximum frequency in the Fourier series of $f(t)$. This connection between the sampling interval and the exact representation of band-limited functions is called the *sampling theorem,* and $\omega_{N/2}$ is called the Nyquist frequency.

By using the periodicity condition, Eq. (4.37), together with Eqs. (4.35) and (4.39), it can be shown that

$$f_m = \sum_{n=0}^{N-1} f_n^{DF} e^{2\pi i n m / N}, \tag{4.41}$$

which is the inverse discrete Fourier transform. Thus Eqs. (4.36) and (4.41) are analogous to the Fourier transform and its inverse, Eqs. (4.13) and (4.14). The difference is that the discrete Fourier transform is defined in terms of a function with a discrete domain.

Data are often obtained at discrete values of time, and in such cases the discrete Fourier transform and its inverse can be applied in the same way the Fourier transform and its inverse are applied to functions defined on continuous domains. Because the discrete transform and its inverse can easily be evaluated by computer, even when we deal with continuous, or *analog* data, it is often advantageous to sample it at discrete times (a process called *analog-to-digital conversion*) and use the discrete transform.

According to the sampling theorem, there is a one-to-one relation between a band-limited function $f(t)$ and its discrete samples f_n if the sampling interval Δ is chosen so that at least two samples occur within each period of the maximum frequency in the spectrum of $f(t)$. Of course this is not possible if a function is not band-limited, but in such cases it may be acceptable to approximate the function by one that is band limited.

When the transform or its inverse is computed as indicated in Eqs. (4.36) and (4.41), the amount of computation is proportional to N^2. When N is a power of 2, a sequence of computations can be chosen so that only $N \log_2 N$ operations are required. This procedure is called the *fast Fourier transform,* or FFT. For example, when $N = 2^{14} = 16384$, the FFT algorithm is 1170 times faster. This algorithm has become the most important computational algorithm in digital electronics and signal processing, and is contained in many types of commercial software.

Examples

1. Half space with displacement boundary condition In Section 4.2, we apply the Fourier transform to a half space of elastic material subjected to the displacement boundary condition

$$u(0,t) = H(t)te^{-bt},$$

where $H(t)$ is the step function and b is a constant. By taking Fourier transforms of the governing wave equation and the boundary condition, we obtain the Fourier transform of the displacement field. The result, Eq. (4.23), is

$$u^F = \frac{1}{(b+i\omega)^2} e^{-i\omega x_1/\alpha}. \tag{4.42}$$

We can begin with this expression and use the inverse discrete Fourier transform to obtain the displacement field u.

Applying the inverse FFT algorithm to this example results in a sequence of N real numbers representing the displacement at N discretely spaced intervals of time and at one position x_1. These times are separated by the sampling interval Δ, and it is up to us to specify not only the number of points N but also Δ and the position x_1.

The solution we want to approximate is a transient function whose amplitude is significant only over a finite range of time. However, the solution obtained with the algorithm is periodic with period $N\Delta$. While we are free to specify any values for N and Δ, the quality of our results suffers if we make bad choices. For example, if Δ is too large, we cannot resolve the profile of the transient wave from the discrete solution. But selecting a very small value for Δ causes another difficulty. The period of the FFT results, and consequently the "time window" within which we can view the results, is given by the product $N\Delta$.

Therefore the choice of a small value for Δ must be offset by a large value of N, which results in large computation times.

For the present example, we can choose a value of Δ based on the functional form of the boundary condition. To obtain sufficient resolution of the transient wave, we select a sampling interval such that the quantity $b\Delta$ is small. This ensures that the part of the propagating wave that is of significant amplitude is sampled by a suitable number of points. Next, we select N large enough to obtain a time window beginning at $t = 0$ sufficiently long to “capture” the solution.

Let the discrete values of the displacement be u_m. The relation between the time t and the index m is $m = t/\Delta$. For convenience, we introduce the parameters

$$\begin{aligned} r &= 1/b\Delta, \\ n &= x_1/\alpha\Delta. \end{aligned}$$

We obtain $u_m^{DF}(n)$ from Eqs. (4.38), (4.40), and (4.42):

$$bu_m^{DF}(n) = \begin{cases} \dfrac{b}{T} u^F(n\alpha\Delta, \omega_m) = \dfrac{e^{-2\pi imn/N}}{bT(1 + 2\pi imr/N)^2}, & -N/2 < m < N/2, \\ 0, & m = N/2. \end{cases} \tag{4.43}$$

The function $u^F(x_1, \omega)$ is non-zero for all frequencies. However, its amplitude decreases monotonically with frequency, and we can approximate $u^F(x_1, \omega)$ by $u_m^{DF}(n)$ if its magnitude at the Nyquist frequency is small compared to its magnitude at zero frequency:

$$\left| \frac{u^F(x_1, 0)}{u^F(x_1, \omega_{N/2})} \right| = |(1 + i\pi r)^2| \gg 1. \tag{4.44}$$

This requires that r be large or, equivalently, that $b\Delta$ be small. This requirement was satisfied by our choice of the sampling interval Δ.

For a given value of the nondimensional position n, we obtain $u_m^{DF}(n)$ from Eq. (4.43) and employ the inverse FFT algorithm to determine $u_m(n)$. The resulting solution for the displacement obtained with $\Delta = 1/b$ is compared to the exact solution in Fig. 4.6. Only the portion of the time window $0 \leq m \leq N$ is shown. Even with this coarse sampling, the discrete solution approximates the actual displacement. By decreasing the sampling interval Δ by a factor of 10, we obtain the comparison shown in Fig. 4.7. The discrete and exact solutions are indistinguishable.

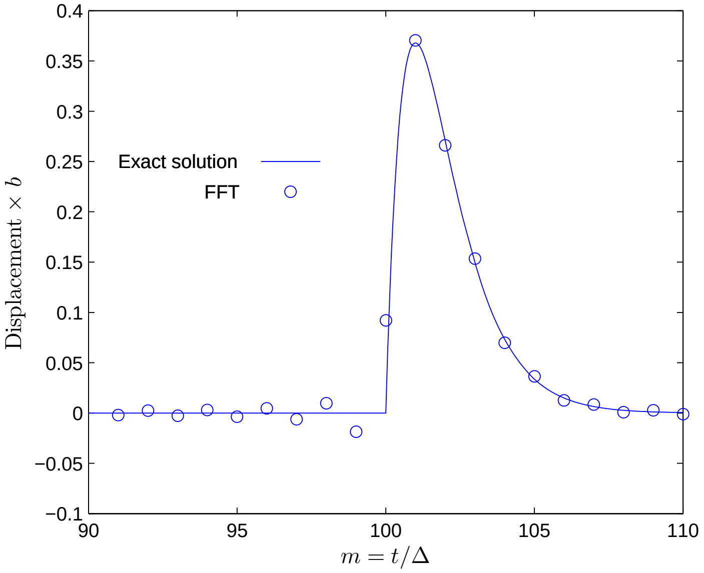


Figure 4.6: Exact and FFT solutions for $N = 256$, $r = 1$, and $n = 100$.

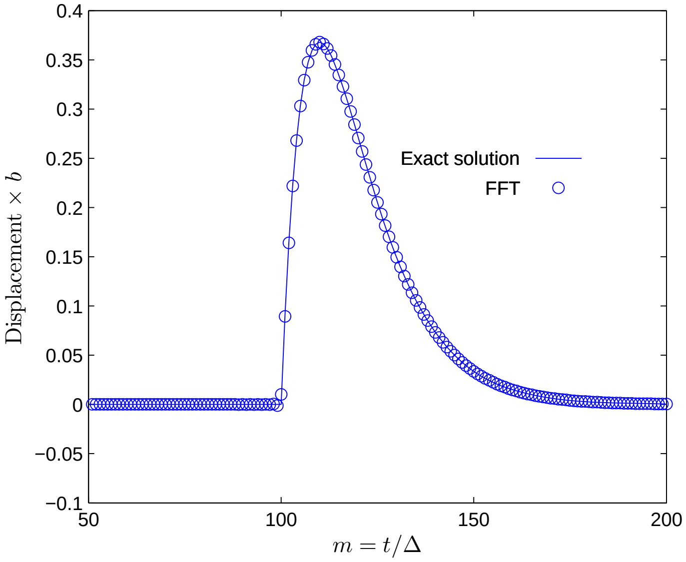


Figure 4.7: Exact and FFT solutions for $N = 1024$, $r = 10$, and $n = 100$.

2. Half-space with embedded layer If the impulse response of a problem is known, its Fourier transform is the solution of the corresponding steady-state problem. Here we demonstrate this procedure by obtaining the steady-state solution for a half space containing a layer of higher-impedance material (Fig. 4.8). We determine the impulse response by the method of characteristics, applying the procedure described in Section 2.7.

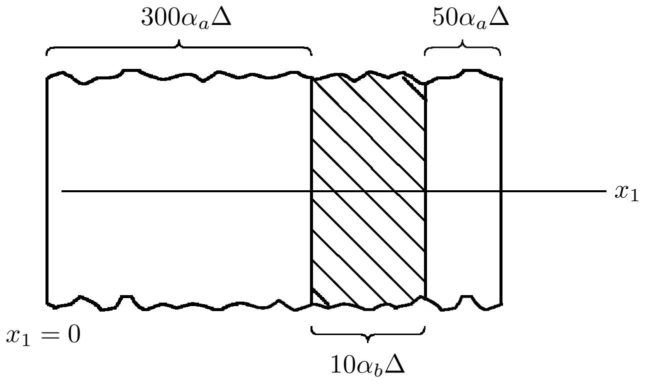


Figure 4.8: Elastic half space with an embedded layer.

The compressional wave velocities in the half space and the layer are denoted by α_a and α_b, respectively, the densities are ρ_a and ρ_b, and the ratio of acoustic impedances is $z_b/z_a = 3$. To apply the method of characteristics, we select a time increment Δ for the calculation that divides the portion of the half space to the left of the embedded layer into 300 sublayers and the embedded layer itself into 10 sublayers. We divide the portion of the half space to the right of the embedded layer into 50 sublayers and impose a condition at the right boundary that prevents the reflection of waves (see page 104).

The method of characteristics solution is evaluated at integer values of time $m = t/\Delta$. We apply an impulse in velocity at the left boundary:

$$v_m = \left\{ \begin{array}{ll} N & m = 0 \\ 0 & 0 < m \leq N - 1 \end{array} \right. ,$$

and terminate the calculation at $m = N - 1$. (In the subsequent application of the FFT algorithm, N is the size of the FFT array.) The velocity calculated at the right "boundary" of the half-space using $N = 1024$ is shown in Fig. 4.9. It consists of a sequence of impulses with diminishing amplitude separated in time by 20Δ, the time required for an impulse to complete two transits across the thickness of the embedded layer.

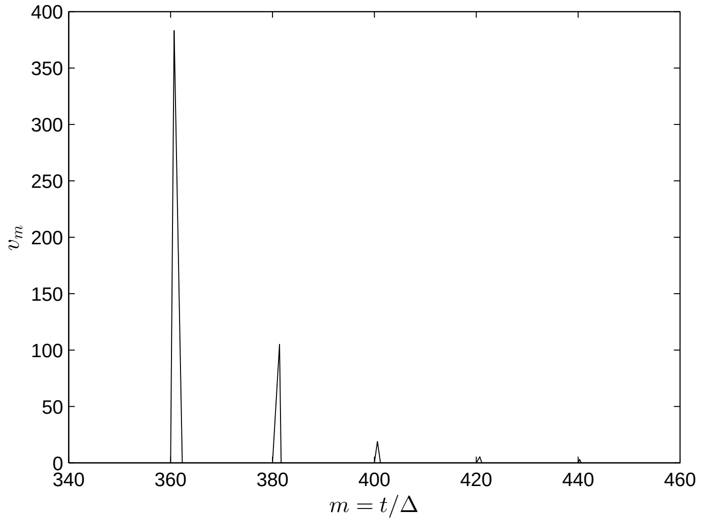


Figure 4.9: Impulse response of the layer.

We apply the FFT algorithm to these results using a sampling interval and sampling length equal, respectively, to the Δ and N used for the method-of-characteristics calculation. The result is the solution for the velocity at the right "boundary" resulting from a steady-state velocity boundary condition (Fig. 4.10). At certain frequencies, waves propagate through the layer without reduction in amplitude. These frequencies occur when an integer multiple of half-wavelengths equals the layer thickness. Between these frequencies, the amplitude is reduced by 60 percent.

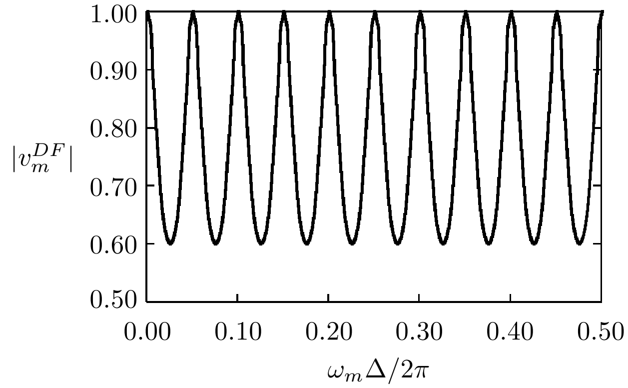


Figure 4.10: FFT of the impulse response, N=1024.

Transient versus steady-state wave analysis

These examples raise an issue often faced when solving problems in elastic wave propagation. In Example 1, we started with a steady-state solution and used the inverse FFT to obtain a transient solution. In Example 2, we started with a transient solution and used the FFT to generate a steady-wave solution. Both approaches are adaptable to large-scale calculations of problems with complex geometries, boundary conditions, and initial data. For problems in two or three spatial dimensions, the approach in Example 1 is usually preferable. For problems with one spatial dimension, the method in Example 2 can produce both transient and steady-state results with much less computation time than a direct steady-state wave analysis.

Exercises

Exercise 4.5 Consider the function of time $f(t) = t$.
(a) If you represent this function as a discrete Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} A_n e^{i\omega_n t}$$

over the interval $0 \leq t \leq 1$, show that the coefficients are

$$A_n = \frac{e^{-i\omega_n}(i\omega_n + 1)}{\omega_n^2} - \frac{1}{\omega_n^2}. \tag{4.45}$$

(b) Show that $A_0 = 1/2$.

Discussion—To do part (b), express the term $e^{-i\omega_n}$ as a Taylor series.

Exercise 4.6 Calculate the results shown in Fig. 4.6. Use $T = 100$ and $\alpha = 1$. Repeat the calculation using N =256, 128, and 64. What causes the solution to change?

Discussion—In carrying out these computations, you need to be aware that commercial implementations of the FFT vary. Some are specialized to real values of f_n, but some accept complex values. The particular implementation used influences the choice of N. Another variation is the placement of the normalization factor $1/N$; it can appear in Eq. (4.36) or in Eq. (4.41), with a corresponding affect on Eq. (4.43).

Exercise 4.7 In Exercise 4.6, assume that there is a second boundary at a distance $x_1 = L = 200\alpha\Delta$ from the existing boundary. Assuming the material is fixed at the new boundary, derive the finite Fourier transform of the displacement field.

Discussion—Accounting for the new boundary requires new expressions for the constants A and B in Eq. (4.21).

Exercise 4.8 The discrete Fourier transform converts N real numbers into N complex numbers. Use the properties of periodicity and symmetry of f_n^{DF} to show that half of the values of the f_n^{DF} are sufficient to determine the other half.

4.4 Transient Waves in Dispersive Media

In Chapter 3 we demonstrated that steady-state longitudinal waves in an elastic layer and steady-state waves in layered materials are dispersive: the phase velocity depends on the frequency of the wave. Dispersive waves commonly occur in problems involving boundaries and interfaces. The Fourier integral theorem expresses a transient wave in terms of superimposed steady-state components with a spectrum of frequencies. If there is dispersion, each component propagates with the phase velocity corresponding to its frequency. As a result, a transient wave tends to spread, or *disperse,* as it propagates. This is the origin of the term dispersion. Because the different frequency components propagate with different velocities, it is not generally possible to define a velocity of a transient wave. However, we will show that a transient wave having a narrow frequency spectrum propagates with a velocity called the *group velocity.*

Group velocity

Let us consider waves propagating in the positive x_1 direction, and suppose that the phase velocity $c_1(\omega) = \omega/k_1(\omega)$ depends on frequency. First we consider a steady-state wave with frequency ω_0:

$$f(x_1, t) = Ae^{i[\omega_0 t - k_1(\omega_0)x_1]} = Ae^{i\omega_0 \xi(\omega_0)}, \tag{4.46}$$

where

$$\xi(\omega) = t - \frac{k_1(\omega)x_1}{\omega} = t - \frac{x_1}{c_1(\omega)}.$$

Its Fourier transform is

$$f^F(x_1, \omega) = 2\pi Ae^{-ik_1(\omega_0)x_1}\delta(\omega - \omega_0),$$

which you can confirm using the Fourier integral

$$f(x_1, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^F(x_1, \omega)e^{i\omega t}\, d\omega. \tag{4.47}$$

Thus the frequency "spectrum" is a single frequency (Fig. 4.11.a).

We now consider a narrow, uniform frequency spectrum (a *narrow-band* spectrum) centered about the frequency ω_0 (Fig. 4.11.b):

$$f^F(x_1, \omega) = \begin{cases} \dfrac{\pi A}{\Omega}e^{-ik_1(\omega)x_1} & -\Omega \le \omega - \omega_0 \le \Omega, \\ 0 & \text{otherwise,} \end{cases} \tag{4.48}$$

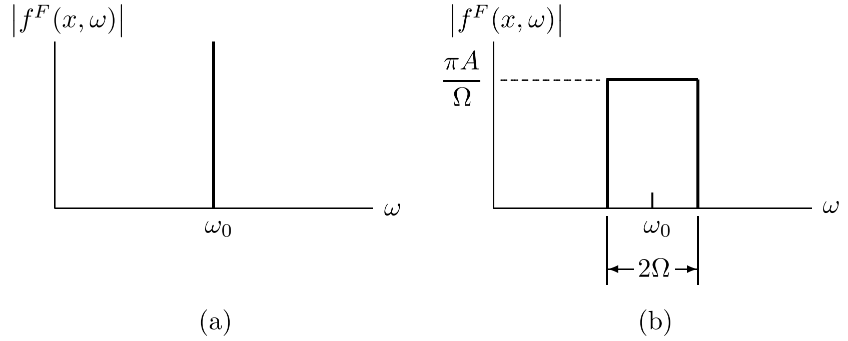


Figure 4.11: (a) Spectrum of a wave with a single frequency ω_0. (b) Spectrum of a narrow-band wave.

where Ω is a constant. We assume Ω to be sufficiently small that we can approximate the wave number by retaining only the first-order term in its Taylor series expansion:

$$k_1(\omega) = k_1(\omega_0) + \frac{dk_1}{d\omega}(\omega_0)(\omega - \omega_0) = k_1(\omega_0) + \frac{\overline{\omega}}{c_g(\omega_0)}, \tag{4.49}$$

where we define

$$\overline{\omega} = \omega - \omega_0$$

and

$$c_g(\omega) = \frac{d\omega}{dk_1}. \tag{4.50}$$

Substituting Eqs. (4.48) and (4.49) into Eq. (4.47), we obtain the time-domain expression for the narrow-band wave:

$$f(x_1, t) = Ae^{i\omega_0 \xi(\omega_0)} \frac{\sin[\Omega \xi_g(\omega_0)]}{\Omega \xi_g(\omega_0)}, \tag{4.51}$$

where

$$\xi_g(\omega) = t - \frac{x_1}{c_g(\omega)}. \tag{4.52}$$

Notice that when $\Omega \to 0$, Eq. (4.51) reduces to Eq. (4.46), which describes a wave with frequency ω_0 propagating at the phase velocity $c_1(\omega_0)$. The wave described by Eq. (4.46) is called the *phase wave.*

The terms $\xi(\omega_0)$ and $\xi_g(\omega_0)$ in Eq. (4.51) are D'Alembert independent variables associated with forward propagating waves. Thus this equation is the product of two waves: the phase wave, propagating with the phase velocity $c_1(\omega_0)$, and a *group wave* propagating with the *group velocity* $c_g(\omega_0)$. In

Fig. 4.12, we plot Eq. (4.51) for $x = 0$ and $\omega_0 = 20\Omega$. The very short-wavelength phase wave is “enveloped” by the group wave. The group wave characterizes the gross shape and propagation velocity of the transient wave. The velocity of the phase wave relative to the group wave is determined by the dispersion curve relating ω to k_1 (Fig. 4.13). When the curve is concave upward, the phase wave propagates more slowly than the group wave. An observer stationary with respect to the group wave sees phase waves being “created” at the front of the group wave, traveling backwards, and disappearing at the back. When the curve is straight, the phase wave and group wave move together. When the curve is concave downward, the observer sees phase waves being created at the back of the group wave, moving forward, and disappearing at the front.

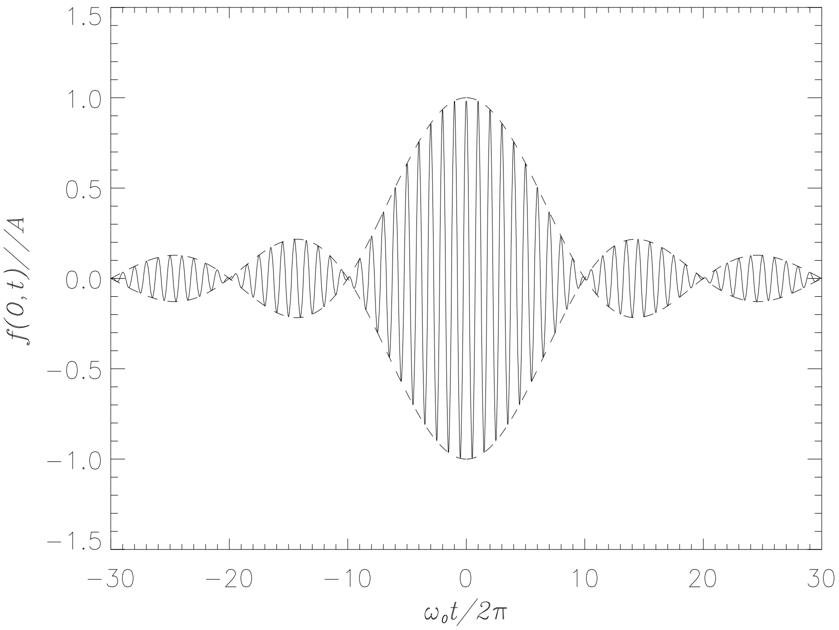


Figure 4.12: Phase wave (—) and group wave (− −) with $\omega_0 = 20\Omega$.

In Fig. 4.14, we compare the phase and group velocities of the first mode for longitudinal waves in an elastic layer for Poisson’s ratio $\nu = 0.3$. (See Section 3.5). The group velocity is approximately equal to the phase velocity at very low frequency, but is smaller than the phase velocity at all frequencies. At high frequency, both the phase and group velocities approach the Rayleigh wave velocity.

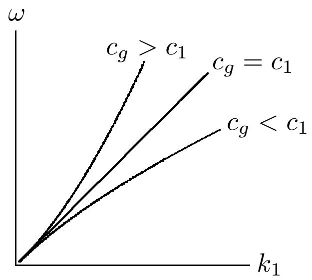


Figure 4.13: Dependence of the group velocity on the relation between frequency and wave number.

Transient waves in layered media

In Section 2.7 we used the method of characteristics to analyze transient waves in layered materials, and we discussed steady-state waves in such materials in Section 3.6. With the FFT and the concepts of Fourier superposition and group velocity, we can now reexamine and interpret the properties of transient waves in layered media. The approach we describe is applicable to other problems involving dispersion.

The medium we consider consists of alternating layers of two elastic materials that form a *unit cell* (Fig. 4.15). We denote the materials by a and b. The ratio of the acoustic impedances is $z_b/z_a = 5$, and the ratio of *transit times* across the layers is $d_a\alpha_b/d_b\alpha_a = 10$. The resulting dispersion curve is shown in Fig. 4.16, where $d = d_a + d_b$ and Δ_d is given by Eq. (3.76). The three solid curves are the first three pass bands, defining ranges of frequency within which steady-state waves propagate without attenuation.

At very low frequency, the dispersion curve is nearly straight. The group and phase velocities are nearly equal and independent of frequency, and waves propagate in the layered material as if it were homogeneous. As the frequency increases, the slope of the dispersion curve in the first pass band decreases. Both the phase and group velocities decrease, but the group velocity decreases more rapidly. At the highest frequency in the first pass band, the dispersion curve has zero slope, so $c_g = 0$. (A wave having zero group velocity is called a *stationary wave.*) At this frequency, the wavelength equals two unit cell widths. The dispersion curve in the second pass band is "flatter" than in the first pass band, which indicates lower group velocities, and $c_1 > c_g$. This flattening trend continues with successively higher pass bands at first, but eventually reverses itself. The layered medium acts as a frequency filter, selectively blocking the

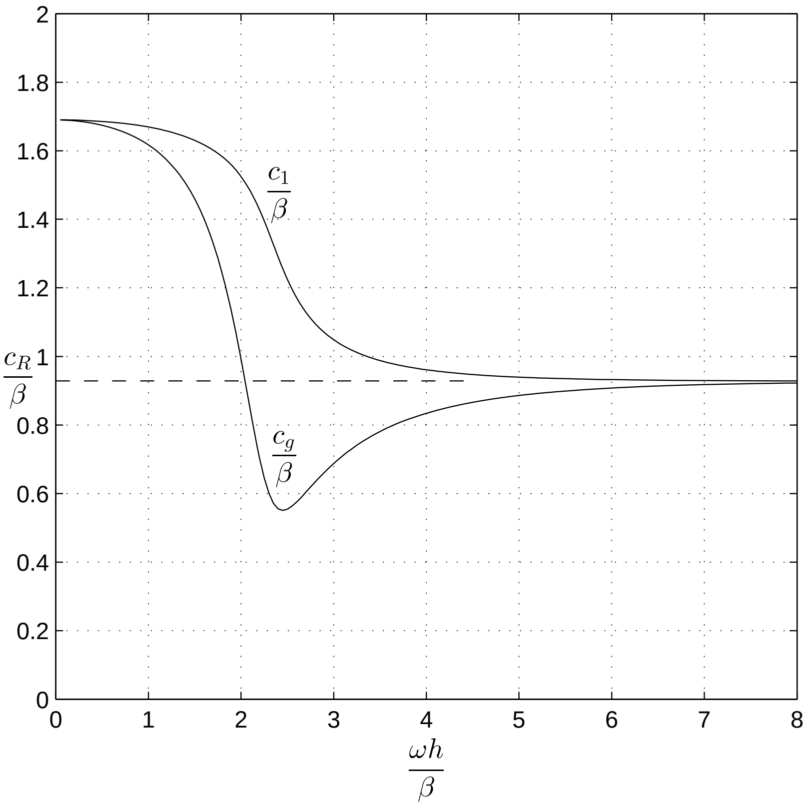


Figure 4.14: First-mode phase and group velocities for symmetric longitudinal waves in a layer of thickness $2h$.

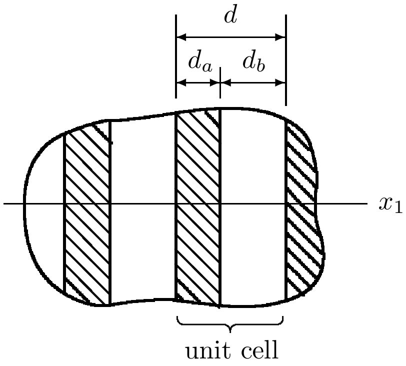


Figure 4.15: Medium of alternating layers of elastic materials a and b.

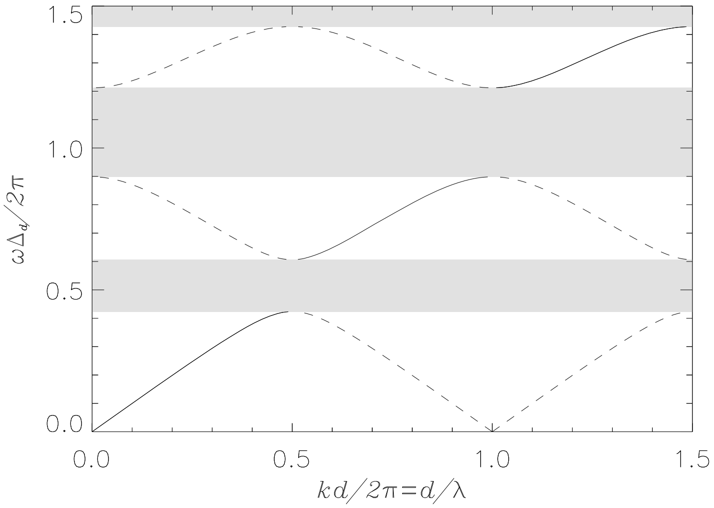


Figure 4.16: Dispersion relation for a layered medium.

propagation of steady-state components with frequencies within stop bands. Within a given pass band, the dispersion increases with frequency. Within each pass band, the group velocity varies from zero at the "edges" to a maximum value near the center.

With these observations in mind, we examine two types of waves. By subjecting the layered medium to an impulse boundary condition, we obtain a wave having a broad spectrum of frequencies. We then apply a boundary condition resulting in a wave with a narrow frequency spectrum. We obtain the solutions for the transient waves by a simple modification of the program given on page 106, dividing a into 10 sublayers and treating b as a single layer. As a result, Eq. (3.76) gives $\Delta_d^2 = 153\Delta^2$.

Impulse boundary condition For the first calculation, the velocity on the left boundary is 1 at $t = 0$ and zero thereafter. The velocity of the layered medium at the left boundary of layer 71 is shown in Fig. 4.17. The results show

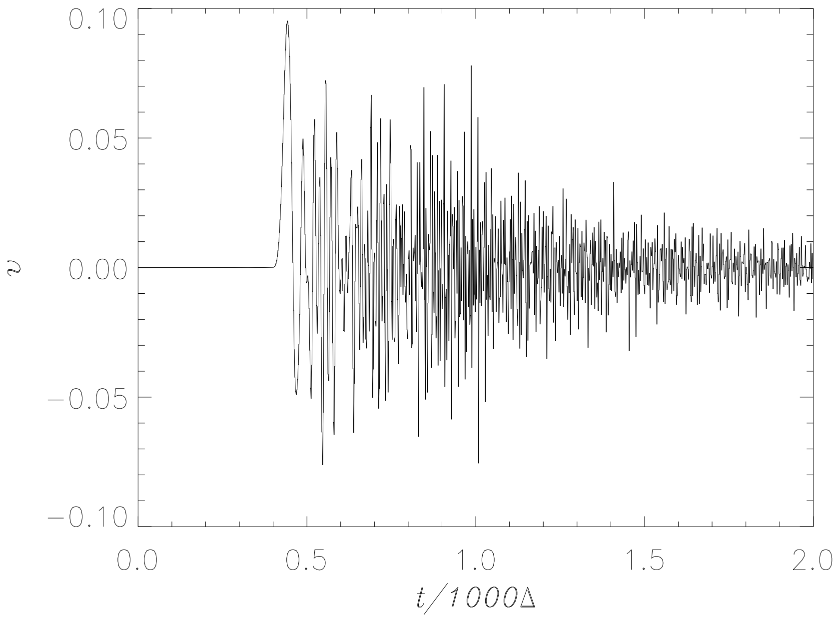


Figure 4.17: Impulse response at a point in the layered medium.

that the impulse, which originally had thickness Δ, has dispersed over many time increments. Another distinctive feature of the impulse response is the low-frequency oscillation near the leading edge of the wave, which is expected because the maximum group velocity occurs at $\omega = 0$. The velocity of the leading edge of the wave should correspond to the maximum group velocity, which is equal to the low-frequency limit of the phase velocity, Eq. (3.75). From this equation, the time required for the wave to travel across 35 unit cells to layer 71 is $35d/(d/\Delta_d) = 433\Delta$, which is verified by Fig. 4.17.

The FFT of the impulse response, Fig. 4.18, clearly exhibits the pass bands and stop bands of the layered medium. The frequencies in this figure are scaled to the parameter Δ_d to allow comparison to the dispersion curves. Notice the agreement between the pass-band boundaries obtained by the two methods.

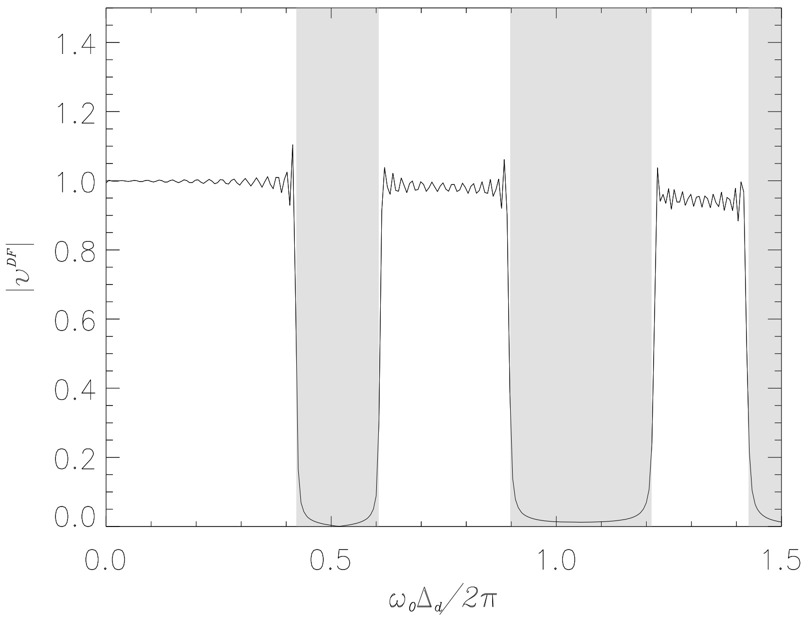


Figure 4.18: The impulse response. The shaded regions indicate the stop bands.

Narrow-band boundary condition To obtain a narrow band of frequencies, we subject the layered medium to a velocity boundary condition described by a finite series of sine waves of frequency $\omega\Delta_d/2\pi = 0.8$ (Fig. 4.19). The dominant portion of the frequency spectrum lies in the second pass band of the medium (Fig. 4.20). The values of the phase and group velocities obtained from Eq. (3.73) are $c_1\Delta/d = 0.0817$, $c_g\Delta/d = 0.0618$.

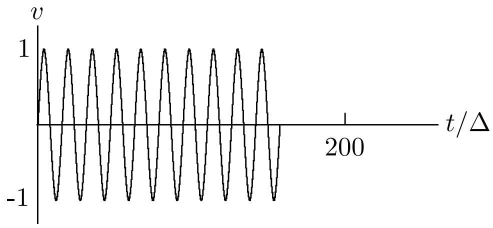


Figure 4.19: A narrow-band boundary condition—ten periods of a sine wave.

Figure 4.21 shows the velocity histories at the left boundary of layer 71 and at the left boundary of layer 75. The distance between these two locations is two unit cells, $2d$, so the time for the group wave to travel between them is $\delta t = 2d/c_g = 32.4\Delta$. Figure 4.22 shows the two velocity histories with the one for layer 75 shifted in time by $-\delta t$. This comparison clearly shows the group-wave envelope propagating at the group velocity c_g. The phase wave, propagating with the higher velocity c_1, is shifted in this comparison.

The method-of-characteristics solution provides another way to illustrate the contrast between the phase and group velocities. To see how, consider Figs. 2.33 through 2.36, where we present solutions of this type as tables of values. By replacing the smallest number in these tables by a white dot, the largest number by a black dot, and intermediate numbers by graduated shades of grey, we can transform them into "images" of the solutions.

To demonstrate this process, we use the velocity boundary condition in Fig. 4.19 extended to 30 periods. The resulting image is shown in Fig. 4.23. Time is the vertical coordinate, $0 \leq t \leq 1500\Delta$, and position is the horizontal coordinate, $0 \leq x \leq 750d/11$. (There are 11 sublayers in a unit cell.) The left ordinate of the image is the left boundary of the layered medium. The layered medium extends well beyond the right ordinate, so no reflections are present.

We have overlayed two solid parallel lines and a dashed line on the image. The slope of the solid lines equals the group velocity predicted by the characteristic equation for the frequency of the sine wave. The group wave is the broad

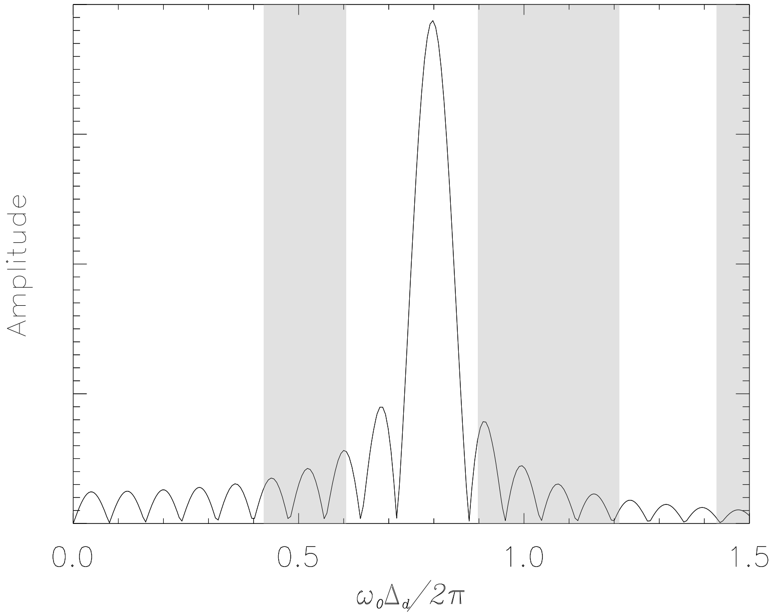


Figure 4.20: Frequency spectrum of the boundary condition superimposed on the pass bands (light areas) and stop bands (shaded areas) of the medium.

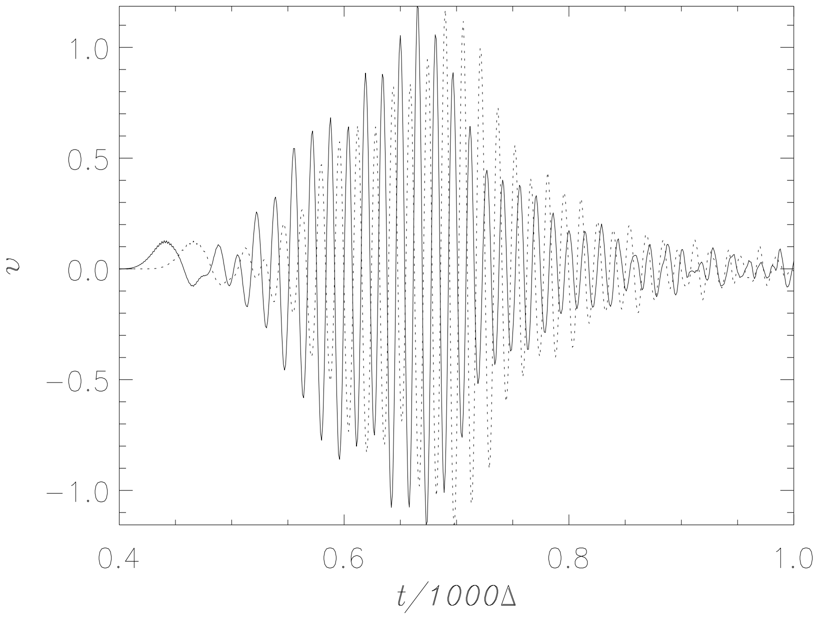


Figure 4.21: Velocity histories at layer 71 (—) and at layer 75 (···).

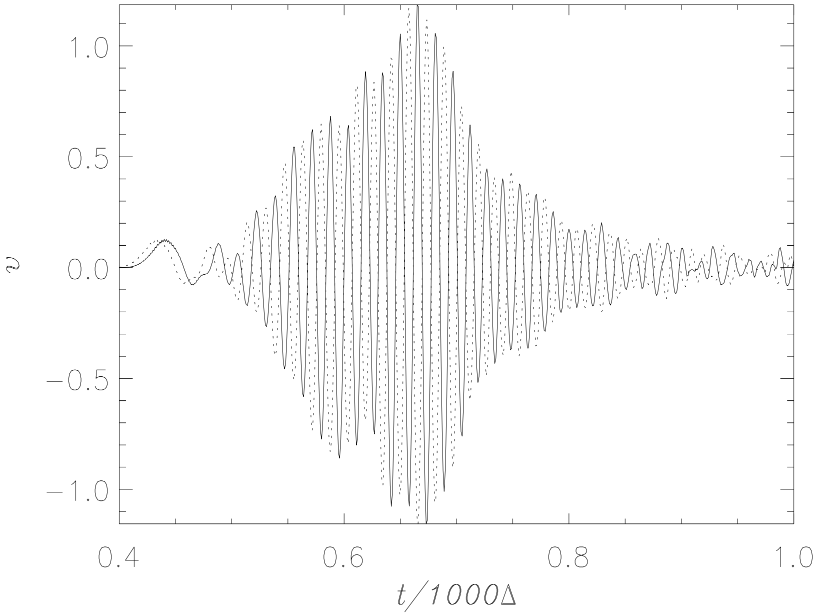


Figure 4.22: The time history at layer 71 (—) compared to the time-shifted history at layer 75 ($\cdots$).

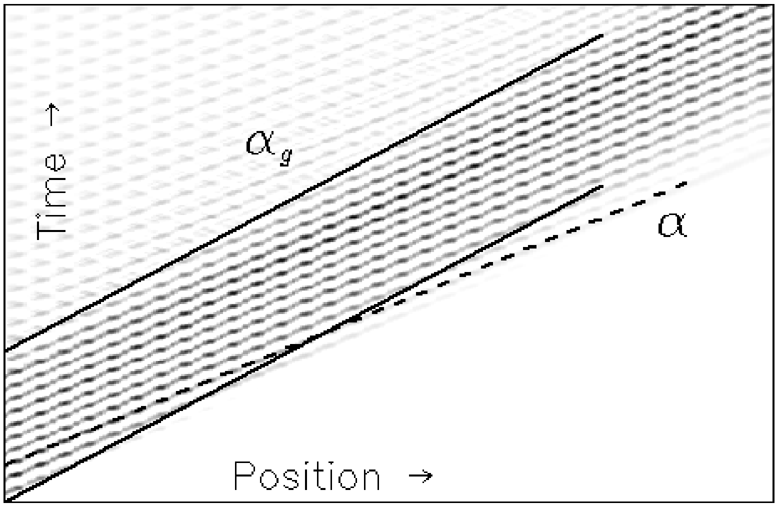


Figure 4.23: Image of the transient wave.

grey band contained between these lines. The slope of the dashed line equals the phase velocity. The propagation of the phase wave appears in the pattern as lines parallel to the dashed line. The phase wave travels faster then the group wave, appearing at the back edge of the group and disappearing at the front edge.

Exercises

Exercise 4.9 Show that the group velocity $c_g = d\omega/dk$ is related to the phase velocity c_1 by

$$c_g = c_1 + k_1 \frac{dc_1}{dk_1} = \frac{c_1}{1 - \dfrac{\omega}{c_1}\dfrac{dc_1}{d\omega}}.$$

Exercise 4.10 In Section 3.5, we analyzed acoustic waves in a channel.
(a) Show that the group velocity of such waves is related to the phase velocity by $c_g c_1 = \alpha^2$.
(b) What is the maximum value of the group velocity?

Answer: (b) α.

Exercise 4.11 Use Eq. (3.73) to derive an expression for the group velocity of a layered medium with periodic layers.

Exercise 4.12 Use the result of Exercise 4.11 to show that the group velocity is zero at the edges of a pass band.

4.5 Cagniard-de Hoop Method

This integral transform method has yielded solutions to many important transient problems in elastic wave propagation. We will demonstrate the procedure by obtaining the solution to a transient problem known as Lamb's problem.

Let us consider a half space of elastic material that is initially stationary. Let a cartesian coordinate system be oriented with the x_1-x_2 plane coincident with the boundary of the half space and the x_3 axis extending into the material. We assume that at $t = 0$ the boundary of the half space is subjected to the stress boundary condition

$$[T_{33}]_{x_3=0} = -T_0 H(t)\delta(x_1),$$

where T_0 is a constant, $H(t)$ is the step function, and $\delta(x_1)$ is a Dirac delta function (Fig. 4.24.a). Thus at $t = 0$ the half space is subjected to a uniform line load along the x_2 axis (Fig. 4.24.b).

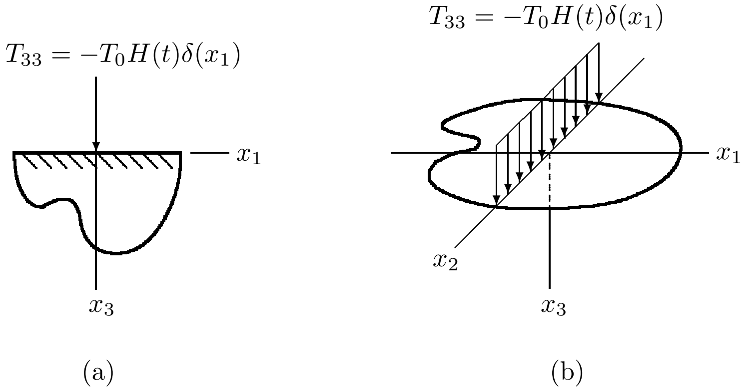


Figure 4.24: Elastic half space subjected to a uniform line load along the x_2 axis.

From the boundary condition we see that no motion occurs in the x_2 direction. The motion is described by the displacement field

$$\begin{aligned} u_1 &= u_1(x_1, x_3, t), \\ u_3 &= u_3(x_1, x_3, t). \end{aligned}$$

For this displacement field, the expressions for the stress components T_{33} and T_{13} in terms of displacements are

$$\begin{aligned} T_{33} &= \lambda \frac{\partial u_1}{\partial x_1} + (\lambda + 2\mu)\frac{\partial u_3}{\partial x_3}. \\ T_{13} &= \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right), \end{aligned} \tag{4.53}$$

In terms of the Helmholtz decomposition

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi},$$

the displacement components u_1 and u_3 are expressed by

$$u_1 = \frac{\partial\phi}{\partial x_1} - \frac{\partial\psi}{\partial x_3}, \qquad u_3 = \frac{\partial\phi}{\partial x_3} + \frac{\partial\psi}{\partial x_1}, \tag{4.54}$$

where ψ is the x_2 component of the vector potential $\boldsymbol{\psi}$. Using these expressions, we can write Eqs. (4.53) in terms of the potentials ϕ and ψ:

$$\begin{aligned} T_{13} &= \mu\left(2\frac{\partial^2\phi}{\partial x_1\partial x_3} + \frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_3^2}\right), \\ T_{33} &= \lambda\frac{\partial^2\phi}{\partial x_1^2} + (\lambda+2\mu)\frac{\partial^2\phi}{\partial x_3^2} + 2\mu\frac{\partial^2\psi}{\partial x_1\partial x_3}. \end{aligned} \tag{4.55}$$

The potentials ϕ and ψ are governed by the wave equations

$$\frac{\partial^2\phi}{\partial t^2} = \alpha^2\left(\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_3^2}\right), \qquad \frac{\partial^2\psi}{\partial t^2} = \beta^2\left(\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_3^2}\right). \tag{4.56}$$

By using Eqs. (4.55), we can write the stress boundary conditions in terms of ϕ and ψ:

$$\begin{aligned} \left[\lambda\frac{\partial^2\phi}{\partial x_1^2} + (\lambda+2\mu)\frac{\partial^2\phi}{\partial x_3^2} + 2\mu\frac{\partial^2\psi}{\partial x_1\partial x_3}\right]_{x_3=0} &= -T_0H(t)\delta(x_1), \\ \left[2\frac{\partial^2\phi}{\partial x_1\partial x_3} + \frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_3^2}\right]_{x_3=0} &= 0. \end{aligned} \tag{4.57}$$

The Laplace transforms of the potentials ϕ and ψ with respect to time are defined by

$$\phi^L = \int_0^\infty \phi e^{-st}\,dt, \qquad \psi^L = \int_0^\infty \psi e^{-st}\,dt.$$

We take the Laplace transforms of Eqs. (4.56),

$$s^2\phi^L = \alpha^2\left(\frac{\partial^2\phi^L}{\partial x_1^2} + \frac{\partial^2\phi^L}{\partial x_3^2}\right), \qquad s^2\psi^L = \beta^2\left(\frac{\partial^2\psi^L}{\partial x_1^2} + \frac{\partial^2\psi^L}{\partial x_3^2}\right), \tag{4.58}$$

and also take the Laplace transforms of the boundary conditions, eqs(4.57):

$$\begin{aligned} \left[\lambda\frac{\partial^2\phi^L}{\partial x_1^2} + (\lambda+2\mu)\frac{\partial^2\phi^L}{\partial x_3^2} + 2\mu\frac{\partial^2\psi^L}{\partial x_1\partial x_3}\right]_{x_3=0} &= -\frac{T_0\delta(x_1)}{s}, \\ \left[2\frac{\partial^2\phi^L}{\partial x_1\partial x_3} + \frac{\partial^2\psi^L}{\partial x_1^2} - \frac{\partial^2\psi^L}{\partial x_3^2}\right]_{x_3=0} &= 0. \end{aligned} \tag{4.59}$$

The next step is the key to the Cagniard-de Hoop method. We define Fourier transforms of ϕ^L and ψ^L with respect to x_1 by

$$\phi^{LF} = \int_{-\infty}^{\infty} \phi^L e^{-iksx_1}\, dx_1, \qquad \psi^{LF} = \int_{-\infty}^{\infty} \psi^L e^{-iksx_1}\, dx_1.$$

Notice that in place of the usual transform variable, we use the product of a parameter k and the Laplace transform variable s. The corresponding inversion integrals are

$$\phi^L = \frac{1}{2\pi}\int_{-\infty}^{\infty} s\phi^{LF} e^{iksx_1}\, dk, \qquad \psi^L = \frac{1}{2\pi}\int_{-\infty}^{\infty} s\psi^{LF} e^{iksx_1}\, dk. \tag{4.60}$$

Using these definitions, we take the Fourier transforms of Eqs. (4.58) and write the resulting equations in the forms

$$\begin{aligned} \frac{d^2\phi^{LF}}{dx_3^2} - (k^2 + v_P^2)s^2\phi^{LF} &= 0, \\ \frac{d^2\psi^{LF}}{dx_3^2} - (k^2 + v_S^2)s^2\psi^{LF} &= 0, \end{aligned} \tag{4.61}$$

where v_P and v_S are the compressional and shear *slownesses* defined by $v_P = 1/\alpha$, $v_S = 1/\beta$. We also take the Fourier transform of the boundary conditions, Eqs. (4.59), obtaining

$$\begin{aligned} \left[-k^2s^2\lambda\phi^{LF} + (\lambda + 2\mu)\frac{d^2\phi^{LF}}{dx_3^2} + 2\mu iks\frac{d\psi^{LF}}{dx_3}\right]_{x_3=0} &= -\frac{T_0}{s}, \\ \left[2iks\frac{d\phi^{LF}}{dx_3} - k^2s^2\psi^{LF} - \frac{d^2\psi^{LF}}{dx_3^2}\right]_{x_3=0} &= 0. \end{aligned} \tag{4.62}$$

The solutions of Eqs. (4.61) are

$$\begin{aligned} \phi^{LF} &= Ae^{-(k^2 + v_P^2)^{1/2}sx_3} + Be^{(k^2 + v_P^2)^{1/2}sx_3}, \\ \psi^{LF} &= Ce^{-(k^2 + v_S^2)^{1/2}sx_3} + De^{(k^2 + v_S^2)^{1/2}sx_3}. \end{aligned} \tag{4.63}$$

Substituting these expressions into the inversion integral for the Laplace transform, we can see that the terms containing A and C represent waves propagating in the positive x_3 direction and the terms containing B and D represent waves propagating in the negative x_3 direction. Because the boundary conditions do not give rise to waves propagating in the negative x_3 direction, we conclude that $B = 0$ and $D = 0$. Substituting Eqs. (4.63) into the boundary conditions, Eqs. (4.62), we obtain two equations we can solve for A and C:

$$A = \frac{-T_0(2k^2 + v_S^2)}{s^3\mu Q}, \qquad C = \frac{T_0 2ik(k^2 + v_P^2)^{1/2}}{s^3\mu Q},$$

where Q is defined by

$$Q = (2k^2 + v_S^2)^2 - 4k^2(k^2 + v_P^2)^{1/2}(k^2 + v_S^2)^{1/2}.$$

(In obtaining these expressions we have used the relation $(\lambda + 2\mu)v_P^2 = \mu v_S^2$.) Substituting these expressions for A and C into Eqs. (4.63), we obtain the solutions for the transformed potentials ϕ^{LF} and ψ^{LF}:

$$\begin{aligned}\phi^{LF} &= \frac{-T_0(2k^2 + v_S^2)}{s^3\mu Q} e^{-(k^2 + v_P^2)^{1/2} s x_3}, \\ \psi^{LF} &= \frac{T_0 2ik(k^2 + v_P^2)^{1/2}}{s^3\mu Q} e^{-(k^2 + v_S^2)^{1/2} s x_3}.\end{aligned}$$

We substitute these expressions into the Fourier inversion integrals, Eqs. (4.60), to obtain the solutions for the transformed potentials ϕ^L and ψ^L:

$$\begin{aligned}\phi^L &= - \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{T_0(2k^2 + v_S^2)}{s^2\mu Q} e^{-[(k^2 + v_P^2)^{1/2} x_3 - ikx_1]s}\, dk, \\ \psi^L &= \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{T_0 2ik(k^2 + v_P^2)^{1/2}}{s^2\mu Q} e^{-[(k^2 + v_S^2)^{1/2} x_3 - ikx_1]s}\, dk.\end{aligned}$$

By using these two results and Eq. (4.55), we obtain the solution for the Laplace transform of the stress component T_{33} in the form of the sum of two terms:

$$T_{33}^L = (T_{33}^L)_1 + (T_{33}^L)_2 = \int_{-\infty}^{\infty} g_1(k)\, dk + \int_{-\infty}^{\infty} g_2(k)\, dk, \tag{4.64}$$

where

$$\begin{aligned}g_1(k) &= -\frac{T_0(2k^2 + v_S^2)^2}{2\pi Q} e^{-[(k^2 + v_P^2)^{1/2} x_3 - ikx_1]s}, \\ g_2(k) &= \frac{2T_0 k^2 (k^2 + v_S^2)^{1/2}(k^2 + v_P^2)^{1/2}}{\pi Q} e^{-[(k^2 + v_S^2)^{1/2} x_3 - ikx_1]s}.\end{aligned}$$

Our objective is to determine the stress component T_{33} by inverting the Laplace transforms $(T_{33}^L)_1$ and $(T_{33}^L)_2$. (The other components of stress can be determined by procedures similar to the one we use to determine T_{33}.) These terms are expressed in terms of integrals along the real axis of the complex k-plane. The Cagniard-de Hoop method involves changing the integration contour in the complex k-plane in such a way that *the integrals assume the form of the definition of the Laplace transform.* By doing so, the terms $(T_{33})_1$ and $(T_{33})_2$ can be determined directly.

Evaluation of $(T_{33})_1$

To make the integral $(T_{33}^L)_1$ assume the form of the definition of the Laplace transform, we define a real parameter t by the expression

$$t = (k^2 + v_P^2)^{1/2} x_3 - ikx_1.$$

We can write this equation in the form

$$(x_1^2 + x_3^2)k^2 - 2itx_1 k + v_P^2 x_3^2 - t^2 = 0. \tag{4.65}$$

We introduce polar coordinates r, θ in the x_1-x_3 plane (Fig. 4.25), so that

$$x_1 = r\cos\theta, \quad x_3 = r\sin\theta.$$

In terms of r and θ, we can write Eq. (4.65) as

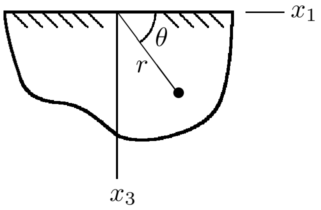


Figure 4.25: Polar coordinates r, θ in the x_1-x_3 plane.

$$r^2k^2 - 2itrk\cos\theta + v_P^2 r^2 \sin^2\theta - t^2 = 0.$$

The solutions of this equation for k are

$$k_\pm = \frac{it}{r}\cos\theta \pm \left(\frac{t^2}{r^2} - v_P^2\right)^{1/2} \sin\theta. \tag{4.66}$$

As the value of the parameter t goes from $v_P r$ to ∞, the values of k given by this equation describe the two paths in the complex k-plane shown in Fig. 4.26, starting from the point $(v_P \cos\theta)i$ and approaching the asymptotes shown in the figure.

Consider the closed contour in the complex k-plane shown in Fig. 4.27. The solution k_+ describes the path labeled B_+ and the solution k_- describes the path labeled B_-. The contour consists of a straight portion C_1 along the real axis from $-R$ to R, closed by two circular arcs C_2 and C_3 and two portions along the contours B_+ and B_-. Our goal is to express the integral $(T_{33}^L)_1$ in terms of integrals along the contours B_+ and B_- by applying the Cauchy

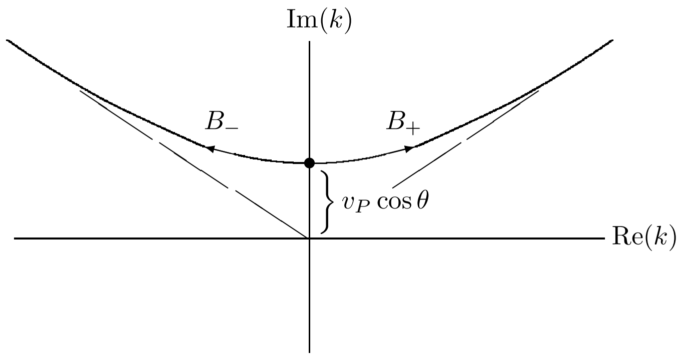


Figure 4.26: The two paths B_+ and B_- in the complex k-plane.

integral theorem to the closed contour. Before we can apply the Cauchy integral theorem, we must determine the poles and branch points of $g_1(k)$.

The function $g_1(k)$ has branch points at $k = \pm v_P i$ and at $k = \pm v_S i$. The only poles of the integrand are the zeros of the function Q. If we set $k = i/c_R$ in the function Q, the equation $Q = 0$ becomes identical to the Rayleigh characteristic equation, Eq. (3.50). Therefore two of the zeros of Q are $k = \pm i/c_R$, where c_R is the Rayleigh wave velocity. We define the Rayleigh wave slowness by $v_R = 1/c_R$ and write these zeros of Q as $k = \pm v_R i$. These poles are not within the closed contour. We show these branch points and poles of $g_1(k)$ in Fig. 4.28, and also show suitable branch cuts to make $g_1(k)$ single-valued. The branch points and poles shown in Fig. 4.28 are not within the closed contour shown in Fig. 4.27, and the contour does not cross the branch cuts. Therefore, we can apply the Cauchy integral theorem to the closed contour:

$$\begin{aligned} &\int_{C_1} g_1(k)\,dk + \int_{C_2} g_1(k)\,dk + \int_{C_3} g_1(k)\,dk \\ &-\int_{B_+} g_1(k)\,dk + \int_{B_-} g_1(k)\,dk = 0. \end{aligned} \tag{4.67}$$

The reason for the minus sign on the integral along B_+ is that the integral is evaluated in the direction shown in Fig. 4.27. The integral $(T_{33}^L)_1$ is

$$(T_{33}^L)_1 = \lim_{R\to\infty} \int_{C_1} g_1(k)\,dk.$$

The integrals along the circular contours C_2 and C_3 vanish as $R \to \infty$. Therefore

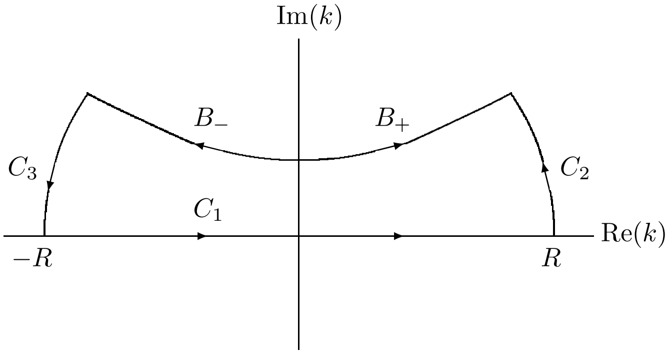


Figure 4.27: Closed contour in the complex k-plane.

Eq. (4.67) states that

$$(T_{33}^L)_1 = \lim_{R\to\infty}\left[\int_{B_+} g_1(k)\,dk - \int_{B_-} g_1(k)\,dk\right].$$

By using the Cauchy integral theorem, we have expressed the integral $(T_{33}^L)_1$ in terms of integrals along the contours shown in Fig. 4.26. We express these integrals in terms of the parameter t by making a change of variable:

$$(T_{33}^L)_1 = \int_{v_P r}^{\infty} g_1(k_+)\frac{\partial k_+}{\partial t}\,dt - \int_{v_P r}^{\infty} g_1(k_-)\frac{\partial k_-}{\partial t}\,dt.$$

Recall that k_+ and k_- are the values of k on B_+ and B_-. By introducing a step function $H(t - v_P r)$, we can change the lower limits of the integrals to zero:

$$(T_{33}^L)_1 = \int_{0}^{\infty} H(t - v_P r) g_1(k_+)\frac{\partial k_+}{\partial t}\,dt - \int_{0}^{\infty} H(t - v_P r) g_1(k_-)\frac{\partial k_-}{\partial t}\,dt.$$

We substitute the explicit expressions for the integrands $g_1(k_+)$ and $g_1(k_-)$ into this equation, obtaining the result

$$\begin{aligned}(T_{33}^L)_1 \quad &= \quad -\frac{T_0}{2\pi}\int_0^\infty H(t - v_P r)\frac{(2k_+^2 + v_S^2)^2}{Q_+}\frac{\partial k_+}{\partial t}e^{-st}\,dt \\ &\quad +\frac{T_0}{2\pi}\int_0^\infty H(t - v_P r)\frac{(2k_-^2 + v_S^2)^2}{Q_-}\frac{\partial k_-}{\partial t}e^{-st}\,dt,\end{aligned} \tag{4.68}$$

where Q_+ and Q_- indicate that the function Q is expressed in terms of k_+ and k_-.

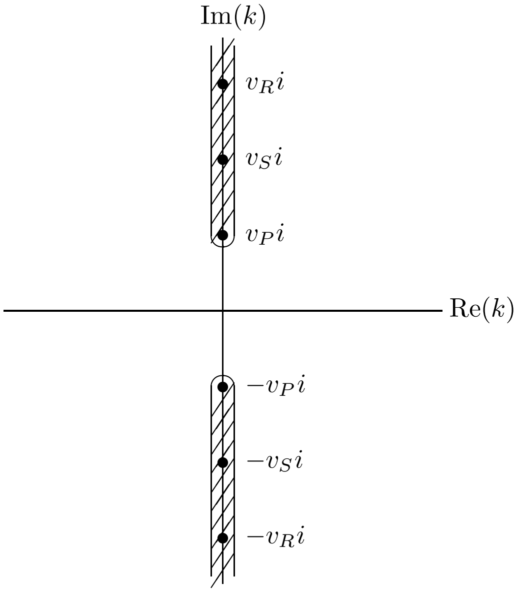


Figure 4.28: Poles and branch points of the function $g_1(k)$ and branch cuts to make it single-valued.

This completes the application of the Cagniard-de Hoop method to the term $(T^L_{33})_1$. The terms on the right of Eq. (4.68) have the form of the definition of the Laplace transform: the transform variable s does not appear anywhere except in the exponential term. Therefore we conclude that the term $(T_{33})_1$ is

$$(T_{33})_1 = \frac{T_0}{2\pi} H(t - v_P r) \left\{ -\frac{(2k_+^2 + v_S^2)^2}{Q_+} \frac{\partial k_+}{\partial t} + \frac{(2k_-^2 + v_S^2)^2}{Q_-} \frac{\partial k_-}{\partial t} \right\}. \qquad (4.69)$$

Because k_+ and k_- are known as functions of position and time, Eq. (4.66), this is a closed-form solution for the term $(T_{33})_1$.

Evaluation of $(T_{33})_2$

To make the integral $(T^L_{33})_2$ in Eq. (4.64) assume the form of the definition of the Laplace transform, we define the real parameter t by

$$t = (k^2 + v_S^2)^{1/2} x_3 - ikx_1.$$

In terms of the polar coordinates r, θ, the solutions of this equation for k are

$$\bar{k}_\pm = \frac{it}{r} \cos\theta \pm \left(\frac{t^2}{r^2} - v_S^2 \right)^{1/2} \sin\theta. \qquad (4.70)$$

As the value of the parameter t goes from $v_S r$ to ∞, the values of k given by this equation describe the two paths in the complex k-plane shown in Fig. 4.29, starting from the point $(v_S \cos\theta)i$ and approaching the asymptotes shown in the figure. The solution $\bar{k}_+$ describes the path labeled B_+ and the solution $\bar{k}_-$ describes the path labeled B_-.

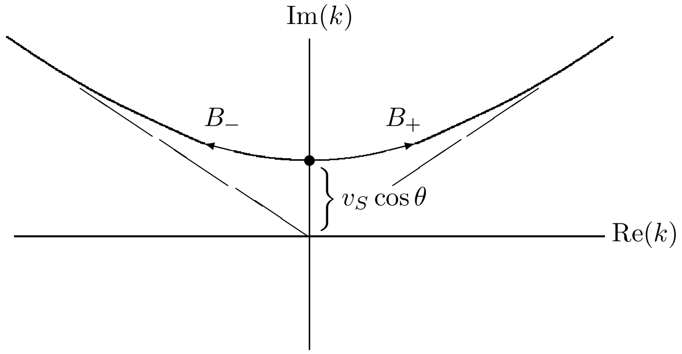


Figure 4.29: The two paths B_+ and B_- in the complex k-plane.

The branch points and poles of $g_2(k)$ are identical to the branch points and poles of the integrand $g_1(k)$ of T_{33}^{1L}. They are shown in Fig. 4.28 together with branch cuts that make $g_2(k)$ single valued. In this case, when we construct the closed contour in the k-plane along the contours B_+ and B_-, the contour intersects the imaginary axis above the branch point $k = v_P i$ when $v_S \cos\theta > v_P$ (Fig 4.30). We therefore modify the closed contour so that it goes around the branch cut by introducing two branch-line contours L_+ and L_-.

As the value of the parameter t goes from $v_S r$ to ∞, the functions $\bar{k}_\pm$ given by Eq. (4.70) describe the paths B_+ and B_- in the k-plane. We can also use the function $\bar{k}_-$ to describe the branch-line contours L_+ and L_-. We write $\bar{k}_-$ in the form

$$\bar{k}_- = \frac{it}{r}\cos\theta - i\left(v_S^2 - \frac{t^2}{r^2}\right)^{1/2}\sin\theta.$$

When $t = v_S r$, $\bar{k}_- = (v_S \cos\theta)i$. As t decreases below the value $v_S r$, we can see that the value of $\bar{k}_-$ moves downward along the imaginary axis, that is, it moves along the branch-line contours L_+ and L_-. To determine the value of t for which $\bar{k}_-$ reaches the bottom of the branch-line contours, we set $\bar{k}_- = v_P i$

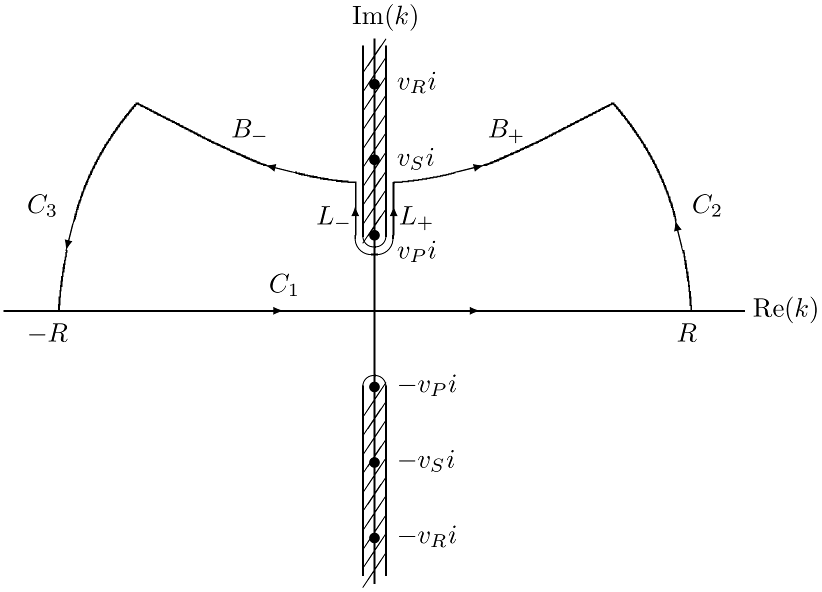


Figure 4.30: Closed contour in the complex k-plane that goes around the branch cut.

and solve for t. We denote the result by $\bar{t}$:

$$\bar{t} = r[v_P \cos\theta + (v_S^2 - v_P^2)^{1/2} \sin\theta]. \tag{4.71}$$

Thus as the value of the parameter t goes from $\bar{t}$ to $v_S r$, the function $\bar{k}_-$ describes the paths L_+ and L_-.

We can apply the Cauchy integral theorem to the closed contour shown in Fig. 4.30:

$$\begin{aligned} &\int_{C_1} g_2(k)\,dk + \int_{C_2} g_2(k)\,dk + \int_{C_3} g_2(k)\,dk \\ &-\int_{B_+} g_2(k)\,dk + \int_{B_-} g_2(k)\,dk \\ &-\int_{L_+} g_2(k)\,dk + \int_{L_-} g_2(k)\,dk = 0. \end{aligned} \tag{4.72}$$

The reason for the minus sign on the integrals along B_+ and L_+ is that we evaluate the integrals in the directions shown in Fig. 4.30. The integral $(T_{33}^L)_2$ is

$$(T_{33}^L)_2 = \lim_{R\to\infty} \int_{C_1} g_2(k)\,dk.$$

The integrals along the circular contours C_2 and C_3 vanish as $R \to \infty$. Therefore

Eq. (4.72) states that

$$(T_{33}^{L})_2 = \lim_{R\to\infty}\left[\int_{B_+} g_2(k)\,dk - \int_{B_-} g_2(k)\,dk + \int_{L_+} g_2(k)\,dk - \int_{L_-} g_2(k)\,dk\right].$$

We can express these integrals in terms of the parameter t by making a change of variable:

$$\begin{aligned}(T_{33}^{L})_2 \quad &= \quad \int_{v_S r}^{\infty} g_2(\bar{k}_+)\frac{\partial \bar{k}_+}{\partial t}\,dt - \int_{v_S r}^{\infty} g_2(\bar{k}_-)\frac{\partial \bar{k}_-}{\partial t}\,dt \\ &\quad + \int_{\bar{t}}^{v_S r}[g_2(\bar{k}_-)]_{L_+}\frac{\partial \bar{k}_-}{\partial t}\,dt - \int_{\bar{t}}^{v_S r}[g_2(\bar{k}_-)]_{L_-}\frac{\partial \bar{k}_-}{\partial t}\,dt.\end{aligned}$$

The subscripts L_+ and L_- mean that the function $g_2(\bar{k}_-)$ is to be evaluated on the right and left sides of the branch cut. By introducing appropriate step functions, we can make the limits of the integrals be from 0 to infinity:

$$\begin{aligned}(T_{33}^{L})_2 \quad &= \quad \int_0^{\infty} H(t - v_S r)g_2(\bar{k}_+)\frac{\partial \bar{k}_+}{\partial t}\,dt - \int_0^{\infty} H(t - v_S r)g_2(\bar{k}_-)\frac{\partial \bar{k}_-}{\partial t}\,dt \\ &\quad + \int_0^{\infty} H(v_s\cos\theta - v_P)H(t-\bar{t})H(v_S r - t)[g_2(\bar{k}_-)]_{L_+}\frac{\partial \bar{k}_-}{\partial t}\,dt \\ &\quad - \int_0^{\infty} H(v_s\cos\theta - v_P)H(t-\bar{t})H(v_S r - t)[g_2(\bar{k}_-)]_{L_-}\frac{\partial \bar{k}_-}{\partial t}\,dt.\end{aligned}$$

We include the step function $H(v_s\cos\theta - v_P)$ in the last two terms because the branch-line contours exist only when $v_s\cos\theta > v_P$. We substitute the explicit expressions for the integrands $g_2(\bar{k}_+)$ and $g_2(\bar{k}_-)$ into this equation, obtaining the result

$$\begin{aligned}(T_{33}^{L})_2 = &\frac{2T_0}{\pi}\int_0^{\infty} H(t - v_S r)\frac{\bar{k}_+^2(\bar{k}_+^2 + v_S^2)^{1/2}(\bar{k}_+^2 + v_P^2)^{1/2}}{\bar{Q}_+}\frac{\partial \bar{k}_+}{\partial t}e^{-st}\,dt \\ &-\frac{2T_0}{\pi}\int_0^{\infty} H(t - v_S r)\frac{\bar{k}_-^2(\bar{k}_-^2 + v_S^2)^{1/2}(\bar{k}_-^2 + v_P^2)^{1/2}}{\bar{Q}_-}\frac{\partial \bar{k}_-}{\partial t}e^{-st}\,dt \\ &+\frac{2T_0}{\pi}\int_0^{\infty} H(v_s\cos\theta - v_P)H(t-\bar{t})H(v_S r - t)\left[\frac{\bar{k}_-^2(\bar{k}_-^2 + v_S^2)^{1/2}(\bar{k}_-^2 + v_P^2)^{1/2}}{\bar{Q}_-}\frac{\partial \bar{k}_-}{\partial t}\right]_{L_+} e^{-st}\,dt \\ &-\frac{2T_0}{\pi}\int_0^{\infty} H(v_s\cos\theta - v_P)H(t-\bar{t})H(v_S r - t)\left[\frac{\bar{k}_-^2(\bar{k}_-^2 + v_S^2)^{1/2}(\bar{k}_-^2 + v_P^2)^{1/2}}{\bar{Q}_-}\frac{\partial \bar{k}_-}{\partial t}\right]_{L_-} e^{-st}\,dt,\end{aligned}$$

where $\bar{Q}_+$ and $\bar{Q}_-$ indicate that the function Q is expressed in terms of $\bar{k}_+$ and $\bar{k}_-$.

This completes the application of the Cagniard-de Hoop method to the term $(T_{33}^L)_2$. The term $(T_{33})_2$ is

$$\begin{aligned}(T_{33})_2 = {} & \frac{2T_0}{\pi}H(t-v_Sr)\left\{\frac{\bar{k}_+^2(\bar{k}_+^2+v_S^2)^{1/2}(\bar{k}_+^2+v_P^2)^{1/2}}{\bar{Q}_+}\frac{\partial\bar{k}_+}{\partial t}\right.\\ & \left.-\frac{\bar{k}_-^2(\bar{k}_-^2+v_S^2)^{1/2}(\bar{k}_-^2+v_P^2)^{1/2}}{\bar{Q}_-}\frac{\partial\bar{k}_-}{\partial t}\right\}\\ & +\frac{2T_0}{\pi}H(v_s\cos\theta-v_P)H(t-\bar{t})H(v_Sr-t)\left\{\left[\frac{\bar{k}_-^2(\bar{k}_-^2+v_S^2)^{1/2}(\bar{k}_-^2+v_P^2)^{1/2}}{\bar{Q}_-}\frac{\partial\bar{k}_-}{\partial t}\right]_{L_+}\right.\\ & \left.-\left[\frac{\bar{k}_-^2(\bar{k}_-^2+v_S^2)^{1/2}(\bar{k}_-^2+v_P^2)^{1/2}}{\bar{Q}_-}\frac{\partial\bar{k}_-}{\partial t}\right]_{L_-}\right\}.\end{aligned} \tag{4.73}$$

Solution for T_{33}

From Eqs. (4.69) and (4.73), the closed-form solution for the stress component T_{33} is

$$\begin{aligned}T_{33} & = (T_{33})_1+(T_{33})_2\\ & = \frac{T_0}{2\pi}H(t-v_Pr)\left\{-\frac{(2k_+^2+v_S^2)^2}{Q_+}\frac{\partial k_+}{\partial t}+\frac{(2k_-^2+v_S^2)^2}{Q_-}\frac{\partial k_-}{\partial t}\right\}\\ & \quad +\frac{2T_0}{\pi}H(t-v_Sr)\left\{\frac{\bar{k}_+^2(\bar{k}_+^2+v_S^2)^{1/2}(\bar{k}_+^2+v_P^2)^{1/2}}{\bar{Q}_+}\frac{\partial\bar{k}_+}{\partial t}\right.\\ & \quad \left.-\frac{\bar{k}_-^2(\bar{k}_-^2+v_S^2)^{1/2}(\bar{k}_-^2+v_P^2)^{1/2}}{\bar{Q}_-}\frac{\partial\bar{k}_-}{\partial t}\right\}\\ & +\frac{2T_0}{\pi}H(v_s\cos\theta-v_P)H(t-\bar{t})H(v_Sr-t)\left\{\left[\frac{\bar{k}_-^2(\bar{k}_-^2+v_S^2)^{1/2}(\bar{k}_-^2+v_P^2)^{1/2}}{\bar{Q}_-}\frac{\partial\bar{k}_-}{\partial t}\right]_{L_+}\right.\\ & \quad \left.-\left[\frac{\bar{k}_-^2(\bar{k}_-^2+v_S^2)^{1/2}(\bar{k}_-^2+v_P^2)^{1/2}}{\bar{Q}_-}\frac{\partial\bar{k}_-}{\partial t}\right]_{L_-}\right\}.\end{aligned}$$

The first term in the solution (the term containing the step function $H(t-v_Pr)$) is a disturbance that propagates with the compressional wave velocity α (Fig. 4.31). The second term (the term containing the step function $H(t-v_Sr)$) is a disturbance that propagates with the shear wave speed β. The wavefront that moves with the compressional wave speed causes a secondary disturbance at the surface that propagates with the shear wave speed, shown as the cross hatched area shown in Fig. 4.31. This disturbance is the third term of the solution, which originates from the branch-line integrals.

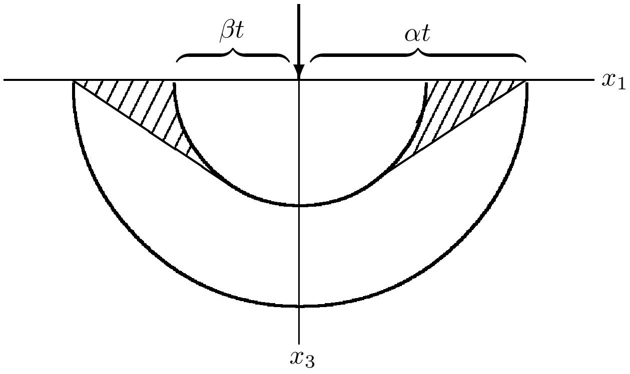


Figure 4.31: The disturbances traveling at the compressional and shear wave velocities.

Figure 4.32 provides insight into the secondary disturbance generated at the surface. We plot the normal stress at the surface as a function of $r/\alpha t$ assuming that the Poisson's ratio $\nu = 0.3$. The term $(T_{33})_1$ is the normal stress due to the disturbance that propagates with the compressional wave speed. This term does not satisfy the boundary condition $[T_{33}]_{x_3=0} = 0$ in the region $\beta t < r < \alpha t$. The term $(T_{33})_2$ is the normal stress due to the secondary disturbance, the cross-hatched area in Fig. 4.31. The sum of these two normal stresses does satisfy the boundary condition. We saw in Chapter 3 that a compressional wave alone cannot satisfy the conditions at a free boundary of an elastic material. The situation here is analogous: the secondary disturbance is "necessary" for the boundary condition to be satisfied.

Figure 4.33 shows a plot of the normal stress T_{33} as a function of $r/\alpha t$ for several values of the angle θ. When the Poisson's ratio $\nu = 0.3$, the front of the disturbance propagating with the shear wave speed is at $r/\alpha t = 0.534$. The front of the secondary disturbance generated at the surface can be seen on the curve for $\theta = 30°$.

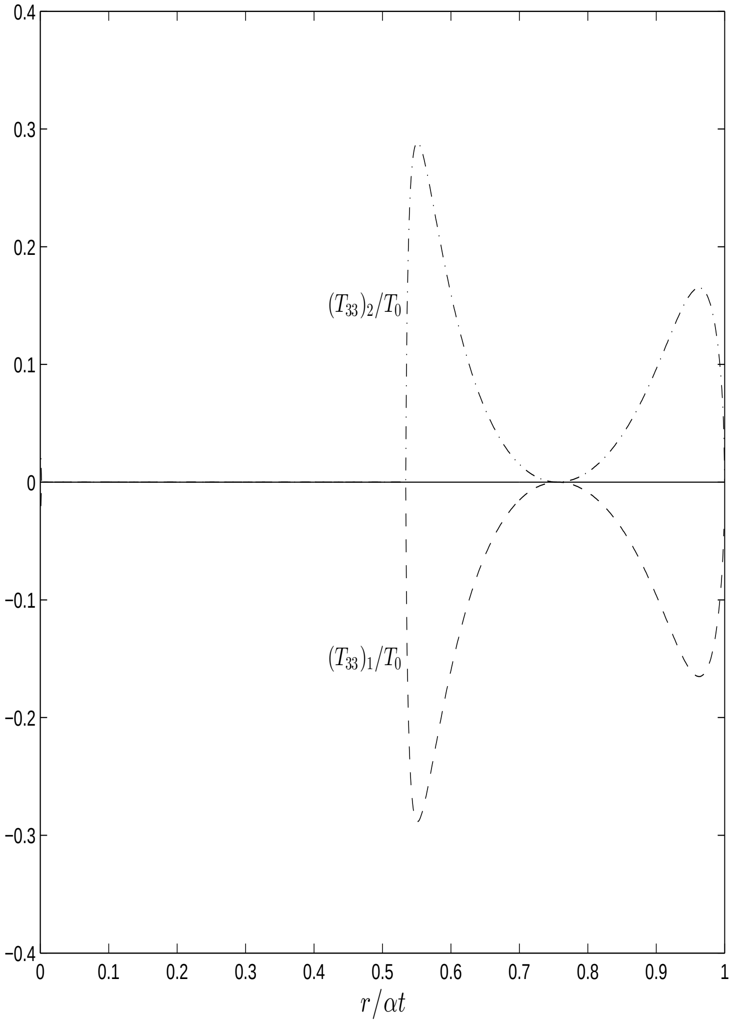


Figure 4.32: The normal stresses at the boundary for Poisson's ratio $\nu = 0.3$.

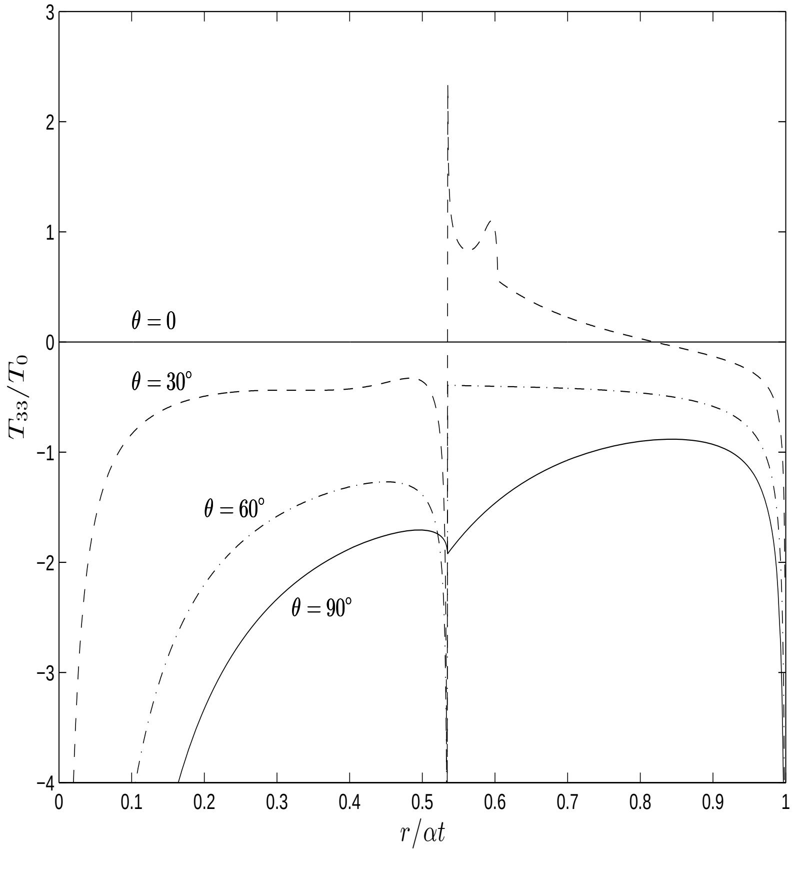


Figure 4.33: The normal stress T_{33} for Poisson's ratio $\nu = 0.3$.

Exercise

Exercise 4.13 Consider Lamb's problem for an acoustic medium. (See the discussion of an acoustic medium on page 44.) Use the Cagniard-de Hoop method to show that the solution for the stress component T_{33} is

$$T_{33} = -\frac{T_0}{\pi} H\left(t - \frac{r}{\alpha}\right) \frac{\sin\theta}{r\left(1 - \dfrac{r^2}{t^2\alpha^2}\right)^{1/2}}.$$

Chapter 5

Nonlinear Wave Propagation

To obtain the equations governing a linear elastic material, we assumed that the stresses in the material were linear functions of the strains and that derivatives of the displacement were sufficiently small that higher-order terms could be neglected. In *large-amplitude waves,* such as those created by high-velocity impacts or explosions, these assumptions are violated, and the equations governing the behavior of the material are nonlinear. In this chapter we discuss the theory of nonlinear wave propagation and apply it to a nonlinear elastic material. We derive the equations of one-dimensional nonlinear elasticity and discuss the theory of characteristics for systems of first-order hyperbolic equations in two independent variables. We give a brief introduction to the theory of singular surfaces and discuss the properties of shock and acceleration waves in elastic materials.

Materials subjected to large-amplitude waves do not usually behave elastically. The stress generally depends upon the temperature and the rate of change of the strain in addition to the strain. Nevertheless, we can demonstrate some of the qualitative features of waves of this type, and methods used to analyze them, using a nonlinear elastic material as an example. The concluding discussion of the steady compressional waves and release waves observed in experimental studies of large amplitude waves will not be limited to elastic materials.

5.1 One-Dimensional Nonlinear Elasticity

Here we summarize the theory of one-dimensional nonlinear elasticity. We first show how the motion and strain of the material are described, then obtain the equations of motion from the postulates for conservation of mass and balance of linear momentum without making approximations or assumptions of linearity.

Motion

To describe the one-dimensional motion of a material, we express the position of a point of the material as a function of the time and its position in the reference state:

$$x = \hat{x}(X, t).$$

This equation gives the position at time t of the point of the material that was at position X in the reference state (Fig. 5.1).

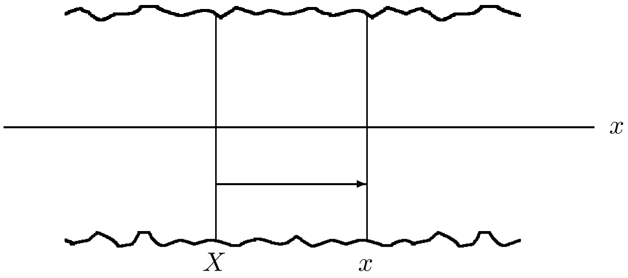


Figure 5.1: The position X of the material in the reference state and the position x at time t.

The displacement and velocity fields are defined by

$$u = \hat{x}(X, t) - X,$$

$$v = \frac{\partial}{\partial t}\hat{x}(X, t).$$

Deformation gradient and longitudinal strain

Consider an element of material that has width dX in the reference state (Fig. 5.2). The width of this element at time t is

$$dx = \hat{x}(X + dX, t) - \hat{x}(X, t) = F\,dX, \tag{5.1}$$

where the term

$$F = \frac{\partial \hat{x}}{\partial X} \tag{5.2}$$

is called the *deformation gradient*.

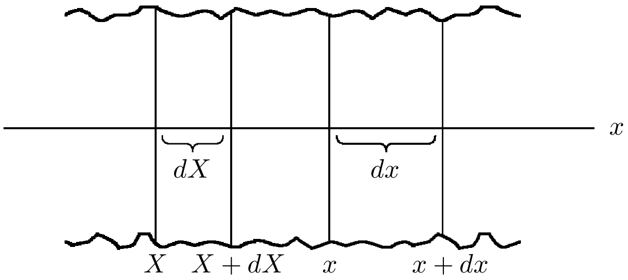


Figure 5.2: An element of the material in the reference state and at time t.

In this chapter we define the longitudinal strain of the material by

$$\varepsilon = -\left(\frac{dx - dX}{dX}\right) = 1 - F. \tag{5.3}$$

This definition, commonly used in studies of nonlinear waves in solids, is the negative of the traditional definition of the strain. It is positive when the length of the element dX decreases. This definition is used because in experimental studies of nonlinear waves in solids the material is most often compressed, so it is convenient to use a definition of strain that is positive when the material is in compression. We see from Eq. (5.3) that the deformation gradient F can also be used as a measure of the longitudinal strain.

Conservation of mass

From Fig. 5.2, we see that conservation of mass requires that

$$\rho_0 dX = \rho\, dx,$$

where ρ is the density of the material and ρ_0 is the density in the reference state. From Eq. (5.1), the deformation gradient is related to the density by

$$F = \frac{\rho_0}{\rho}.$$

We take the partial derivative of this equation with respect to time, obtaining the expression

$$\frac{\partial \hat{v}}{\partial X} = -\frac{\rho_0}{\rho^2}\frac{\partial \hat{\rho}}{\partial t}, \tag{5.4}$$

where we use a hat $\hat{}$ to indicate that a variable is expressed in terms of the material, or Lagrangian, variables X, t. Using the chain rule, we can write the left side of this equation as

$$\frac{\partial \hat{v}}{\partial X} = \frac{\partial \hat{x}}{\partial X}\frac{\partial v}{\partial x} = F\frac{\partial v}{\partial x} = \frac{\rho_0}{\rho}\frac{\partial v}{\partial x}. \tag{5.5}$$

We can also write the partial derivative of $\hat{\rho}(X,t)$ with respect to time as

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}v,$$

where ρ is the spatial, or Eulerian representation for the density. Using this expression and Eq. (5.5), we can write Eq. (5.4) as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \tag{5.6}$$

which is the equation of conservation of mass for the material, Eq. (1.46).

Balance of linear momentum

Figure 5.3 shows the free-body diagram at time t of an element of material that has width dX in the reference state. $\hat{T}$ is the normal stress.

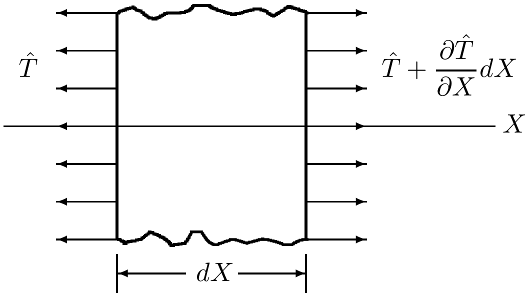


Figure 5.3: Free-body diagram of an element of material.

Newton's second law for the element is

$$\rho_0\,dX\frac{\partial \hat{v}}{\partial t} = \left(\hat{T} + \frac{\partial \hat{T}}{\partial X}\,dX\right) - \hat{T}.$$

Thus we obtain the equation of balance of linear momentum for the material in Lagrangian form:

$$\rho_0 \frac{\partial \hat{v}}{\partial t} = \frac{\partial \hat{T}}{\partial X}. \tag{5.7}$$

Applying the chain rule to the right side of this equation,

$$\frac{\partial \hat{T}}{\partial X} = \frac{\partial \hat{x}}{\partial X}\frac{\partial T}{\partial x} = \frac{\rho_0}{\rho}\frac{\partial T}{\partial x},$$

and using the result

$$\frac{\partial \hat{v}}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}v,$$

we obtain the equation of balance of linear momentum in the spatial, or Eulerian form:

$$\rho\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}v\right) = \frac{\partial T}{\partial x}. \tag{5.8}$$

Stress-strain relation

To obtain a model for a nonlinear elastic material, we assume that the stress T is a function of the deformation gradient:

$$T = \tilde{T}(F). \tag{5.9}$$

Lagrangian and Eulerian formulations

The material, or Lagrangian, form of the equation of balance of linear momentum is Eq. (5.7),

$$\rho_0 \frac{\partial \hat{v}}{\partial t} = \frac{\partial \hat{T}}{\partial X}.$$

Using Eq. (5.9), we can write the equation of balance of linear momentum as

$$\rho_0 \frac{\partial \hat{v}}{\partial t} = \tilde{T}_F \frac{\partial F}{\partial X}, \tag{5.10}$$

where $\tilde{T}_F = d\tilde{T}/dF$. From the definition of the deformation gradient F, Eq. (5.2), we see that

$$\frac{\partial F}{\partial t} = \frac{\partial \hat{v}}{\partial X}. \tag{5.11}$$

This completes the Lagrangian formulation of one-dimensional nonlinear elasticity. Equations (5.10) and (5.11) are two first-order differential equations in the two dependent variables $\hat{v}$ and F. We can write these equations in the matrix form

$$\begin{bmatrix} \rho_0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{v} \\ F \end{bmatrix}_t + \begin{bmatrix} 0 & -\tilde{T}_F \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \hat{v} \\ F \end{bmatrix}_X = 0, \tag{5.12}$$

where the subscripts t and X denote partial derivatives.

The spatial, or Eulerian form of the equation of balance of linear momentum, Eq. (5.8), can be written

$$\rho\left(\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}v\right)=-\frac{\partial p}{\partial x}, \tag{5.13}$$

where $p=-T$ is the pressure. For an elastic material, the pressure is a function of the longitudinal strain:

$$p=\tilde{p}(\varepsilon), \tag{5.14}$$

where

$$\varepsilon=1-\frac{\rho_0}{\rho}. \tag{5.15}$$

The motion of the material is also governed by the equation of conservation of mass, Eq. (5.6):

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0. \tag{5.16}$$

This completes the Eulerian formulation of one-dimensional nonlinear elasticity. Equations (5.13), (5.14), (5.15), and (5.16) are four equations in the four dependent variables v, p, ε, and ρ.

5.2 Hyperbolic Systems and Characteristics

In Section 5.1 we showed that the equations governing one-dimensional nonlinear elasticity can be written as a system of first-order differential equations in two independent variables, Eq. (5.12). Let us consider a generalization of that system of equations:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}_t+\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}_x=0,$$

where the dependent variables $u_1, u_2, \ldots, u_n$ are functions of two independent variables x, t. The subscripts t and x denote partial derivatives. We assume that the coefficients $a_{11}, a_{12}, \ldots, a_{nn}$ and $b_{11}, b_{12}, \ldots, b_{nn}$ may be functions of the dependent variables, but do not depend on derivatives of the dependent variables. We write this system of equations as

$$\mathbf{A}\mathbf{u}_t+\mathbf{B}\mathbf{u}_x=0. \tag{5.17}$$

Characteristics

Suppose that $\mathbf{u}$ is a solution of Eq. (5.17) in some region R of the x, t plane. We assume that $\mathbf{u}$ is continuous in R and that the derivatives $\mathbf{u}_t$ and $\mathbf{u}_x$ are continuous in R except across a smooth curve described by a function $\eta(x, t) = 0$ (Fig. 5.4.a). We define new independent variables η, ζ such that the curve $\eta(x, t) = 0$ is a coordinate line (Fig. 5.4.b). Using the chain rule, we can write the derivatives $\mathbf{u}_t$ and $\mathbf{u}_x$ as

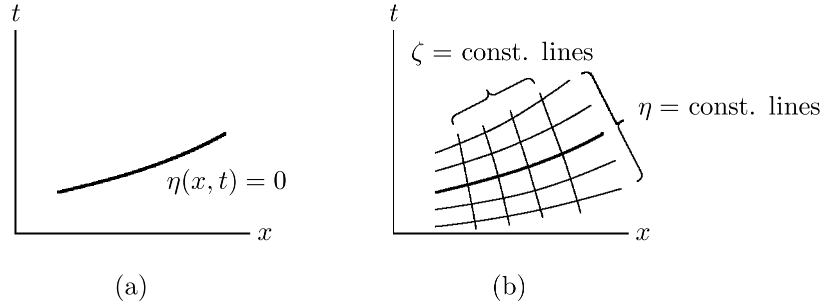


Figure 5.4: (a) The curve $\eta(x, t) = 0$. (b) Curvilinear coordinates η, ζ.

$$\begin{aligned}\mathbf{u}_t &= \eta_t \mathbf{u}_\eta + \zeta_t \mathbf{u}_\zeta, \\ \mathbf{u}_x &= \eta_x \mathbf{u}_\eta + \zeta_x \mathbf{u}_\zeta.\end{aligned}$$

With these expressions we can write Eq. (5.17) in terms of the variables η, ζ:

$$(\eta_t \mathbf{A} + \eta_x \mathbf{B})\mathbf{u}_\eta + (\zeta_t \mathbf{A} + \zeta_x \mathbf{B})\mathbf{u}_\zeta = 0. \tag{5.18}$$

By evaluating the left side of this equation at two points p_- and p_+ on either side of the curve $\eta(x, t) = 0$ and taking the limit as the two points approach a point p on the curve (Fig. 5.5), we obtain the equation

$$(\eta_t \mathbf{A} + \eta_x \mathbf{B})[\![\mathbf{u}_\eta]\!] = 0. \tag{5.19}$$

The *jump* $[\![\mathbf{u}_\eta]\!]$ of $\mathbf{u}_\eta$ is defined by

$$[\![\mathbf{u}_\eta]\!] = (\mathbf{u}_\eta)_+ - (\mathbf{u}_\eta)_-,$$

where the symbol $(\mathbf{u}_\eta)_-$ means the limit of $\mathbf{u}_\eta$ evaluated at p_- as $p_- \to p$ and $(\mathbf{u}_\eta)_+$ means the limit of $\mathbf{u}_\eta$ evaluated at p_+ as $p_+ \to p$.

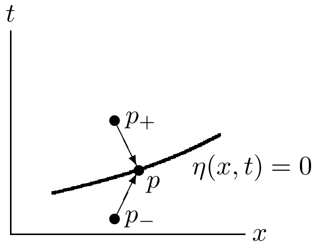


Figure 5.5: Points p_- and p_+ on either side of the curve $\eta(x,t) = 0$.

Along the curve $\eta(x,t) = 0$,

$$d\eta = \eta_t \, dt + \eta_x \, dx = 0.$$

The slope of the curve is

$$c = \frac{dx}{dt} = -\frac{\eta_t}{\eta_x}.$$

Dividing Eq. (5.19) by η_x, we can express it in terms of the slope c:

$$(-c\mathbf{A} + \mathbf{B})[\![\mathbf{u}_\eta]\!] = 0. \tag{5.20}$$

This is a homogeneous system of equations for the components of the vector $[\![\mathbf{u}_\eta]\!]$. It has a nontrivial solution, that is, the derivatives of $\mathbf{u}$ can be discontinuous across the curve $\eta(x,t) = 0$, only if the determinant of the matrix of the coefficients equals zero:

$$\det(-c\mathbf{A} + \mathbf{B}) = 0. \tag{5.21}$$

Equation (5.21) defines the slope of a curve in the x,t plane across which the derivatives $\mathbf{u}_t$ and $\mathbf{u}_x$ may be discontinuous. A curve in the x,t plane whose slope c is given by this equation is called a *characteristic* of Eq. (5.17).

Interior equation and hyperbolic equations

Suppose that the curve $\eta(x,t) =$ constant is a characteristic of Eq. (5.17). That is, it is a curve whose slope is given by Eq. (5.21). We write Eq. (5.18) as

$$\mathbf{D}\mathbf{u}_\eta + \mathbf{E}\mathbf{u}_\zeta = 0, \tag{5.22}$$

where $\mathbf{D}$ and $\mathbf{E}$ are

$$\mathbf{D} = \eta_t\mathbf{A} + \eta_x\mathbf{B}, \qquad \mathbf{E} = \zeta_t\mathbf{A} + \zeta_x\mathbf{B}. \tag{5.23}$$

Equation (5.21) implies that the determinant of the matrix of $\mathbf{D}$ is zero. Because the determinant of a square matrix is equal to the determinant of its transpose, $\det(\mathbf{D}^T) = 0$. This is the necessary condition for the existence of a non-zero vector $\mathbf{r}$ such that

$$\mathbf{D}^T \mathbf{r} = 0. \tag{5.24}$$

The product of Eq. (5.22) with the transpose of $\mathbf{r}$ is

$$\mathbf{r}^T \mathbf{D} \mathbf{u}_\eta + \mathbf{r}^T \mathbf{E} \mathbf{u}_\zeta = 0. \tag{5.25}$$

The first term in this equation is equal to zero as a consequence of Eq. (5.24):

$$\mathbf{r}^T \mathbf{D} \mathbf{u}_\eta = \mathbf{u}_\eta^T \mathbf{D}^T \mathbf{r} = 0;$$

therefore Eq. (5.25) reduces to

$$\mathbf{r}^T \mathbf{E} \mathbf{u}_\zeta = 0. \tag{5.26}$$

This differential equation governs the variation of the solution $\mathbf{u}$ along a characteristic. It is called the *interior equation*, and is the foundation of the method of characteristics. Dividing Eq. (5.24) by η_x, we obtain a more convenient equation for determining the vector $\mathbf{r}$:

$$(-c\mathbf{A}^T + \mathbf{B}^T)\mathbf{r} = 0. \tag{5.27}$$

When the roots of Eq. (5.21) are real and there are n real, linearly independent vectors $\mathbf{r}$ that satisfy Eq. (5.27) in a region R of the x, t plane, Eqs. (5.17) are said to be *hyperbolic* in R.

Examples

A first-order hyperbolic equation

Consider the first-order partial differential equation

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = 0, \tag{5.28}$$

where $\alpha = \tilde{\alpha}(u)$ is a function of the dependent variable u. When α is a constant, it is easy to show that the solution of this differential equation is the D'Alembert solution $u = f(x - \alpha t)$. When α depends on u, the differential equation is nonlinear. Although it is a very simple equation, its solution has some of the same qualitative features exhibited by waves in nonlinear elastic materials.

Equation 5.28 is of the form of Eq. (5.17). Because there is only one equation, the matrices reduce to scalars: $A = 1$ and $B = \alpha$, and Eq. (5.21) for the slope c becomes the scalar equation

$$-c(1) + \alpha = 0.$$

Thus $c = \alpha = \tilde{\alpha}(u)$, and we see that the slope of a characteristic depends on the value of u.

We use the variable ζ in the interior equation, Eq. (5.26), to measure distance along a characteristic. In this example we can simply let $\zeta = t$. From Eq. (5.23), the scalar $E = 1$. Substituting the values of A and B into (5.27), it becomes the scalar equation

$$[-c(1) + \alpha]r = 0,$$

which is satisfied for any scalar r since $c = \alpha$. As a result, the interior equation reduces to the scalar equation

$$u_\zeta = 0.$$

Thus u is constant along a characteristic. Because the slopes of the characteristics depend only on u, the characteristics are straight lines.

Let us assume that the value of $\tilde{\alpha}(u)$ decreases monotonically with u (Fig. 5.6). Suppose that at $t = 0$, the value of u as a function of x has the distribution shown in Fig. 5.7.a. Thus we know the slopes of the characteristics $c = \tilde{\alpha}(u)$ at $t = 0$. This information is sufficient to determine the characteristics since they are straight lines (Fig. 5.7.b). The slopes dx/dt of the characteristics increase with increasing x because the initial distribution of u decreases with increasing x. Because the value of u is constant along a characteristic, we can use the characteristics to determine the solution at any time (Figs. 5.7.c, d, and e). The distribution of u as a function of x changes as a function of time; the wave "spreads" with increasing time.

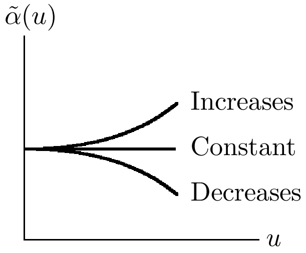


Figure 5.6: The function $\tilde{\alpha}(u)$.

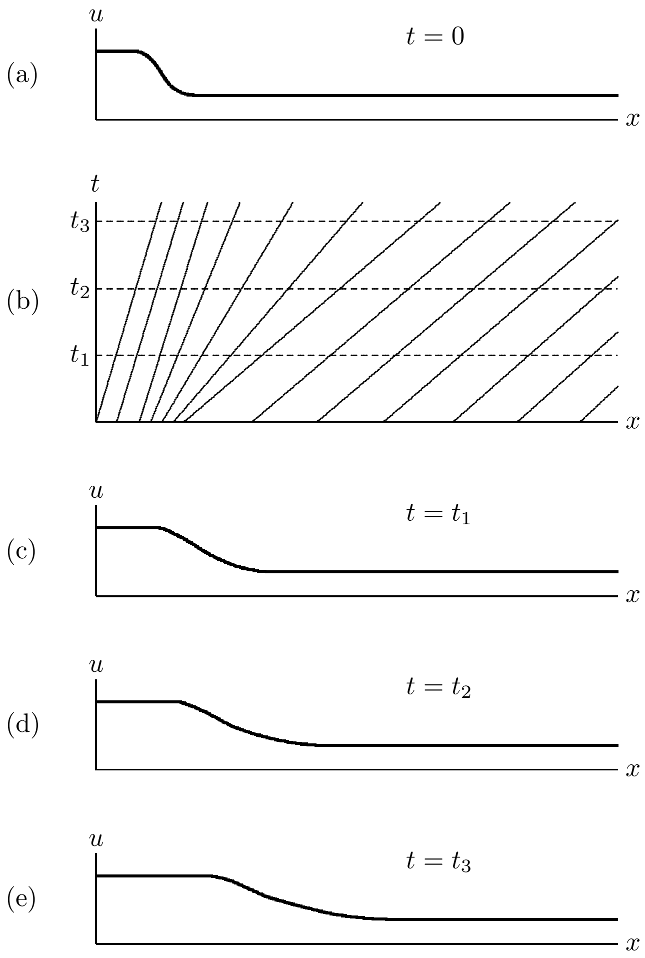


Figure 5.7: A spreading wave. (a) The initial condition. (b) The characteristics. (c),(d),(e) The distribution of u at progressive values of time.

Now let us assume that the value of $\tilde{\alpha}(u)$ increases monotonically with u (Fig. 5.6), and that at $t = 0$ the value of u as a function of x has the distribution shown in Fig. 5.8.a. The characteristics are shown in Fig. 5.7.b. The slopes dx/dt of the characteristics now decrease with increasing x. The distribution of u is shown in Figs. 5.8.c, d, and e. In this case the wave "steepens" with increasing time. As shown in Fig. 5.8.e, if we interpret the characteristics literally, the wave eventually "breaks" and u is no longer single valued. How-

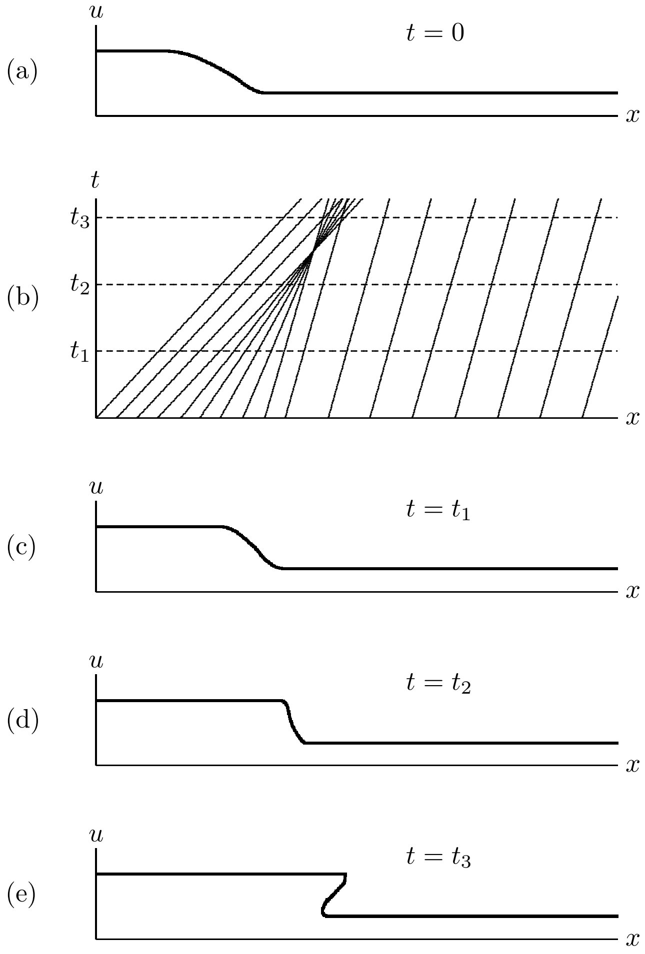


Figure 5.8: A steepening wave. (a) The initial condition. (b) The characteristics. (c),(d),(e) The distribution of u at progressive values of time.

ever, when the slope $\partial u/\partial x$ of the wave becomes unbounded, we can no longer trust Eq. (5.28).

Simple waves in elastic materials

The Lagrangian equations governing the one-dimensional motion of a nonlinear elastic material, Eqs. (5.12), are

$$\left[\begin{array}{cc} \rho_0 & 0 \\ 0 & 1 \end{array}\right]\left[\begin{array}{c} \hat{v} \\ F \end{array}\right]_t + \left[\begin{array}{cc} 0 & -\tilde{T}_F \\ -1 & 0 \end{array}\right]\left[\begin{array}{c} \hat{v} \\ F \end{array}\right]_X = 0. \tag{5.29}$$

These equations are of the form of Eq. (5.17), where

$$A_{km} = \left[\begin{array}{cc} \rho_0 & 0 \\ 0 & 1 \end{array}\right], \qquad B_{km} = \left[\begin{array}{cc} 0 & -\tilde{T}_F \\ -1 & 0 \end{array}\right].$$

Equation (5.21) for the slope c of a characteristic is

$$\det(-c\mathbf{A} + \mathbf{B}) = \det\left[\begin{array}{cc} -c\rho_0 & -\tilde{T}_F \\ -1 & -c \end{array}\right] = 0.$$

This equation has two roots for c:

$$c = \pm\left(\frac{\tilde{T}_F}{\rho_0}\right)^{1/2}.$$

Thus there are two *families* of characteristics with slopes $+\alpha$ and $-\alpha$ in the X-t plane, where $\alpha = (\tilde{T}_F/\rho_0)^{1/2}$. If $\tilde{T}_F$ depends on F (that is, if the material is not linear elastic), the slopes of the characteristics depend on the value of F.

In a special class of solutions called *simple waves*, the dependent variables are assumed to be constant along one family of characteristics. Here we describe an important example of a simple wave in a nonlinear elastic material. Suppose that a half space of material is initially undisturbed and at $t = 0$ the boundary is subjected to a constant velocity v_0 (Fig. 5.9). We can obtain the solution to this problem by assuming that the dependent variables $\hat{v}$ and F are constant along the right-running characteristics in the X-t plane.

Let us assume that the dependent variables are constants along the right-running characteristics shown in Fig. 5.10. A characteristic "fan" radiates from the origin. Because the slopes of the characteristics depend only on the value of F, the right-running characteristics are straight lines. The fan separates two regions in which the values of the dependent variables are assumed to be uniform. In region 1, "ahead" of the wave, the material is stationary ($\hat{v} = 0$) and undeformed ($F = 1$). In region 2, "behind" the wave, the velocity matches

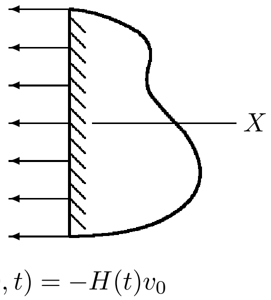


Figure 5.9: Half space subjected to a velocity boundary condition.

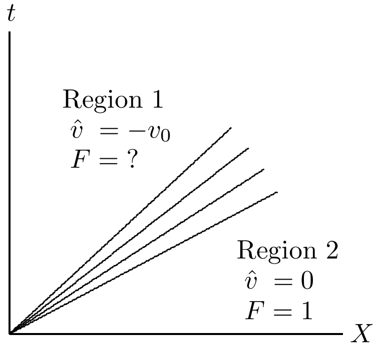


Figure 5.10: A characteristic fan radiating from the origin.

the boundary condition ($\hat{v} = -v_0$) and F is unknown. To determine the extent of the fan of characteristics (that is, to determine the boundary between the fan and region 2) we must determine the value of F in region 2.

We can determine the value of F in region 2 and also the variation of the velocity $\hat{v}$ within the characteristic fan by determining the values of the dependent variables along a left-running characteristic (Fig. 5.11). To write the interior equation, Eq. (5.26), for the variation of the dependent variables along a left-running characteristic, we define $\zeta = t$. From Eq. (5.23), the matrix of $\mathbf{E}$ is

$$E_{km} = \zeta_t A_{km} + \zeta_x B_{km} = \begin{bmatrix} \rho_0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We determine the vector $\mathbf{r}$ in the interior equation from Eq. (5.27):

$$(-cA^T_{km} + B^T_{km})r_m = \begin{bmatrix} \alpha\rho_0 & -1 \\ -\tilde{T}_F & \alpha \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 0.$$

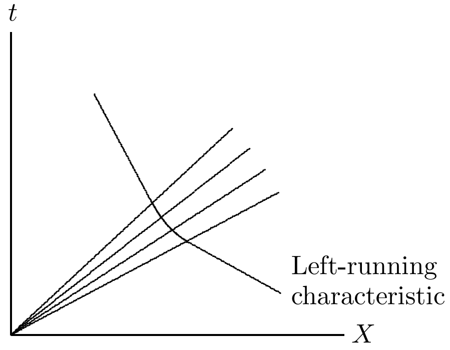


Figure 5.11: A left-running characteristic.

It is easy to see that this equation is satisfied by the vector

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha\rho_0 \end{bmatrix}.$$

Therefore we obtain the interior equation

$$\mathbf{r}^T\mathbf{E}\mathbf{u}_\zeta = \begin{bmatrix} 1 & \alpha\rho_0 \end{bmatrix} \begin{bmatrix} \rho_0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{v} \\ F \end{bmatrix}_\zeta = 0,$$

which reduces to the simple equation

$$\hat{v}_\zeta = -\alpha F_\zeta.$$

Integrating this equation, we determine the velocity as a function of F:

$$\hat{v} = -\int_1^F \alpha \, dF. \tag{5.30}$$

This result completes the solution. When the stress-strain relation $T = \tilde{T}(F)$ of the material is known, we can determine $\alpha = (\tilde{T}_F/\rho_0)^{1/2}$ as a function of F. Therefore we can evaluate Eq. (5.30), numerically if necessary, to determine the velocity $\hat{v}$ as a function of F. In this way, we determine the value of F in region 2 since the velocity in region 2 is known. Also, since the velocity $\hat{v}$ is then known as a function of the characteristic slope α, we know the velocity at any point within the characteristic fan.

Weak waves

On page 218, we introduced characteristics by assuming that the solution $\mathbf{u}$ of Eq. (5.17) was continuous in a region R of the x, t plane and that the derivatives $\mathbf{u}_t$ and $\mathbf{u}_x$ were continuous in R except across a smooth curve described by the function $\eta(x, t) = 0$ (Fig. 5.12). Such a "wave front" across which the dependent variables are continuous but their derivatives are discontinuous is called a *weak wave*.

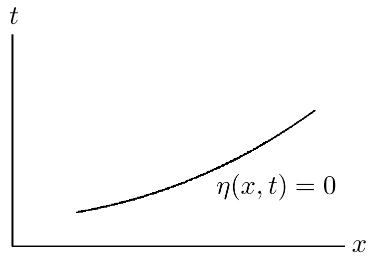


Figure 5.12: Path of a weak wave in the x-t plane.

The curve describing the path of a weak wave in the x-t plane is a characteristic. Therefore the velocity $c = dx/dt$ of a weak wave is given by Eq. (5.21):

$$\det(-c\mathbf{A} + \mathbf{B}) = 0. \tag{5.31}$$

Because $\mathbf{A}$ and $\mathbf{B}$ generally depend on the solution $\mathbf{u}$, using this expression to determine the wave velocity requires that we know the solution at the wave front. One case in which this is true is when the weak wave propagates into an undisturbed region.

Evolution of the amplitude

We can define the amplitude of a weak wave to be the value of the discontinuity in one of the dependent variables across the wave. In some cases, we can determine the rate of change of the amplitude with respect to time. Consider Eq. (5.22):

$$\mathbf{D}\mathbf{u}_\eta + \mathbf{E}\mathbf{u}_\zeta = 0.$$

In index notation, this equation is

$$D_{ij}\frac{\partial u_j}{\partial \eta} + E_{ij}\frac{\partial u_j}{\partial \zeta} = 0.$$

The derivative of this equation with respect to η is

$$\frac{\partial}{\partial \eta}\left(D_{ij}\frac{\partial u_j}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(E_{ij}\frac{\partial u_j}{\partial \zeta}\right) = 0.$$

The terms D_{ij} and E_{ij} may depend on the dependent variables u_j. Therefore we use the chain rule to express this expression in the form

$$D_{ij}\frac{\partial^2 u_j}{\partial\eta^2}+\frac{\partial D_{ij}}{\partial u_k}\frac{\partial u_k}{\partial\eta}\frac{\partial u_j}{\partial\eta}+E_{ij}\frac{\partial}{\partial\zeta}\left(\frac{\partial u_j}{\partial\eta}\right)+\frac{\partial E_{ij}}{\partial u_k}\frac{\partial u_k}{\partial\eta}\frac{\partial u_j}{\partial\zeta}=0.$$

Our next step is to determine the jump of this equation by evaluating the left side at two points p_- and p_+ on either side of the curve $\eta(x,t)=0$ and taking the limit as the two points approach a point p on the curve (Fig. 5.5). By using the identity

$$\begin{aligned}\left[\!\left[\frac{\partial u_k}{\partial\eta}\frac{\partial u_j}{\partial\eta}\right]\!\right] &= \left[\!\left[\frac{\partial u_k}{\partial\eta}\right]\!\right]\left[\!\left[\frac{\partial u_j}{\partial\eta}\right]\!\right]+\left[\!\left[\frac{\partial u_k}{\partial\eta}\right]\!\right]\left(\frac{\partial u_j}{\partial\eta}\right)_-\\ &\quad+\left(\frac{\partial u_k}{\partial\eta}\right)_-\left[\!\left[\frac{\partial u_j}{\partial\eta}\right]\!\right],\end{aligned}$$

we obtain the result

$$\begin{aligned}&D_{ij}\left[\!\left[\frac{\partial^2 u_j}{\partial\eta^2}\right]\!\right]+\frac{\partial D_{ij}}{\partial u_k}\left[\!\left[\frac{\partial u_k}{\partial\eta}\right]\!\right]\left[\!\left[\frac{\partial u_j}{\partial\eta}\right]\!\right]+\frac{\partial D_{ij}}{\partial u_k}\left[\!\left[\frac{\partial u_k}{\partial\eta}\right]\!\right]\left(\frac{\partial u_j}{\partial\eta}\right)_-\\ &+\frac{\partial D_{ij}}{\partial u_k}\left(\frac{\partial u_k}{\partial\eta}\right)_-\left[\!\left[\frac{\partial u_j}{\partial\eta}\right]\!\right]+E_{ij}\frac{d}{d\zeta}\left[\!\left[\frac{\partial u_j}{\partial\eta}\right]\!\right]+\frac{\partial E_{ij}}{\partial u_k}\left[\!\left[\frac{\partial u_k}{\partial\eta}\right]\!\right]\frac{\partial u_j}{\partial\zeta}=0.\end{aligned}$$

If we take the dot product of this vector equation with the vector $\mathbf{r}$ defined by Eq. (5.24), the first term vanishes and we can write the resulting equation in the form

$$\begin{aligned}r_iE_{ij}\frac{d}{d\zeta}\left[\!\left[\frac{\partial u_j}{\partial\eta}\right]\!\right] &= -r_i\frac{\partial D_{ij}}{\partial u_k}\left[\!\left[\frac{\partial u_k}{\partial\eta}\right]\!\right]\left[\!\left[\frac{\partial u_j}{\partial\eta}\right]\!\right]\\ &\quad -r_i\frac{\partial D_{ij}}{\partial u_k}\left[\!\left[\frac{\partial u_k}{\partial\eta}\right]\!\right]\left(\frac{\partial u_j}{\partial\eta}\right)_- - r_i\frac{\partial D_{ij}}{\partial u_k}\left(\frac{\partial u_k}{\partial\eta}\right)_-\left[\!\left[\frac{\partial u_j}{\partial\eta}\right]\!\right]\\ &\quad -r_i\frac{\partial E_{ij}}{\partial u_k}\left[\!\left[\frac{\partial u_k}{\partial\eta}\right]\!\right]\frac{\partial u_j}{\partial\zeta}.\end{aligned}\tag{5.32}$$

Because the variable ζ measures distance along the path of the weak wave in the x-t plane, this equation governs the evolution of the values of the discontinuities in the dependent variables as the weak wave propagates. This is only a single equation, but the values of the discontinuities are not independent. They are related by Eq. (5.20):

$$(-cA_{ij}+B_{ij})\left[\!\left[\frac{\partial u_j}{\partial\eta}\right]\!\right]=0.\tag{5.33}$$

Weak waves in elastic materials

Consider the one-dimensional motion of a nonlinear elastic material described by Eq. (5.29). The matrices of **A** and **B** are

$$A_{km} = \begin{bmatrix} \rho_0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B_{km} = \begin{bmatrix} 0 & -\tilde{T}_F \\ -1 & 0 \end{bmatrix}.$$

Suppose that a weak wave propagates into undisturbed material. From Eq. (5.29), the velocity of the wave is

$$c = \alpha_0 = \left(\frac{\tilde{T}_F}{\rho_0} \right)^{1/2}. \tag{5.34}$$

The velocity α_0 of the wave is constant because the deformation gradient is uniform ($F = 1$) ahead of the wave (Fig. 5.13). To obtain a coordinate η that is constant along the path of the wave in the X-t plane, we define

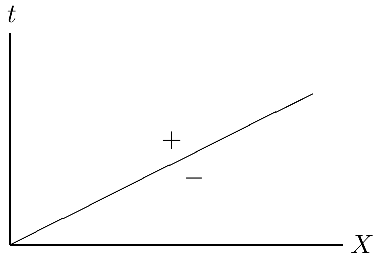


Figure 5.13: Path of the weak wave in the X-t plane.

$$\eta = t - \frac{X}{\alpha_0}.$$

To obtain a coordinate ζ that measures distance along the path of the wave, we define $\zeta = t$. From Eq. (5.23), the matrices of **D** and **E** are

$$D_{km} = \eta_t \mathbf{A} + \eta_X \mathbf{B} = \begin{bmatrix} \rho_0 & \tilde{T}_F/\alpha_0 \\ 1/\alpha_0 & 1 \end{bmatrix},$$

$$E_{km} = \zeta_t \mathbf{A} + \zeta_X \mathbf{B} = \begin{bmatrix} \rho_0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The vector **r** is defined by Eq. (5.27):

$$(-c\mathbf{A}^T + \mathbf{B}^T)\mathbf{r} = \begin{bmatrix} -\alpha_0\rho_0 & -1 \\ -\tilde{T}_F & -\alpha_0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 0.$$

This equation is satisfied by the vector

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\alpha_0\rho_0 \end{bmatrix}.$$

If we let the $-$ and $+$ subscripts denote values ahead of and behind the wave respectively (Fig. 5.13), the terms $(\partial u_j/\partial\eta)_-$ in Eq. (5.32) are zero because the material is undisturbed ahead of the wave. Also, the terms $\partial u_j/\partial\zeta$ vanish since the velocity and the deformation gradient are uniform ahead of the wave ($v = 0$ and $F = 1$). As a result, Eq. (5.32) reduces to

$$r_i E_{ij} \frac{d}{d\zeta} \left[\!\left[\frac{\partial u_j}{\partial\eta} \right]\!\right] = -r_i \frac{\partial D_{ij}}{\partial u_k} \left[\!\left[\frac{\partial u_k}{\partial\eta} \right]\!\right] \left[\!\left[\frac{\partial u_j}{\partial\eta} \right]\!\right].$$

Substituting the expressions for the matrices of $\mathbf{D}$ and $\mathbf{E}$ and the components of $\mathbf{r}$ into this equation, we obtain the expression

$$\rho_0 \frac{d}{d\zeta}[\![v_\eta]\!] - \alpha_0\rho_0 \frac{\partial}{\partial\zeta}[\![F_\eta]\!] = -\frac{\tilde{T}_{FF}}{\alpha_0}[\![F_\eta]\!]^2, \tag{5.35}$$

where $\tilde{T}_{FF} = d^2\tilde{T}/dF^2$. The values of the discontinuities are related by Eq. (5.33):

$$(-cA_{ij} + B_{ij}) \left[\!\left[\frac{\partial u_j}{\partial\eta} \right]\!\right] = \begin{bmatrix} -\alpha_0\rho_0 & -\tilde{T}_F \\ -1 & -\alpha_0 \end{bmatrix} \begin{bmatrix} [\![v_\eta]\!] \\ [\![F_\eta]\!] \end{bmatrix} = 0.$$

From this equation we see that the discontinuities $[\![v_\eta]\!]$ and $[\![F_\eta]\!]$ are related by

$$[\![v_\eta]\!] + \alpha_0[\![F_\eta]\!] = 0.$$

With this expression we can eliminate $[\![F_\eta]\!]$ in Eq. (5.35), obtaining the equation

$$\frac{d}{d\zeta}[\![v_\eta]\!] = -\left(\frac{\tilde{T}_{FF}}{2\rho_0\alpha_0^3} \right) [\![v_\eta]\!]^2. \tag{5.36}$$

This equation governs the evolution of $[\![v_\eta]\!]$ as the weak wave propagates. Evaluating the jump of the chain rule expression

$$\hat{v}_t = \eta_t v_\eta + \zeta_t v_\zeta,$$

we see that in this example $[\![v_\eta]\!]$ is equal to the discontinuity in the acceleration:

$$[\![\hat{v}_t]\!] = \eta_t[\![v_\eta]\!] = [\![v_\eta]\!].$$

Therefore, recalling that $\zeta = t$, we can write Eq. (5.36) as

$$\frac{d}{dt}[\![\hat{v}_t]\!] = -\left(\frac{\tilde{T}_{FF}}{2\rho_0\alpha_0^3} \right) [\![\hat{v}_t]\!]^2. \tag{5.37}$$

We have obtained an equation that governs the evolution of the discontinuity, or jump, in the acceleration across a weak wave. Suppose that at $t = 0$, the value of the jump in the acceleration across the wave is $[\![\hat{v}_t]\!]_0$. We can integrate Eq. (5.37) to determine the jump in the acceleration as a function of time:

$$[\![\hat{v}_t]\!] = \frac{1}{\dfrac{1}{[\![\hat{v}_t]\!]_0} + \left(\dfrac{\tilde{T}_{FF}}{2\rho_0\alpha_0^3}\right)t}. \tag{5.38}$$

In a linear elastic material, the slope $\tilde{T}_F$ is constant, so $\tilde{T}_{FF} = 0$. In this case, we see that $[\![\hat{v}_t]\!]$ does not change and the amplitude of a weak wave is constant. Suppose that $\tilde{T}_{FF} \neq 0$. The signs of the two terms in the denominator of Eq. (5.38) depend on the signs of the initial value of the jump in the acceleration and the term $\tilde{T}_{FF}$. If the two terms have the same sign, the magnitude of the wave decreases monotonically with time and the amplitude of the weak wave *decays*. If the two terms are of opposite sign, the amplitude of the weak wave *grows* to infinity when

$$t = \frac{-2\rho_0\alpha_0^3}{[\![\hat{v}_t]\!]_0\tilde{T}_{FF}}.$$

When this occurs, the wave is no longer weak. It becomes a *shock wave*.

Exercises

Exercise 5.1 Consider the one-dimensional linear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2\frac{\partial^2 u}{\partial x^2},$$

where α is constant. Introducing the variables $v = \partial u/\partial t$ and $w = \partial u/\partial x$, it can be written as the system of first-order equations

$$\begin{aligned}\frac{\partial v}{\partial t} &= \alpha^2\frac{\partial w}{\partial x},\\ \frac{\partial v}{\partial x} &= \frac{\partial w}{\partial t}.\end{aligned}$$

(a) Use Eq. (5.21) to show that there are two families of right-running and left-running characteristics in the x-t plane and that the characteristics are straight lines.
(b) Show that the system of first-order equations is hyperbolic.

Exercise 5.2 Consider the system of first-order equations in Exercise 5.1. Use the interior equation, Eq. (5.26), to show that $v - \alpha w$ is constant along a right-running characteristic and $v + \alpha w$ is constant along a left-running characteristic.

Exercise 5.3 Consider the simple wave solution beginning on page 224. Suppose that the relationship between the stress and the deformation gradient is $\tilde{T}(F) = E_0 \ln F$, where E_0 is a constant.

(a) Show that within the characteristic fan, the velocity is given in terms of the deformation gradient by

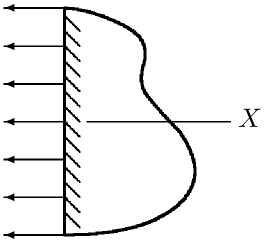
$$v = -2\alpha_0(F^{1/2} - 1),$$

where $\alpha_0 = (E_0/\rho_0)^{1/2}$.

(b) Show that within the characteristic fan, the deformation gradient and velocity are given as functions of X, t by

$$F = \frac{\alpha_0^2 t^2}{X^2}, \qquad v = -2\alpha_0 \left(\frac{\alpha_0 t}{X} - 1) \right).$$

Exercise 5.4



$T(0,t) = H(t)T_0$

A half space of elastic material is initially undisturbed. At $t = 0$ the boundary is subjected to a uniform constant stress T_0. The stress is related to the deformation gradient by the logarithmic expression $T = \tilde{T}(F) = E_0 \ln F$, where E_0 is a constant. Show that the resulting velocity of the boundary of the half space is

$$\left(\frac{E_0}{\rho_0} \right)^{1/2} \left[e^{(T_0/2E_0)} - 1 \right].$$

Discussion—See the discussion of simple waves in elastic materials on page 224.

Exercise 5.5 Consider a weak wave propagating into an undisturbed nonlinear elastic material. Show that the equation governing the rate of change of the discontinuity in $\hat{v}_X$ is

$$\frac{d}{dt}[\![\hat{v}_X]\!] = \left(\frac{\tilde{T}_{FF}}{2\rho_0 \alpha_0^2} \right) [\![\hat{v}_X]\!]^2.$$

5.3 Singular Surface Theory

A *singular surface* is a surface across which a field (a function of position and time) is discontinuous. In this section we show how the theory of singular surfaces can be used to analyze the propagation of waves in materials. For particular classes of waves called shock and acceleration waves, we derive the velocity and the rate of change, or *growth and decay* of the amplitude for a nonlinear elastic material.

Kinematic compatibility condition

Consider a scalar field $\phi(x,t)$. Suppose that at time t, the field is discontinuous at a point x_s (Fig. 5.14). We denote the regions to the left and right of x_s by $+$ and $-$, and define ϕ_+ and ϕ_- to be the limits of $\phi(x,t)$ as x approaches x_s from the $+$ and $-$ sides:

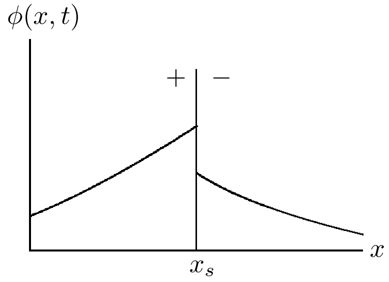


Figure 5.14: A discontinuous field.

$$\phi_+ = \lim_{(x \to x_s)_+} \phi(x,t), \qquad \phi_- = \lim_{(x \to x_s)_-} \phi(x,t).$$

The *jump* of $\phi(x,t)$ is defined by

$$[\![\phi]\!] = \phi_+ - \phi_-.$$

Now let us assume that during some interval of time the field $\phi(x,t)$ is discontinuous at a *moving* point x_s given by a function

$$x_s = x_s(t).$$

The velocity of the point x_s along the x axis is

$$v_s = \frac{dx_s(t)}{dt}.$$

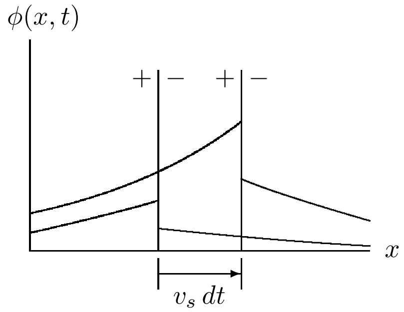


Figure 5.15: The field $\phi(x,t)$ at time t and at time $t+dt$.

During the interval of time from t to $t+dt$, the point x_s moves a distance $v_s\,dt$ (Fig. 5.15). We can express the value of ϕ_+ at time $t+dt$ in terms of its value at time t:

$$\phi_+(t+dt) = \phi_+(t) + \left(\frac{\partial\phi}{\partial t}\right)_+ (t)\,dt + \left(\frac{\partial\phi}{\partial x}\right)_+ (t)\,v_s\,dt.$$

From this expression we see that the rate of change of ϕ_+ can be expressed in terms of the limits of the derivatives of ϕ_+ at x_s:

$$\frac{d}{dt}\phi_+ = \left(\frac{\partial\phi}{\partial t}\right)_+ + \left(\frac{\partial\phi}{\partial x}\right)_+ v_s. \tag{5.39}$$

We can derive a corresponding expression for the rate of change of ϕ_-:

$$\frac{d}{dt}\phi_- = \left(\frac{\partial\phi}{\partial t}\right)_- + \left(\frac{\partial\phi}{\partial x}\right)_- v_s. \tag{5.40}$$

Subtracting this equation from Eq. (5.39), we obtain an equation for the rate of change of the jump of $\phi(x,t)$:

$$\frac{d}{dt}[\![\phi]\!] = \left[\!\!\left[\frac{\partial\phi}{\partial t}\right]\!\!\right] + \left[\!\!\left[\frac{\partial\phi}{\partial x}\right]\!\!\right] v_s. \tag{5.41}$$

This equation is called the *kinematic compatibility condition*. When the field $\phi(x,t)$ is continuous at x_s during an interval of time, $[\![\phi]\!] = 0$ and the kinematic compatibility condition states that

$$\left[\!\!\left[\frac{\partial\phi}{\partial t}\right]\!\!\right] + \left[\!\!\left[\frac{\partial\phi}{\partial x}\right]\!\!\right] v_s = 0 \qquad \text{(when } \phi \text{ is continuous).} \tag{5.42}$$

Acceleration waves

The one-dimensional motion of a nonlinear elastic material is governed by the equation of balance of linear momentum,

$$\rho_0 \frac{\partial \hat{v}}{\partial t} = \tilde{T}_F \frac{\partial F}{\partial X}. \tag{5.43}$$

The deformation gradient and the velocity are related by

$$\frac{\partial F}{\partial t} = \frac{\partial \hat{v}}{\partial X}. \tag{5.44}$$

An *acceleration wave* is a propagating singular surface across which the velocity v and strain F of a material are continuous but the first partial derivatives of the velocity and strain are discontinuous. Thus an acceleration wave is identical to the weak wave we discuss on page 226. Here we use the kinematic compatibility condition (5.41) to determine the velocity and the rate of change of the amplitude of an acceleration wave in an elastic material.

Velocity

Because v and F are continuous across an acceleration wave, by substituting them into Eq. (5.41) we obtain the relations

$$\left[\!\left[\frac{\partial \hat{v}}{\partial t}\right]\!\right] + \left[\!\left[\frac{\partial \hat{v}}{\partial X}\right]\!\right] v_s = 0 \tag{5.45}$$

and

$$\left[\!\left[\frac{\partial F}{\partial t}\right]\!\right] + \left[\!\left[\frac{\partial F}{\partial X}\right]\!\right] v_s = 0. \tag{5.46}$$

From Eq. (5.44) we see that

$$\left[\!\left[\frac{\partial F}{\partial t}\right]\!\right] = \left[\!\left[\frac{\partial \hat{v}}{\partial X}\right]\!\right]. \tag{5.47}$$

The jump of the equation of balance of linear momentum, Eq. (5.43), is

$$\rho_0 \left[\!\left[\frac{\partial \hat{v}}{\partial t}\right]\!\right] = \tilde{T}_F \left[\!\left[\frac{\partial F}{\partial X}\right]\!\right].$$

Using Eqs. (5.45)-(5.47) we can write this equation in the form

$$\left(v_s^2 - \frac{\tilde{T}_F}{\rho_0}\right) \left[\!\left[\frac{\partial \hat{v}}{\partial t}\right]\!\right] = 0.$$

Thus the velocity of an acceleration wave in an elastic material is

$$v_s = \left(\frac{\tilde{T}_F}{\rho_0}\right)^{1/2}. \tag{5.48}$$

This result is identical to Eq. (5.34).

Rate of change of the amplitude

We can use the jump in the acceleration of the material as a measure of the amplitude of an acceleration wave. Substituting the acceleration $\partial\hat{v}/\partial t$ and $\partial F/\partial t$ into Eq. (5.41), we obtain the relations

$$\begin{aligned}\frac{d}{dt}\left[\!\!\left[\frac{\partial\hat{v}}{\partial t}\right]\!\!\right] &= \left[\!\!\left[\frac{\partial^2\hat{v}}{\partial t^2}\right]\!\!\right] + \left[\!\!\left[\frac{\partial^2\hat{v}}{\partial t\partial X}\right]\!\!\right] v_s \\ &= \left[\!\!\left[\frac{\partial^2\hat{v}}{\partial t^2}\right]\!\!\right] + \left[\!\!\left[\frac{\partial^2 F}{\partial t^2}\right]\!\!\right] v_s\end{aligned} \tag{5.49}$$

and

$$\frac{d}{dt}\left[\!\!\left[\frac{\partial F}{\partial t}\right]\!\!\right] = \left[\!\!\left[\frac{\partial^2 F}{\partial t^2}\right]\!\!\right] + \left[\!\!\left[\frac{\partial^2 F}{\partial t\partial X}\right]\!\!\right] v_s. \tag{5.50}$$

From Eqs. (5.45) and (5.47),

$$\left[\!\!\left[\frac{\partial F}{\partial t}\right]\!\!\right] = -\frac{1}{v_s}\left[\!\!\left[\frac{\partial\hat{v}}{\partial t}\right]\!\!\right].$$

The time derivative of this equation is

$$\frac{d}{dt}\left[\!\!\left[\frac{\partial F}{\partial t}\right]\!\!\right] = -\frac{1}{v_s}\frac{d}{dt}\left[\!\!\left[\frac{\partial\hat{v}}{\partial t}\right]\!\!\right] + \frac{1}{v_s^2}\left[\!\!\left[\frac{\partial\hat{v}}{\partial t}\right]\!\!\right]\frac{dv_s}{dt}.$$

With this expression and Eq. (5.50), we obtain the relation

$$\left[\!\!\left[\frac{\partial^2 F}{\partial t^2}\right]\!\!\right] = -\frac{1}{v_s}\frac{d}{dt}\left[\!\!\left[\frac{\partial\hat{v}}{\partial t}\right]\!\!\right] + \frac{1}{v_s^2}\left[\!\!\left[\frac{\partial\hat{v}}{\partial t}\right]\!\!\right]\frac{dv_s}{dt} - \left[\!\!\left[\frac{\partial^2 F}{\partial t\partial X}\right]\!\!\right] v_s.$$

We substitute this result into Eq. (5.49), obtaining the expression

$$2\frac{d}{dt}\left[\!\!\left[\frac{\partial\hat{v}}{\partial t}\right]\!\!\right] = \left[\!\!\left[\frac{\partial^2\hat{v}}{\partial t^2}\right]\!\!\right] - \left[\!\!\left[\frac{\partial^2 F}{\partial t\partial X}\right]\!\!\right] v_s^2 + \frac{1}{v_s}\left[\!\!\left[\frac{\partial\hat{v}}{\partial t}\right]\!\!\right]\frac{dv_s}{dt}. \tag{5.51}$$

We next take the partial derivative of Eq. (5.43) with respect to time:

$$\rho_0\frac{\partial^2\hat{v}}{\partial t^2} = \tilde{T}_F\frac{\partial^2 F}{\partial t\partial X} + \tilde{T}_{FF}\frac{\partial F}{\partial t}\frac{\partial F}{\partial X},$$

where $\tilde{T}_{FF} = \partial^2\tilde{T}/\partial F^2$. Taking the jump of this equation, we obtain the result

$$\left[\!\!\left[\frac{\partial^2\hat{v}}{\partial t^2}\right]\!\!\right] - \left[\!\!\left[\frac{\partial^2 F}{\partial t\partial X}\right]\!\!\right] v_s^2 = \frac{\tilde{T}_{FF}}{\rho_0}\left[\!\!\left[\frac{\partial F}{\partial t}\frac{\partial F}{\partial X}\right]\!\!\right].$$

Using this result, we can write Eq. (5.51) in the form

$$2\frac{d}{dt}\left[\!\!\left[\frac{\partial\hat{v}}{\partial t}\right]\!\!\right] - \frac{1}{v_s}\left[\!\!\left[\frac{\partial\hat{v}}{\partial t}\right]\!\!\right]\frac{dv_s}{dt} = \frac{\tilde{T}_{FF}}{\rho_0}\left[\!\!\left[\frac{\partial F}{\partial t}\frac{\partial F}{\partial X}\right]\!\!\right]. \tag{5.52}$$

Consider an acceleration wave that propagates into undisturbed material. In this case the velocity v_s is constant and

$$\left[\!\left[\frac{\partial F}{\partial t}\frac{\partial F}{\partial X}\right]\!\right] = \left[\!\left[\frac{\partial F}{\partial t}\right]\!\right]\left[\!\left[\frac{\partial F}{\partial X}\right]\!\right].$$

From Eqs. (5.45)-(5.47),

$$\left[\!\left[\frac{\partial F}{\partial t}\right]\!\right]\left[\!\left[\frac{\partial F}{\partial X}\right]\!\right] = -\frac{1}{v_s^3}\left[\!\left[\frac{\partial \hat{v}}{\partial t}\right]\!\right]^2.$$

Thus Eq. (5.52) is

$$\frac{d}{dt}\left[\!\left[\frac{\partial \hat{v}}{\partial t}\right]\!\right] = -\frac{\tilde{T}_{FF}}{2\rho_0 v_s^3}\left[\!\left[\frac{\partial \hat{v}}{\partial t}\right]\!\right]^2. \tag{5.53}$$

This equation governs the amplitude of an acceleration wave propagating into an undisturbed elastic material. It is identical to Eq. (5.37).

Shock waves

A *shock wave* in an elastic material can be modeled as a propagating singular surface across which the motion $\hat{x}(X,t)$ of the material is continuous but the velocity v and strain F are discontinuous. Shock waves occur commonly in supersonic gas flows, and can also occur in solid materials subjected to strong disturbances.

Suppose that X_s is the position of a shock wave, and let us draw a free-body diagram of the material between positions X_L and X_R to the left and right of X_s (Fig. 5.16). The equation of balance of linear momentum for the material

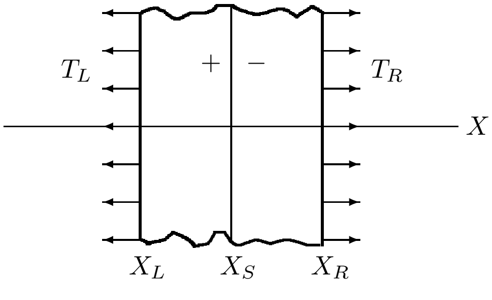


Figure 5.16: Free-body diagram of the material between positions X_L and X_R.

is

$$T_R - T_L = \frac{d}{dt}\int_{X_L}^{X_s}\rho_0\hat{v}\,dX + \frac{d}{dt}\int_{X_s}^{X_R}\rho_0\hat{v}\,dX.$$

Recalling that the point X_s moves with velocity v_s, we can write this expression as

$$T_R - T_L = \rho_0 \hat{v}_+ v_s + \int_{X_L}^{X_s} \rho_0 \frac{\partial \hat{v}}{\partial t}\, dX - \rho_0 \hat{v}_- v_s + \int_{X_s}^{X_R} \rho_0 \frac{\partial \hat{v}}{\partial t}\, dX.$$

From the limit of this equation as $X_S \to X_s$ and $X_R \to X_s$ we obtain the result

$$[\![T]\!] + \rho_0 [\![\hat{v}]\!] v_s = 0. \tag{5.54}$$

This equation relates the jump in the stress to the jump in the velocity across a shock wave. Using it together with the kinematic compatibility condition, Eq. (5.41), we can determine the velocity and the rate of change of the amplitude of a shock wave in a nonlinear elastic material.

Velocity

Because the motion $\hat{x}(X,t)$ is continuous across a shock wave, by substituting it into Eq. (5.41) we obtain the relation

$$[\![v]\!] + [\![F]\!] v_s = 0. \tag{5.55}$$

From this relation and Eq. (5.54), we find that the velocity of a shock wave is given by

$$v_s = \left(\frac{[\![T]\!]}{\rho_0 [\![F]\!]} \right)^{1/2}. \tag{5.56}$$

This result is independent of the stress-strain relation of the material.

Rate of change of the amplitude

Substituting the velocity and the deformation gradient into Eq. (5.41), we obtain the relations

$$\begin{aligned} \frac{d}{dt}[\![v]\!] &= \left[\!\!\left[\frac{\partial \hat{v}}{\partial t} \right]\!\!\right] + \left[\!\!\left[\frac{\partial \hat{v}}{\partial X} \right]\!\!\right] v_s \\ &= \left[\!\!\left[\frac{\partial \hat{v}}{\partial t} \right]\!\!\right] + \left[\!\!\left[\frac{\partial F}{\partial t} \right]\!\!\right] v_s \end{aligned} \tag{5.57}$$

and

$$\frac{d}{dt}[\![F]\!] = \left[\!\!\left[\frac{\partial F}{\partial t} \right]\!\!\right] + \left[\!\!\left[\frac{\partial F}{\partial X} \right]\!\!\right] v_s. \tag{5.58}$$

The jump of the equation of balance of linear momentum, Eq. (5.43), is

$$\rho_0 \left[\!\!\left[\frac{\partial \hat{v}}{\partial t} \right]\!\!\right] = \left[\!\!\left[\frac{\partial T}{\partial X} \right]\!\!\right].$$

We substitute this relation into Eq. (5.57) to obtain

$$\frac{d}{dt}[\![v]\!] = \frac{1}{\rho_0}\left[\!\!\left[\frac{\partial T}{\partial X}\right]\!\!\right] + \left[\!\!\left[\frac{\partial F}{\partial t}\right]\!\!\right] v_s. \tag{5.59}$$

Taking the time derivative of Eq. (5.55), we write the result in the form

$$\frac{d}{dt}[\![v]\!] = -v_s\frac{d}{dt}[\![F]\!] - [\![F]\!]\frac{dv_s}{dt}.$$

We set this expression equal to Eq. (5.59) to obtain

$$\left[\!\!\left[\frac{\partial F}{\partial t}\right]\!\!\right] = -\frac{d}{dt}[\![F]\!] - \frac{1}{v_s}[\![F]\!]\frac{dv_s}{dt} - \frac{1}{\rho_0 v_s}\left[\!\!\left[\frac{\partial T}{\partial X}\right]\!\!\right],$$

and we substitute this result into Eq. (5.58), which yields the equation

$$2v_s\frac{d}{dt}[\![F]\!] + [\![F]\!]\frac{dv_s}{dt} = \left[\!\!\left[\frac{\partial F}{\partial X}\right]\!\!\right] v_s^2 - \frac{1}{\rho_0}\left[\!\!\left[\frac{\partial T}{\partial X}\right]\!\!\right]. \tag{5.60}$$

Consider a shock wave that propagates into undisturbed material. In this case

$$\left[\!\!\left[\frac{\partial T}{\partial X}\right]\!\!\right] = \left(\frac{\partial T}{\partial X}\right)_+ = \left(\frac{d\tilde{T}}{dF}\right)_+\left(\frac{\partial F}{\partial X}\right)_+ = \left(\tilde{T}_F\right)_+\left[\!\!\left[\frac{\partial F}{\partial X}\right]\!\!\right],$$

and by taking the time derivative of Eq. (5.56), we find that

$$[\![F]\!]\frac{dv_s}{dt} = \frac{\left[\frac{1}{\rho_0}(\tilde{T}_F)_+ - v_s^2\right]}{2v_s}\frac{d}{dt}[\![F]\!].$$

Therefore we can write Eq. (5.60) in the form

$$\frac{d}{dt}[\![F]\!] = \frac{-[(\tilde{T}_F)_+ - \rho_0 v_s^2]\left[\!\!\left[\frac{\partial F}{\partial X}\right]\!\!\right]}{2\rho_0 v_s\left[1 + \frac{(\tilde{T}_F)_+ - \rho_0 v_s^2}{4\rho_0 v_s^2}\right]}. \tag{5.61}$$

This equation governs the rate of change of the jump in the deformation gradient of a shock wave propagating into an undisturbed elastic material. It determines whether the amplitude of the wave grows or decays given the value of the deformation gradient (or the density) and the value of the gradient of the deformation gradient (or the gradient of the density) behind the wave.

Exercises

Exercise 5.6 (a) Determine the relation $T = \tilde{T}(F)$ for a linear elastic material.
(b) What is the velocity of an acceleration wave in a linear elastic material?
(c) What is the velocity of a shock wave in a linear elastic material?

Answer: (a) $T = (\lambda + 2\mu)(F - 1)$. (b) $\alpha = \sqrt{(\lambda + 2\mu)/\rho_0}$. (c) $\alpha = \sqrt{(\lambda + 2\mu)/\rho_0}$.

Exercise 5.7 The relation between the stress and the deformation gradient in a material is $T = \tilde{T}(F) = E_0 \ln F$, where E_0 is a constant. The density of the undeformed material is ρ_0.
(a) What is the velocity of an acceleration wave in the undeformed material?
(b) If the material is homogeneously compressed so that its density is $2\rho_0$, what is the velocity of an acceleration wave?

Answer: (a) $v_s = (E_0/\rho_0)^{1/2}$. (b) $v_s = (2E_0/\rho_0)^{1/2}$.

Exercise 5.8 The relation between the stress and the deformation gradient in a material is $T = \tilde{T}(F) = E_0 \ln F$, where E_0 is a constant. The density of the undeformed material is ρ_0. An acceleration wave propagates into the undeformed material. At $t = 0$, the acceleration of the material just behind the wave is a_0 in the direction of propagation. What is the acceleration of the material just behind the wave at time t?

Answer:

$$\frac{1}{\dfrac{1}{a_0} - \left(\dfrac{\rho_0}{4E_0}\right)^{1/2} t}$$

Exercise 5.9 If a gas behaves isentropically, the relation between the pressure and the density is

$$\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma},$$

where γ, the ratio of specific heats, is a positive constant. In the one-dimensional problems we are discussing, the stress $T = -p$.
(a) Show that the velocity of an acceleration wave propagating into undisturbed gas with pressure p_0 and density ρ_0 is

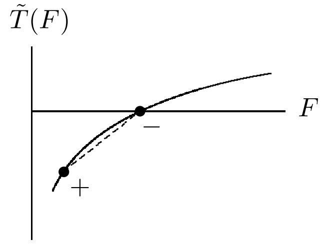
$$v_s = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}.$$

(b) If the amplitude $[\![\partial \hat{v}/\partial t]\!]$ of an acceleration wave propagating into undisturbed gas with pressure p_0 and density ρ_0 is positive, show that the amplitude increases with time.

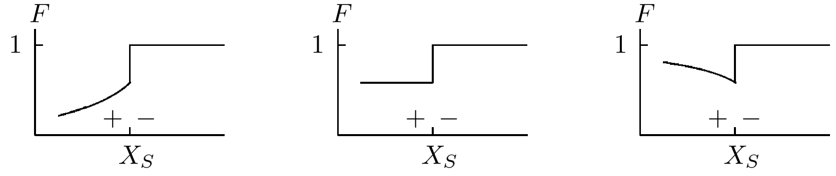
Exercise 5.10 The relation between the stress and the deformation gradient in a material is $T = \tilde{T}(F) = E_0 \ln F$, where E_0 is a constant. The density of the undeformed material is ρ_0. A shock wave propagates into the undeformed material. If the density of the material just behind the wave is $2\rho_0$ at a particular time, show that the velocity of the wave at that time is

$$v_s = \left[\frac{2E_0 \ln 2}{\rho_0}\right]^{1/2}.$$

Exercise 5.11



The stress-strain relation of an elastic material is given by the solid curve. The point labeled minus ($-$) is the state ahead of a shock wave and the point labeled plus ($+$) is the state just behind the wave at a time t. Consider three possibilities for the variation of F as a function of X at time t:



(a) (b) (c)

Determine whether $[\![F]\!]$ increases, remains the same, or decreases as a function of time in cases (a), (b), and (c).

Discussion—Notice that $[\![T]\!]/[\![F]\!]$ is the slope of the straight dashed line and $(\tilde{T}_F)_+$ is the slope of the solid curve at point ($+$).

Answer: (a) Decrease. (b) Remain the same. (c) Increase.

5.4 Loading and Release Waves

In the examples we have used to analyze nonlinear wave propagation in the previous sections of this chapter, we assumed that the material was elastic; that is, we assumed there was a one-to-one relationship between the stress and the strain. In some applications, waves are created that involve deformations so large and rapid that materials no longer behave elastically. In this section we discuss an important example.

We consider a type of experiment in which a large, nearly constant pressure p is applied to the plane boundary of a sample of initially undisturbed material, causing a compressional *loading wave* to propagate into the material (Fig. 5.17). After a period of time, the pressure at the boundary decreases, causing an expansional *unloading wave* or *release wave* to propagate into the material. A graph of the strain of the material at a time t shows two distinct waves (Fig. 5.18). The strain rises due to the arrival of the loading wave. The material is in an approximately constant or equilibrium state until the arrival of the release wave.

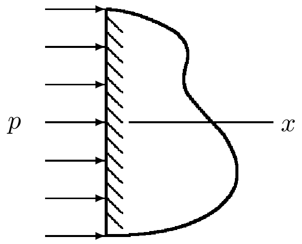


Figure 5.17: A pressure p applied to a sample of a material.

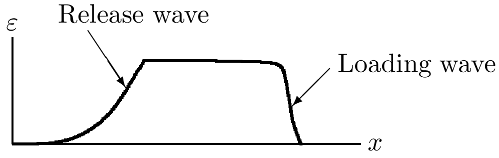


Figure 5.18: Loading and release waves resulting from a step increase in pressure at the boundary of a sample.

Loading wave

Experiments have shown that after an initial transient period, the loading wave propagates with nearly constant velocity and its "shape," or waveform, is approximately constant. It is then referred to as a *steady wave.* During this period, we can express the dependent variables in terms of a single independent variable

$$\zeta = x - Ut,$$

where U is the velocity of the steady wave.

The equation of conservation of mass of the material, Eq. (5.16), is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0. \tag{5.62}$$

By using the chain rule, we can express this equation in terms of ζ as

$$\frac{d}{d\zeta}[\rho(v - U)] = 0.$$

We see that in the steady wave, conservation of mass requires that

$$\rho(v - U) = \text{ constant.} \tag{5.63}$$

The equation of balance of linear momentum, Eq. (5.13), is

$$\rho\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}v\right) = -\frac{\partial p}{\partial x}. \tag{5.64}$$

In terms of ζ, it is

$$\rho(v - U)\frac{dv}{d\zeta} = -\frac{dp}{d\zeta}.$$

Using Eq. (5.63), we can write this equation in the form

$$\frac{d}{d\zeta}[\rho(v - U)v + p] = 0.$$

Thus in the steady wave, balance of linear momentum requires that

$$\rho(v - U)v + p = \text{ constant.} \tag{5.65}$$

Suppose that the steady loading wave propagates into undisturbed material with density ρ_0 and pressure $p = 0$. From Eqs. (5.63) and (5.65), we obtain the relations

$$\rho(v - U) = -\rho_0 U \tag{5.66}$$

and

$$\rho(v - U)v + p = 0. \tag{5.67}$$

Using Eq. (5.66), we can write Eq. (5.67) in the form

$$\rho_0 U v = p. \tag{5.68}$$

The strain ε of the material is related to its density by Eq. (5.15):

$$\varepsilon = 1 - \frac{\rho_0}{\rho}.$$

By using this relation to eliminate the density from Eq. (5.66), we obtain a very simple relation between the strain, the wave velocity, and the velocity of the material in the steady wave:

$$\varepsilon = \frac{v}{U}. \tag{5.69}$$

With this relation we can express Eq. (5.68) in the form

$$p = \rho_0 U^2 \varepsilon. \tag{5.70}$$

After the passage of the steady loading wave, we assume the material is in a homogeneous, equilibrium state until the arrival of the release wave. Let v_e be the velocity of the material in the equilibrium state. Experiments have shown that for many (but not all) materials, the velocity v_e is related to the wave velocity U by a linear relation

$$U = c + s v_e, \tag{5.71}$$

where c and s are material constants. The equilibrium state of the material at the end of the passage of the steady wave must also satisfy Eqs. (5.69) and (5.70):

$$\begin{aligned} \varepsilon_e &= \frac{v_e}{U}, \\ p_e &= \rho_0 U^2 \varepsilon_e. \end{aligned} \tag{5.72}$$

Using these two relations and Eq. (5.71), we find that

$$p_e = \frac{\rho_0 c^2 \varepsilon_e}{(1 - s\varepsilon_e)^2}. \tag{5.73}$$

Once the constants c and s have been determined from experiments for a particular material, the pressure p_e in the equilibrium state following the passage of the steady loading wave can be plotted as a function of the strain ε_e in the equilibrium state by using Eq. (5.73). The resulting graph is called a *Hugoniot curve* (Fig. 5.19). The pressure p_e approaches infinity at the finite strain $\varepsilon_e = 1/s$. This is an artifact of the empirical Eq. (5.71). Each point on the Hugoniot curve for a material defines an equilibrium state following the

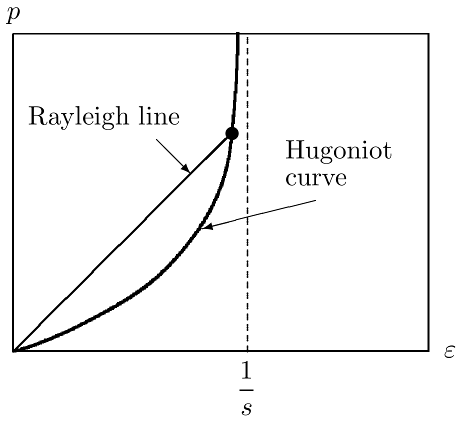


Figure 5.19: The Hugoniot curve and a Rayleigh line.

passage of a steady loading wave. If any one of the four variables U, v_e, p_e, ε_e is measured following the passage of the wave, the other three can be determined from the Hugoniot curve and Eqs. (5.72).

From Eq. (5.70), we see that the pressure is linearly related to the strain during the passage of the steady loading wave. The resulting straight line is called a *Rayleigh line* (Fig. 5.19). The intersection of the Rayleigh line with the Hugoniot curve is the equilibrium state following the passage of the wave.

Release wave

In some materials, experiments have shown that the pressure of the material and the strain approximately follow the Hugoniot curve during the passage of the release wave. That is, the loading of the material occurs along the Rayleigh line and the unloading occurs (approximately) along the Hugoniot curve (Fig. 5.20).

During the passage of the release wave, the state of the material changes relatively slowly in comparison to the rapid change of state in the loading wave. As a result, the unloading process approximately follows the Hugoniot curve, which represents equilibrium states of the material. That is, during the passage of the release wave the material can be modeled approximately as an elastic material whose stress-strain relation is given by Eq. (5.73). By doing so, we can model the unloading wave as a simple wave. (The analysis of simple waves is discussed beginning on page 224.) This means that in the unloading wave the dependent variables are constant along right-running characteristics in the x-t

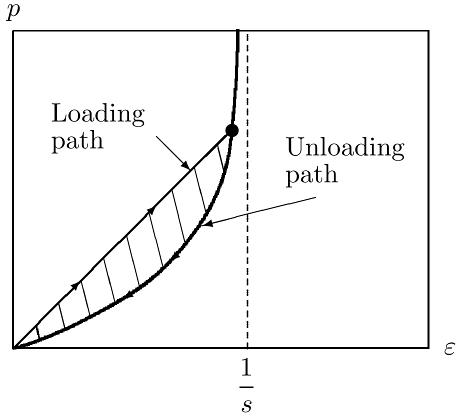


Figure 5.20: The loading and unloading paths.

plane.

We can express the equations of conservation of mass, Eq. (5.62), and balance of linear momentum, Eq. (5.64), in matrix form as

$$\begin{bmatrix} 1 & 0 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \rho \\ v \end{bmatrix}_t + \begin{bmatrix} v & \rho \\ \dfrac{dp}{d\rho} & \rho v \end{bmatrix} \begin{bmatrix} \rho \\ v \end{bmatrix}_x = 0.$$

From Eq. (5.21), the slope c of a right-running characteristic is given by

$$c = v + \left(\frac{dp}{d\rho}\right)^{1/2} = v + \left[\frac{(1-\varepsilon)^2}{\rho_0}\frac{dp}{d\varepsilon}\right]^{1/2}.$$

This expression relates the slopes of the characteristics defining the motion of the release wave to the slope of the Hugoniot curve. We can arrive at two conclusions from this relation and Fig. 5.20. First, since the slope of the Hugoniot decreases as the strain decreases, the high-strain (leading) part of the release wave travels faster than the low-strain (trailing) part. As a result, the release wave "spreads" with increasing time. Second, because the slope of the Hugoniot curve at high strain is larger than the slope of the Rayleigh line, we see from the second of Eqs. (5.72) that the high-strain part of the release wave travels faster than the loading wave. Eventually (if the sample is large enough) the release wave overtakes the loading wave. As a result, the amplitude of the loading wave is attenuated.

The integral of the pressure with respect to the strain as the loading and unloading waves pass through the material is the cross-hatched area between the Rayleigh line and the Hugoniot curve in Fig. 5.20. This integral is the net work per unit volume done to the material by the passage of the wave. This work creates thermal energy in the material, and is called *shock heating.*

Flyer plate experiment

The first shock wave experiments in solid materials involved striking a stationary target material with a "flyer plate" set into motion by explosives (Fig. 5.21). As the flyer plate approached the target, its velocity v_0 was measured by two electrical "shorting pins." Another shorting pin measured the arrival time of the shock wave at the back surface of the target. From this the velocity of the shock wave passing through the target was determined. Therefore, these first *Hugoniot experiments* were simply time measurements.

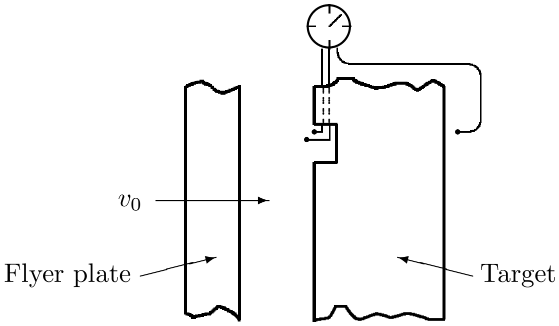


Figure 5.21: Flyer plate experiment.

When the flyer plate strikes the target, a steady loading wave propagates into the undisturbed target material. From Eqs. (5.68) and (5.71), the pressure of the equilibrium state behind the loading wave is

$$p_T = \rho_T(c_T + s_T v_T)v_T, \tag{5.74}$$

where ρ_T is the density of the undisturbed target material and v_T is the velocity of the material in the equilibrium state behind the wave. This relationship between p_T and v_T is a form of the Hugoniot relationship, which is the locus of all possible equilbrium states behind a loading wave. The loading path to a given equilibrium state is the straight line between the origin and the point (v_T, p_T).

When the flyer plate strikes the target a steady loading wave also propagates into the flyer plate material. Equation (5.74) can be written for the pressure of the equilibrium state behind the loading wave in the flyer plate,

$$p_F = \rho_F(c_F + s_F v_F)v_F, \tag{5.75}$$

where v_F is the velocity of the material in the equilibrium state *toward the left, relative to the undisturbed flyer plate.* Relative to a stationary reference frame, the velocity of the flyer-plate material in the equilibrium state *toward the right* is $v_0 - v_F$, where v_0 is the initial velocity of the flyer plate. *Because the flyer plate and target are in contact,*

$$p_T = p_F \tag{5.76}$$

and

$$v_T = v_0 - v_F. \tag{5.77}$$

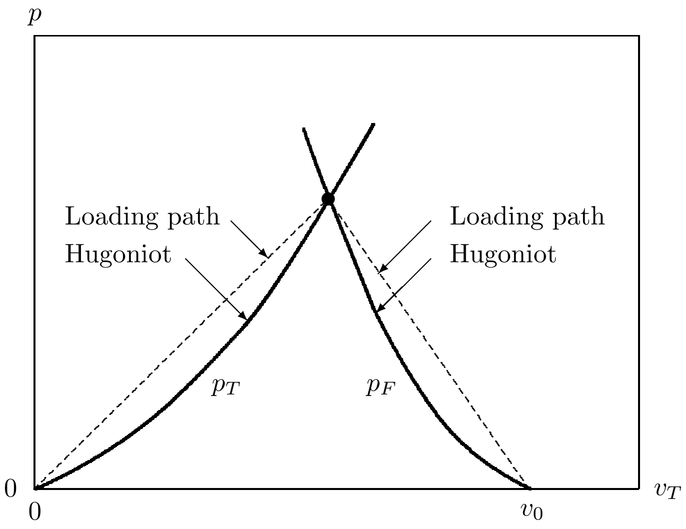


Figure 5.22: Hugoniot curves for a flyer plate experiment.

Figure 5.22 shows the locus of equilibrium states for the target material given by Eq. (5.74). Using Eqs. (5.75) and (5.77), we also show the locus of equilibrium states for the flyer plate material as a function of v_T. The point where these curves intersect determines the velocity and pressure of the materials in the equilibrium regions behind the steady loading waves. The dashed straight lines are the Rayleigh lines, the loading paths followed by the two materials in changing from their undisturbed states to the equilibrium state.

A Little History

As we have mentioned, large-amplitude steady loading waves are also called shock waves, and the pivotal relationship to the development of the field of shock physics was Eq. 5.70:

$$p = \rho_0 U^2 (1 - \rho_0/\rho).$$

This relationship was derived in 1859 by the Reverend S. Earnshaw. Unfortunately, he assumed that all steady waves must be elastic phenomena. He concluded that steady waves could only exist in materials with a one-to-one relationship between stress and strain given by this expression. As he believed that air obeyed Boyle's Law,

$$p = p_0 \rho/\rho_0,$$

he concluded that, because these two expressions do not agree, shock waves could not exist in air. As anyone who has heard a supersonic aircraft pass knows, Earnshaw's conclusion was wrong. Indeed shock waves will propagate in nearly all materials, because they induce behaviors that are not elastic. As we have observed, shock waves induce heating in materials.

The study of thermal phenomena was undergoing fundamental change during Earnshaw's day. The old caloric theory of heat was being abandoned and replaced by the modern laws of thermodynamics. We now understand that energy is conserved while being converted in part from a mechanical form to heat. And in fact, shock waves cause entropy to increase. In contrast, as the release wave passes, the material response is isentropic. While we have approximated the release path by the Hugoniot, the path is actually an isentrope (path of constant entropy) that falls slightly above the Hugoniot and accounts for the effects of thermal expansion in the material (Fig. 5.23).

It would be another decade before Rankine analyzed the shock wave and concluded that "there must be both a change of temperature and a conduction of heat" in order that steady waves might exist. And it was not until the late 1880's that Hugoniot eliminated the need for heat conduction from the theory of shock waves. See Drumheller (1998).

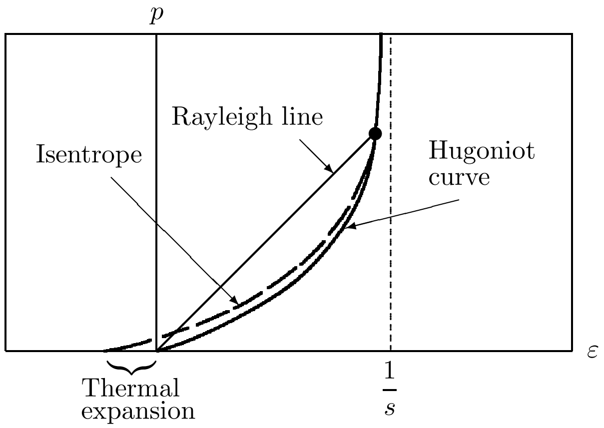


Figure 5.23: The isentrope and resulting thermal expansion.

Exercises

Exercise 5.12 Consider a flyer plate experiment in which the flyer plate and the target are identical materials with mass density 2.79 Mg/m^3, the measured shock wave velocity is 6.66 km/s, and the measured flyer-plate velocity is 2 km/s. What is the pressure in the equilibrium region behind the loading wave?

Answer: 18.6 GPa.

Exercise 5.13 (a) Suppose that in the experiment of Exercise 5.12, the flyer plate is aluminum with properties $\rho_0 = 2.79$ Mg/m^2, $c = 5.33$ km/s, and $s = 1.40$, the target is stainless steel with properties $\rho_0 = 7.90$ Mg/m^2, $c = 4.57$ km/s, and $s = 1.49$, and the velocity of the flyer plate is 2 km/s. What is the pressure in both materials in the equilibrium region behind the loading wave, and what is the velocity of their interface?
(b) If the materials are interchanged so that the flyer plate is stainless steel and the target is aluminum, what is the pressure in both materials in the equilibrium region behind the loading wave and what is the velocity of their interface?

Answer: (a) Pressure is 27 GPa, velocity is 630 m/s.

Exercise 5.14 In Exercise 5.13, assume that the target is much thicker than the flyer plate. After the loading wave in the flyer plate reaches the back surface of the flyer plate, it is reflected as a release wave and returns to the interface between the materials. Determine whether the plates separate when the release wave reaches the interface: (a) if the flyer plate is aluminum and the target is stainless steel; (b) if the flyer plate is stainless steel and the target is aluminum.

Answer: (a) Yes. (b) No.

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Index

Solutions to the Exercises

Chapter 1. Linear Elasticity

Exercise 1.1 Show that $\delta_{km}u_k v_m = u_k v_k$.

Solution—In terms of the numerical values of the indices, the left side of the equation is

$$\begin{aligned}\delta_{km}u_k v_m &= \delta_{11}u_1v_1 + \delta_{12}u_1v_2 + \delta_{13}u_1v_3 \\ &\quad +\delta_{21}u_2v_1 + \delta_{22}u_2v_2 + \delta_{23}u_2v_3 \\ &\quad +\delta_{31}u_3v_1 + \delta_{32}u_3v_2 + \delta_{33}u_3v_3 \\ &= (1)u_1v_1 + (0)u_1v_2 + (0)u_1v_3 \\ &\quad +(0)u_2v_1 + (1)u_2v_2 + (0)u_2v_3 \\ &\quad +(0)u_3v_1 + (0)u_3v_2 + (1)u_3v_3 \\ &= u_1v_1 + u_2v_2 + u_3v_3.\end{aligned}$$

The right side of the equation is

$$u_k v_k = u_1v_1 + u_2v_2 + u_3v_3.$$

Exercise 1.2
(a) Show that $\delta_{km}e_{kmn} = 0$.
(b) Show that $\delta_{km}\delta_{kn} = \delta_{mn}$.

Solution—
(a)

$$\begin{aligned}\delta_{km}e_{kmn} &= \delta_{11}e_{11n} + \delta_{12}e_{12n} + \delta_{13}e_{13n} \\ &\quad +\delta_{21}e_{21n} + \delta_{22}e_{22n} + \delta_{23}e_{23n} \\ &\quad +\delta_{31}e_{31n} + \delta_{32}e_{32n} + \delta_{33}e_{33n} \\ &= e_{11n} + e_{22n} + e_{33n}.\end{aligned}$$

Because $e_{kmn} = 0$ if any two of its indices are equal, $e_{11n} = 0$, $e_{22n} = 0$, and $e_{33n} = 0$ for any value of n.

(b) Let $m = 1$ and $n = 1$.

$$\begin{aligned} \delta_{k1}\delta_{k1} &= \delta_{11}\delta_{11} + \delta_{21}\delta_{21} + \delta_{31}\delta_{31} \\ &= \delta_{11}. \end{aligned}$$

The procedure is the same for each value of m and n.

Exercise 1.3 Show that $T_{km}\delta_{mn} = T_{kn}$.

Solution—Let $k = 1$ and $n = 1$.

$$\begin{aligned} T_{1m}\delta_{m1} &= T_{11}\delta_{11} + T_{12}\delta_{21} + T_{13}\delta_{31} \\ &= T_{11}. \end{aligned}$$

The procedure is the same for each value of k and n.

Exercise 1.4 If $T_{km}e_{kmn} = 0$, show that $T_{km} = T_{mk}$.

Solution—Let $n = 1$ and recall that $e_{kmn} = 0$ if any two of the subscripts k, m, and n are equal. Thus

$$T_{km}e_{km1} = T_{23}e_{231} + T_{32}e_{321} = T_{23} - T_{32} = 0,$$

so $T_{32} = T_{23}$. By letting $n = 2$ and then $n = 3$, the same procedure shows that $T_{13} = T_{31}$ and $T_{12} = T_{21}$.

Exercise 1.5 Consider the equation

$$a_k = \frac{\partial v_k}{\partial t} + \frac{\partial v_k}{\partial x_m} v_m.$$

(a) How many equations result when this equation is written explicitly in terms of the numerical values of the indices?
(b) Write the equations explicitly in terms of the numerical values of the indices.

Solution—
(a) The equation holds for each value of k, so there are three equations.

(b) The equations are

$$\begin{aligned} a_1 &= \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_1} v_1 + \frac{\partial v_1}{\partial x_2} v_2 + \frac{\partial v_1}{\partial x_3} v_3, \\ a_2 &= \frac{\partial v_2}{\partial t} + \frac{\partial v_2}{\partial x_1} v_1 + \frac{\partial v_2}{\partial x_2} v_2 + \frac{\partial v_2}{\partial x_3} v_3, \\ a_3 &= \frac{\partial v_3}{\partial t} + \frac{\partial v_3}{\partial x_1} v_1 + \frac{\partial v_3}{\partial x_2} v_2 + \frac{\partial v_3}{\partial x_3} v_3. \end{aligned}$$

Exercise 1.6 Consider the equation

$$T_{km} = c_{kmij} E_{ij}.$$

(a) How many equations result when this equation is written explicitly in terms of the numerical values of the indices?
(b) Write the equation for T_{11} explicitly in terms of the numerical values of the indices.

Solution—
(a) The equation holds for each value of k and for each value of m, so there are $3 \times 3 = 9$ equations.

(b)

$$\begin{aligned} T_{11} \quad = \quad & c_{1111}E_{11} + c_{1112}E_{12} + c_{1113}E_{13} \\ & + c_{1121}E_{21} + c_{1122}E_{22} + c_{1123}E_{23} \\ & + c_{1131}E_{31} + c_{1132}E_{32} + c_{1133}E_{33}. \end{aligned}$$

Exercise 1.7 The "del operator" is defined by

$$\begin{aligned}\nabla(\cdot) &= \frac{\partial(\cdot)}{\partial x_1}\,\mathbf{i}_1 + \frac{\partial(\cdot)}{\partial x_2}\,\mathbf{i}_2 + \frac{\partial(\cdot)}{\partial x_3}\,\mathbf{i}_3 \\ &= \frac{\partial(\cdot)}{\partial x_k}\,\mathbf{i}_k.\end{aligned} \tag{1.78}$$

The expression for the gradient of a scalar field ϕ in terms of cartesian coordinates can be obtained in a formal way by applying the del operator to the scalar field:

$$\nabla\phi = \frac{\partial\phi}{\partial x_k}\,\mathbf{i}_k.$$

(a) By taking the dot product of the del operator with a vector field $\mathbf{v}$, obtain the expression for the divergence $\nabla\cdot\mathbf{v}$ in terms of cartesian coordinates. (b) By taking the cross product of the del operator with a vector field $\mathbf{v}$, obtain the expression for the curl $\nabla\times\mathbf{v}$ in terms of cartesian coordinates.

Solution—

(a) The divergence is

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \left[\frac{\partial(\cdot)}{\partial x_k}\,\mathbf{i}_k\right]\cdot[v_m\,\mathbf{i}_m] \\ &= \frac{\partial v_m}{\partial x_k}(\mathbf{i}_k\cdot\mathbf{i}_m) \\ &= \frac{\partial v_m}{\partial x_k}\delta_{km} \\ &= \frac{\partial v_k}{\partial x_k}.\end{aligned}$$

(b) The curl is

$$\begin{aligned}\nabla\times\mathbf{v} &= \left[\frac{\partial(\cdot)}{\partial x_k}\,\mathbf{i}_k\right]\times[v_m\,\mathbf{i}_m] \\ &= \frac{\partial v_m}{\partial x_k}(\mathbf{i}_k\times\mathbf{i}_m) \\ &= \frac{\partial v_m}{\partial x_k}e_{kmn}\,\mathbf{i}_n.\end{aligned}$$

Exercise 1.8 By expressing the dot and cross products in terms of index notation, show that for any two vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0.$$

Solution—

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= (u_r\, \mathbf{i}_r) \cdot (u_k v_m e_{kmn}\, \mathbf{i}_n) \\ &= u_r u_k v_m e_{kmn} (\mathbf{i}_r \cdot \mathbf{i}_n) \\ &= u_r u_k v_m e_{kmn} \delta_{rn} \\ &= u_n u_k v_m e_{kmn}. \end{aligned}$$

The term $u_n u_k e_{kmn} = 0$. For example, let $m = 1$. Then

$$u_n u_k e_{k1n} = u_2 u_3 e_{312} + u_3 u_2 e_{213} = u_2 u_3 - u_3 u_2 = 0.$$

Exercise 1.9 Show that

$$u_k v_m e_{kmn} \mathbf{i}_n = \det \begin{bmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

Solution—The $\mathbf{i}_1$ component of the left side of the equation is

$$u_k v_m e_{km1} = u_2 v_3 e_{231} + u_3 v_2 e_{321} = u_2 v_3 - u_3 v_2.$$

The $\mathbf{i}_1$ component of the determinant is

$$u_2 v_3 - u_3 v_2.$$

In the same way, the $\mathbf{i}_2$ and $\mathbf{i}_3$ components can be shown to be equal.

Exercise 1.10 By expressing the divergence and curl in terms of index notation, show that for any vector field $\mathbf{v}$ for which the indicated derivatives exist,

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0.$$

Solution—The curl of $\mathbf{v}$ is

$$\nabla \times \mathbf{v} = \frac{\partial v_m}{\partial x_k} e_{kmn}\,\mathbf{i}_n.$$

Therefore

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{v}) &= \frac{\partial}{\partial x_n}\left(\frac{\partial v_m}{\partial x_k} e_{kmn}\right)\\ &= \frac{\partial^2 v_m}{\partial x_n \partial x_k} e_{kmn}.\end{aligned}$$

Let $m = 1$. Then

$$\begin{aligned}\frac{\partial^2 v_1}{\partial x_n \partial x_k} e_{k1n} &= \frac{\partial^2 v_1}{\partial x_2 \partial x_3} e_{312} + \frac{\partial^2 v_1}{\partial x_3 \partial x_2} e_{213}\\ &= \frac{\partial^2 v_1}{\partial x_2 \partial x_3} - \frac{\partial^2 v_1}{\partial x_3 \partial x_2}\\ &= 0.\end{aligned}$$

The result can be shown for $m = 2$ and $m = 3$ in the same way.

EXERCISE 1.11 By expressing the operations on the left side of the equation in terms of index notation, show that for any vector field $\mathbf{v}$,

$$\nabla(\nabla \cdot \mathbf{v}) - \nabla \times (\nabla \times \mathbf{v}) = \frac{\partial^2 v_k}{\partial x_m \partial x_m}\,\mathbf{i}_k.$$

Solution—The first term on the left is

$$\nabla(\nabla \cdot \mathbf{v}) = \frac{\partial}{\partial x_k}\left(\frac{\partial v_m}{\partial x_m}\right)\mathbf{i}_k = \frac{\partial^2 v_m}{\partial x_k \partial x_m}\,\mathbf{i}_k.$$

The curl of $\mathbf{v}$ is

$$\nabla \times \mathbf{v} = e_{kmn} \frac{\partial v_m}{\partial x_k}\,\mathbf{i}_n,$$

so

$$\nabla \times (\nabla \times \mathbf{v}) = e_{pnr} \frac{\partial}{\partial x_p}\left(e_{kmn} \frac{\partial v_m}{\partial x_k}\right)\mathbf{i}_r = e_{pnr} e_{kmn} \frac{\partial^2 v_m}{\partial x_p \partial x_k}\,\mathbf{i}_r.$$

Interchanging the indices k and r in this expression, it becomes

$$\nabla \times (\nabla \times \mathbf{v}) = e_{pnk} e_{rmn} \frac{\partial^2 v_m}{\partial x_p \partial x_r}\,\mathbf{i}_k.$$

Therefore we must show that

$$\frac{\partial^2 v_m}{\partial x_m \partial x_k} - e_{pnk} e_{rmn} \frac{\partial^2 v_m}{\partial x_p \partial x_r} = \frac{\partial^2 v_k}{\partial x_m \partial x_m}. \quad (*)$$

This equation must hold for each value of k. Let $k = 1$. The right side of the equation is

$$\frac{\partial^2 v_1}{\partial x_m \partial x_m} = \frac{\partial^2 v_1}{\partial x_1 \partial x_1} + \frac{\partial^2 v_1}{\partial x_2 \partial x_2} + \frac{\partial^2 v_1}{\partial x_3 \partial x_3}.$$

On the left side of Eq. $(*)$, let us sum over the index n:

$$\frac{\partial^2 v_m}{\partial x_m \partial x_1} - e_{pn1} e_{rmn} \frac{\partial^2 v_m}{\partial x_p \partial x_r} = \frac{\partial^2 v_m}{\partial x_m \partial x_1} - (e_{p11} e_{rm1} + e_{p21} e_{rm2} + e_{p31} e_{rm3}) \frac{\partial^2 v_m}{\partial x_p \partial x_r}.$$

The term $e_{p11} = 0$. Also, e_{p21} is zero unless $p = 3$ and e_{p31} is zero unless $p = 2$. Therefore this expression is

$$\begin{aligned} \frac{\partial^2 v_m}{\partial x_m \partial x_1} + e_{rm2} \frac{\partial^2 v_m}{\partial x_3 \partial x_r} - e_{rm3} \frac{\partial^2 v_m}{\partial x_2 \partial x_r} &= \frac{\partial^2 v_1}{\partial x_1 \partial x_1} + \frac{\partial^2 v_2}{\partial x_2 \partial x_1} + \frac{\partial^2 v_3}{\partial x_3 \partial x_1} + \frac{\partial^2 v_1}{\partial x_3 \partial x_3} \\ &\quad - \frac{\partial^2 v_3}{\partial x_3 \partial x_1} - \frac{\partial^2 v_2}{\partial x_2 \partial x_1} + \frac{\partial^2 v_1}{\partial x_2 \partial x_2} \\ &= \frac{\partial^2 v_1}{\partial x_1 \partial x_1} + \frac{\partial^2 v_1}{\partial x_2 \partial x_2} + \frac{\partial^2 v_1}{\partial x_3 \partial x_3}. \end{aligned}$$

Thus the left side of Eq. $(*)$ equals the right side for $k = 1$. The procedure is the same for $k = 2$ and $k = 3$.

EXERCISE 1.12 Consider a smooth, closed surface S with outward-directed unit vector $\mathbf{n}$. Show that

$$\int_S \mathbf{n}\, dS = 0.$$

Solution—The x_1 component of the integral is

$$\int_S n_1\, dS.$$

This integral is of the form

$$\int_S v_k n_k\, dS,$$

where $v_1 = 1$, $v_2 = 0$, and $v_3 = 0$. Let us define a vector field v_k on the volume V of the surface S by $v_1 = 1$, $v_2 = 0$, and $v_3 = 0$. Because the v_k are constants, the Gauss theorem states that

$$\int_S v_k n_k \, dS = \int_V \frac{\partial v_k}{\partial x_k} \, dV = 0.$$

By the same procedure, the x_2 and x_3 components of the integral can be shown to equal zero.

Exercise 1.13 An object of volume V is immersed in a stationary liquid. The pressure in the liquid is $p_o - \gamma x_2$, where p_o is the pressure at $x = 0$ and γ is the weight per unit volume of the liquid. The force exerted by the pressure on an element dS of the surface of the object is $-(p_o - \gamma x_2)\,\mathbf{n}\,dS$, where $\mathbf{n}$ is a unit vector perpendicular to the element dS. Show that the total force exerted on the object by the pressure of the water is $\gamma V\,\mathbf{i}_2$.

Solution—The x_2 component of the force exerted on the object by the pressure is

$$\int_S -\,(p_o - \gamma x_2) n_2 \, dS.$$

This is of the form

$$\int_S v_k n_k \, dS,$$

where $v_1 = 0$, $v_2 = -(p_o - \gamma x_2)$, and $v_3 = 0$. We define a vector field v_k on the volume V by $v_1 = 0$, $v_2 = -(p_o - \gamma x_2)$, and $v_3 = 0$. The Gauss theorem states that

$$\int_S -\,(p_o - \gamma x_2) n_2 \, dS = \int_V \frac{\partial[-(p_o - \gamma x_2)]}{\partial x_2} \, dV = \int_V \gamma \, dV = \gamma V.$$

By the same procedure, it can be shown that the x_1 and x_3 components of the force equal zero. Therefore the total force is $\gamma V\,\mathbf{i}_2$.

EXERCISE 1.14 A bar of material 1 meter long is subjected to the "stretching" motion

$$x_1 = X_1(1+t^2), \quad x_2 = X_2, \quad x_3 = X_3.$$

(a) Determine the inverse motion of the bar.
(b) Determine the material description of the displacement.
(c) Determine the spatial description of the displacement.
(d) What is the displacement of a point at the right end of the bar when $t = 2$ seconds?

Solution—
(a) Solving the motion for X_1, X_2, and X_3 in terms of x_1, x_2, x_3, and t, we obtain

$$X_1 = x_1/(1+t^2), \;\; X_2 = x_2, \;\; \text{and} \;\; X_3 = x_3.$$

(b) By using the motion, the three components of the displacement are

$$\begin{aligned} &u_1 = x_1 - X_1 = X_1(1+t^2) - X_1 = X_1 t^2, \\ &u_2 = x_2 - X_2 = X_2 - X_2 = 0, \\ &u_3 = x_3 - X_3 = X_3 - X_3 = 0. \end{aligned}$$

(c) From the results of Parts (a) and (b),

$$u_1 = X_1 t^2 = \frac{x_1}{(1+t^2)} t^2, \;\; u_2 = 0, \;\; \text{and} \;\; u_3 = 0.$$

(d) At the right end of the bar, $X_1 = 1$ m. From the result of Part (b),

$$u_1 = X_1 t^2 = (1)(2)^2 = 4 \text{ m}, \;\; u_2 = 0, \;\; \text{and} \;\; u_3 = 0.$$

EXERCISE 1.15 Consider the motion of the bar described in Exercise 1.14.
(a) Determine the material description of the velocity.
(b) Determine the spatial description of the velocity.
(c) Determine the material description of the acceleration.
(d) Determine the spatial description of the acceleration by substituting the inverse motion into the result of Part (c).
(e) Determine the spatial description of the acceleration by using Eq. (1.13).

Solution—

(a) From the motion,

$$\begin{aligned} v_1 &= \frac{\partial \hat{x}_1}{\partial t} = \frac{\partial}{\partial t}[X_1(1+t^2)] = 2X_1 t, \\ v_2 &= \frac{\partial \hat{x}_2}{\partial t} = \frac{\partial}{\partial t}(X_2) = 0, \\ v_3 &= \frac{\partial \hat{x}_3}{\partial t} = \frac{\partial}{\partial t}(X_3) = 0. \end{aligned}$$

(b) By using the inverse motion and the expressions we obtained in Part (a),

$$v_1 = 2X_1 t = 2x_1 t/(1+t^2), \quad v_2 = 0, \quad \text{and} \quad v_3 = 0.$$

(c)

$$\begin{aligned} a_1 &= \frac{\partial \hat{v}_1}{\partial t} = \frac{\partial}{\partial t}[2X_1 t] = 2X_1, \\ a_2 &= \frac{\partial \hat{v}_2}{\partial t} = 0, \\ a_3 &= \frac{\partial \hat{v}_3}{\partial t} = 0. \end{aligned}$$

(d)

$$a_1 = 2X_1 = \frac{2x_1}{1+t^2}, \quad a_2 = 0, \quad \text{and} \quad a_3 = 0.$$

(e)

$$\begin{aligned} a_1 &= \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_1} v_1 + \frac{\partial v_1}{\partial x_2} v_2 + \frac{\partial v_1}{\partial x_3} v_3 \\ &= \frac{\partial}{\partial t}\left(\frac{2x_1 t}{1+t^2}\right) + \left[\frac{\partial}{\partial x_1}\left(\frac{2x_1 t}{1+t^2}\right)\right]\left(\frac{2x_1 t}{1+t^2}\right) \\ &= \frac{2x_1}{(1+t^2)} - \frac{2x_1 t}{(1+t^2)^2}(2t) + \frac{2t}{(1+t^2)}\,\frac{2x_1 t}{(1+t^2)} \\ &= \frac{2x_1}{1+t^2}. \end{aligned}$$

EXERCISE 1.16 A 1 meter cube of material is subjected to the "shearing" motion

$$x_1 = X_1 + X_2^2 t^2, \quad x_2 = X_2, \quad x_3 = X_3.$$

(a) Determine the inverse motion of the cube.
(b) Determine the material description of the displacement.
(c) Determine the spatial description of the displacement.
(d) What is the displacement of a point at the upper right edge of the cube when $t = 2$ seconds?

Solution—
(a) Solving the motion for X_1, X_2, and X_3 in terms of x_1, x_2, x_3, and t, we obtain

$$X_1 = x_1 - X_2^2 t^2 = x_1 - x_2^2 t^2, \quad X_2 = x_2, \quad \text{and} \quad X_3 = x_3.$$

(b) By using the motion, the three components of the displacement are

$$\begin{aligned} u_1 &= x_1 - X_1 = X_1 + X_2^2 t^2 - X_1 = X_2^2 t^2, \\ u_2 &= x_2 - X_2 = X_2 - X_2 = 0, \\ u_3 &= x_3 - X_3 = X_3 - X_3 = 0. \end{aligned}$$

(c) From the results of Parts (a) and (b),

$$u_1 = X_2^2 t^2 = x_2^2 t^2, \quad u_2 = 0, \quad \text{and} \quad u_3 = 0.$$

(d) At the upper right edge of the cube, $X_1 = 1$ m and $X_2 = 1$ m. From the result of Part (b),

$$u_1 = X_2^2 t^2 = (1)^2 (2)^2 = 4 \text{ m}, \quad u_2 = 0, \quad \text{and} \quad u_3 = 0.$$

Exercise 1.17 Consider the motion of the cube described in Exercise 1.16.
(a) Determine the material description of the velocity.
(b) Determine the spatial description of the velocity.
(c) Determine the material description of the acceleration.
(d) Determine the spatial description of the acceleration by substituting the inverse motion into the result of Part (c).
(e) Determine the spatial description of the acceleration by using Eq. (1.13).

Solution—

(a) From the motion,

$$\begin{aligned} v_1 &= \frac{\partial \hat{x}_1}{\partial t} = \frac{\partial}{\partial t}[X_1 + X_2^2 t^2] = 2X_2^2 t, \\ v_2 &= \frac{\partial \hat{x}_2}{\partial t} = \frac{\partial}{\partial t}(X_2) = 0, \\ v_3 &= \frac{\partial \hat{x}_3}{\partial t} = \frac{\partial}{\partial t}(X_3) = 0. \end{aligned}$$

(b) By using the inverse motion and the expressions we obtained in Part (a),

$$v_1 = 2X_2^2 t = 2x_2^2 t, \;\; v_2 = 0, \;\; \text{and} \;\; v_3 = 0.$$

(c)

$$\begin{aligned} a_1 &= \frac{\partial \hat{v}_1}{\partial t} = \frac{\partial}{\partial t}[2X_2^2 t] = 2X_2^2, \\ a_2 &= \frac{\partial \hat{v}_2}{\partial t} = 0, \\ a_3 &= \frac{\partial \hat{v}_3}{\partial t} = 0. \end{aligned}$$

(d)

$$a_1 = 2X_2^2 = 2x_2^2, \;\; a_2 = 0, \;\; \text{and} \;\; a_3 = 0.$$

(e)

$$\begin{aligned} a_1 &= \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_1} v_1 + \frac{\partial v_1}{\partial x_2} v_2 + \frac{\partial v_1}{\partial x_3} v_3 \\ &= \frac{\partial}{\partial t}(2x_2^2 t) \\ &= 2x_2^2. \end{aligned}$$

EXERCISE 1.18 Show that the acceleration is given in terms of derivatives of the spatial description of the velocity by

$$a_k = \frac{\partial v_k}{\partial t} + \frac{\partial v_k}{\partial x_n} v_n.$$

Solution—By substituting the motion into the spatial description of the velocity field, we obtain the expression

$$v_k = v_k(x_n, t) = v_k(\hat{x}_n(X_m, t), t) = \hat{v}_k(X_m, t).$$

The acceleration is the time derivative of this expression with $\mathbf{X}$ held fixed:

$$a_k = \frac{\partial}{\partial t}\hat{v}_k(X_m, t) = \frac{\partial v_k}{\partial t} + \frac{\partial v_k}{\partial x_n}\frac{\partial \hat{x}_n}{\partial t}.$$

Because

$$\frac{\partial \hat{x}_n}{\partial t} = v_n,$$

we obtain the result

$$a_k = \frac{\partial v_k}{\partial t} + \frac{\partial v_k}{\partial x_n}v_n.$$

Exercise 1.19 A bar of material 1 meter long is subjected to the "stretching" motion

$$x_1 = X_1(1+t^2), \quad x_2 = X_2, \quad x_3 = X_3.$$

(a) Determine the Lagrangian strain tensor of the bar at time t.
(b) Use the motion to determine the length of the bar when $t = 2$ seconds.
(c) Use the Lagrangian strain tensor to determine the length of the bar when $t = 2$ seconds.

Solution—
(a) The material description of the displacement field is

$$\begin{aligned}
u_1 &= x_1 - X_1 = X_1(1+t^2) - X_1 = X_1 t^2,\\
u_2 &= x_2 - X_2 = X_2 - X_2 = 0,\\
u_3 &= x_3 - X_3 = X_3 - X_3 = 0.
\end{aligned}$$

The Lagrangian strain tensor is

$$E_{mn} = \frac{1}{2}\left(\frac{\partial \hat{u}_m}{\partial X_n} + \frac{\partial \hat{u}_n}{\partial X_m} + \frac{\partial \hat{u}_k}{\partial X_m}\frac{\partial \hat{u}_k}{\partial X_n}\right)$$

The only non-zero component is

$$\begin{aligned}
E_{11} &= \frac{1}{2}\left(\frac{\partial \hat{u}_1}{\partial X_1} + \frac{\partial \hat{u}_1}{\partial X_1} + \frac{\partial \hat{u}_1}{\partial X_1}\frac{\partial \hat{u}_1}{\partial X_1}\right)\\
&= \tfrac{1}{2}[t^2 + t^2 + (t^2)(t^2)]\\
&= t^2 + \tfrac{1}{2}t^4.
\end{aligned}$$

(b) From the motion we can see that at $t = 2$ seconds the left end of the bar is located at

$$x_1 = X_1(1+t^2) = (0)[1+(2)^2] = 0$$

and the right end of the bar is located at

$$x_1 = X_1(1 + t^2) = (1)[1 + (2)^2] = 5 \text{ meters.}$$

Therefore the length of the bar is 5 meters.

(c) From Part (a), at $t = 2$ seconds

$$E_{11} = t^2 + \tfrac{1}{2}t^4 = (2)^2 + \tfrac{1}{2}(2)^4 = 12.$$

We can determine the longitudinal strain ϵ of the material from the equation

$$2\epsilon + \epsilon^2 = 2E_{mn}n_m n_n.$$

Let us determine the longitudinal strain in the x_1 direction by setting $n_1 = 1$, $n_2 = 0$, and $n_3 = 0$:

$$2\epsilon + \epsilon^2 = 2E_{11}(1)(1) = 24.$$

The solution of this equation is $\epsilon = 4$. (The root $\epsilon = -6$ can be ignored since the longitudinal strain must be greater than -1.)

At $t = 2$ seconds, the length ds of an element of the bar parallel to the x_1 axis is related to its length dS in the reference state by

$$\frac{ds - dS}{dS} = \epsilon = 4.$$

Therefore

$$ds = (1 + 4)\, dS = 5\, dS,$$

and the length of the bar at $t = 2$ seconds is

$$\int_0^1 5\, dS = 5 \, \text{meters.}$$

EXERCISE 1.20 An object consists of a 1 meter cube in the reference state. At time t, the value of the linear strain tensor at each point of the object is

$$[E_{km}] = \begin{bmatrix} 0.001 & -0.001 & 0 \\ -0.001 & 0.002 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) What is the length of the edge AB at time t?

(b) What is the length of the diagonal AC at time t?

(c) What is the angle θ at time t?

Solution—

(a) In linear elasticity, the longitudinal strain ϵ is given by $\epsilon = E_{mn}n_m n_n$. To determine the longitudinal strain in the direction of the edge AB, let $n_1 = 1$, $n_2 = 0$, and $n_3 = 0$:

$$\epsilon = E_{11}(1)(1) = E_{11} = 0.001.$$

Therefore at time t the length ds of an element of the edge AB is related to its length dS in the reference state by

$$\frac{ds - dS}{dS} = \epsilon = 0.001,$$

so

$$ds = (1 + 0.001)\, dS$$

The length of the edge AB at time t is

$$\int_0^1 (1 + 0.001)\, dS = 1.001\, \text{meter}.$$

(b) To determine the longitudinal strain in the direction of the diagonal AC, we let $n_1 = 1/\sqrt{2}$, $n_2 = 1/\sqrt{2}$, and $n_3 = 0$:

$$\begin{aligned}\epsilon &= E_{11}n_1n_1 + E_{12}n_1n_2 + E_{21}n_2n_1 + E_{22}n_2n_2\\ &= (0.001)(1/\sqrt{2})(1/\sqrt{2}) + 2(-0.001)(1/\sqrt{2})(1/\sqrt{2}) + (0.002)(1/\sqrt{2})(1/\sqrt{2})\\ &= 0.0005.\end{aligned}$$

Therefore at time t the length ds of an element of the diagonal AC is related to its length dS in the reference state by

$$\frac{ds - dS}{dS} = \epsilon = 0.0005,$$

so

$$ds = (1 + 0.0005)\, dS$$

The length of the diagonal AC at time t is

$$\int_0^{\sqrt{2}} (1 + 0.001)\, dS = 1.0005\sqrt{2}\, \text{meter}.$$

(c) At time t, the angle between line elements originally parallel to the x_1 and x_2 axes is given in terms of the linear strain tensor by $\pi/2 - 2E_{12}$. Therefore the angle θ at time t is

$$\theta = \pi/2 + 2E_{12} = (\pi/2 - 0.002)\, \text{rad}.$$

Exercise 1.21 Show that in linear elasticity,

$$\frac{\partial \hat{u}_k}{\partial X_m} = \frac{\partial u_k}{\partial x_m}.$$

Discussion—Use the chain rule to write

$$\frac{\partial \hat{u}_k}{\partial X_m} = \frac{\partial u_k}{\partial x_n}\frac{\partial \hat{x}_n}{\partial X_m},$$

and use Eq. (1.5).

Solution—The displacement is

$$u_k = u_k(x_n, t) = u_k(\hat{x}_n(X_m, t), t) = \hat{u}_k(X_m, t).$$

Therefore

$$\frac{\partial \hat{u}_k}{\partial X_m} = \frac{\partial u_k}{\partial x_n}\frac{\partial \hat{x}_n}{\partial X_m}. \quad (*)$$

From the relation

$$x_n = X_n + u_n$$

we obtain the result

$$\frac{\partial \hat{x}_n}{\partial X_m} = \delta_{nm} + \frac{\partial \hat{u}_n}{\partial X_m}.$$

Substituting this into Eq. (∗) gives

$$\frac{\partial \hat{u}_k}{\partial X_m} = \frac{\partial u_k}{\partial x_m} + \frac{\partial u_k}{\partial x_n}\frac{\partial \hat{u}_n}{\partial X_m}. \quad (**)$$

This equation contains the derivative $\partial \hat{u}_n/\partial X_m$ on the right side. By changing indices, we can write the equation as

$$\frac{\partial \hat{u}_n}{\partial X_m} = \frac{\partial u_n}{\partial x_m} + \frac{\partial u_n}{\partial x_r}\frac{\partial \hat{u}_r}{\partial X_m}.$$

We can substitute this expression into the right side of Eq. (∗∗). Continuing in this way, we obtain the infinite series

$$\frac{\partial \hat{u}_k}{\partial X_m} = \frac{\partial u_k}{\partial x_m} + \frac{\partial u_k}{\partial x_n}\frac{\partial u_n}{\partial x_m} + \frac{\partial u_k}{\partial x_n}\frac{\partial u_n}{\partial x_r}\frac{\partial u_r}{\partial x_m} + \cdots$$

In linear elasticity, we neglect higher order terms in the derivatives of the displacement and this equation reduces to

$$\frac{\partial \hat{u}_k}{\partial X_m} = \frac{\partial u_k}{\partial x_m}.$$

Exercise 1.22 Consider Eq. (1.27):

$$2E_{12} = (1 + 2E_{11})^{\frac{1}{2}}(1 + 2E_{22})^{\frac{1}{2}} \cos(\pi/2 - \gamma_{12}).$$

In linear elasticity, the terms E_{11}, E_{22}, E_{12}, and the shear strain γ_{12} are small. Show that in linear elasticity this relation reduces to

$$\gamma_{12} = 2E_{12}.$$

Solution—Let us expand the term $(1 + 2E_{11})^{\frac{1}{2}}$ in a Taylor series in terms of the small term E_{11}:

$$\begin{aligned}(1 + 2E_{11})^{\frac{1}{2}} &= [(1 + 2E_{11})^{\frac{1}{2}}]_{E_{11}=0} + \left\{ \frac{d}{dE_{11}}[(1 + 2E_{11})^{\frac{1}{2}}] \right\}_{E_{11}=0} E_{11} + \cdots \\ &= 1 + E_{11} + \cdots\end{aligned}$$

Next, we expand the term $\cos(\pi/2 - \gamma_{12})$ in a Taylor series in terms of the small term γ_{12}:

$$\begin{aligned}\cos(\pi/2 - \gamma_{12}) &= [\cos(\pi/2 - \gamma_{12})]_{\gamma_{12}=0} + \left\{ \frac{d}{d\gamma_{11}}[\cos(\pi/2 - \gamma_{12})] \right\}_{\gamma_{12}=0} \gamma_{12} + \cdots \\ &= \gamma_{12} + \cdots\end{aligned}$$

Therefore the equation can be written

$$2E_{12} = (1 + E_{11} + \cdots)(1 + E_{22} + \cdots)(\gamma_{12} + \cdots).$$

If we neglect higher order terms, the equation reduces to

$$2E_{12} = \gamma_{12}.$$

Exercise 1.23 By using Eqs. (1.30) and (1.32), show that in linear elasticity the density ρ is related to the density ρ_0 in the reference state by Eq. (1.33).

Solution—From the equations

$$\rho_0 \, dV_0 = \rho \, dV$$

and

$$dV = (1 + E_{kk})\ dV_0,$$

we see that

$$\frac{\rho}{\rho_0} = (1 + E_{kk})^{-1}.$$

We expand the expression on the right in terms of the small term E_{kk}:

$$\begin{aligned}(1 + E_{kk})^{-1} &= [(1 + E_{kk})^{-1}]_{E_{kk}=0} + \left\{\frac{d}{dE_{kk}}[(1 + E_{kk})^{-1}]\right\}_{E_{kk}=0} E_{kk} + \cdots \\ &= 1 - E_{kk} + \cdots\end{aligned}$$

Therefore in linear elasticity

$$\frac{\rho}{\rho_0} = 1 - E_{kk}.$$

Exercise 1.24 A stationary square bar is acted upon by uniform normal tractions σ_0 at the ends (Fig. 1). As a result, the components of the stress tensor at each point of the material are

$$[T_{km}] = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

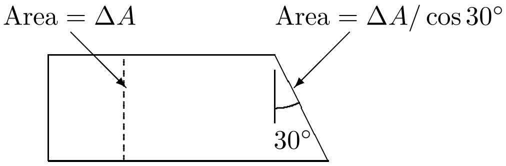
(a) Determine the magnitudes of the normal and shear stresses acting on the plane shown in Fig. 2 by writing equilibrium equations for the free body diagram shown.

(b) Determine the magnitudes of the normal and shear stresses acting on the plane shown in Fig. 2 by using Eq. (1.36).

Solution—

(a) The normal stress σ and shear stress τ acting on the plane are shown:

If we denote the cross sectional area of the bar by ΔA, the area of the plane the stesses σ and τ act on is $\Delta A/\cos 30°$:



The sum of the forces in the direction normal to the plane must equal zero:

$$\sigma(\Delta A/\cos 30°) - (\sigma_0 \Delta A)\cos 30° = 0.$$

Therefore $|\sigma| = \sigma_0 \cos^2 30°$.

The sum of the forces in the direction parallel to the plane must equal zero:

$$\tau(\Delta A/\cos 30°) + (\sigma_0 \Delta A)\sin 30° = 0.$$

From this equation we see that $|\tau| = \sigma_0 \sin 30° \cos 30°$.

(b) In terms of the coordinate system shown, the components of the unit vector $\mathbf{n}$ normal to the plane are $n_1 = \cos 30°$, $n_2 = \sin 30°$, and $n_3 = 0$. Therefore the components of the traction vector acting on the plane are

$$\begin{aligned} t_1 &= T_{1m}n_m = T_{11}n_1 + T_{12}n_2 + T_{13}n_3 = \sigma_0 \cos 30°, \\ t_2 &= T_{2m}n_m = T_{21}n_1 + T_{22}n_2 + T_{23}n_3 = \quad 0, \\ t_3 &= T_{3m}n_m = T_{31}n_1 + T_{32}n_2 + T_{33}n_3 = \quad 0. \end{aligned}$$

The normal stress acting on the plane is

$$\sigma = \mathbf{t}\cdot\mathbf{n} = t_1 n_1 + t_2 n_2 + t_3 n_3 = \sigma_0 \cos^2 30°.$$

The shear stress is the component of the vector $\mathbf{t}$ parallel to the plane. Its magnitude is

$$|t_1 \sin 30° - t_2 \cos 30°| = \sigma_0 \sin 30° \cos 30°.$$

EXERCISE 1.25 The components of the stress tensor at each point of the cube of material shown in Fig. 1 are

$$[T_{km}] = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix}.$$

Determine the magnitudes of the normal and shear stresses acting on the plane shown in Fig. 2.

Solution—The components of the unit vector normal to the plane are $n_1 = \cos\theta$, $n_2 = \sin\theta$, and $n_3 = 0$. The components of the traction vector acting on the plane are

$$\begin{aligned} t_1 &= T_{1m}n_m = T_{11}n_1 + T_{12}n_2 + T_{13}n_3 = T_{11}\cos\theta + T_{12}\sin\theta, \\ t_2 &= T_{2m}n_m = T_{21}n_1 + T_{22}n_2 + T_{23}n_3 = T_{12}\cos\theta + T_{22}\sin\theta, \\ t_3 &= T_{3m}n_m = T_{31}n_1 + T_{32}n_2 + T_{33}n_3 = \qquad 0. \end{aligned}$$

The normal stress acting on the plane is

$$\begin{aligned} \sigma = \mathbf{t}\cdot\mathbf{n} &= t_1n_1 + t_2n_2 + t_3n_3 \\ &= (T_{11}\cos\theta + T_{12}\sin\theta)(\cos\theta) + (T_{12}\cos\theta + T_{22}\sin\theta)(\sin\theta) \\ &= T_{11}\cos^2\theta + 2T_{12}\sin\theta\cos\theta + T_{22}\sin^2\theta. \end{aligned}$$

The shear stress is the component of the vector $\mathbf{t}$ parallel to the plane. Its magnitude is

$$|t_1\sin\theta - t_2\cos\theta| = |T_{12}(\cos^2\theta - \sin^2\theta) - (T_{11} - T_{22})\sin\theta\cos\theta|.$$

EXERCISE 1.26 The components of the stress tensor at each point of the cube of material shown in Fig. 1 are

$$[T_{km}] = \begin{bmatrix} 100 & -100 & 0 \\ -100 & 100 & 0 \\ 0 & 0 & 300 \end{bmatrix} \text{ Pa.}$$

A pascal (Pa) is 1 newton/meter^2. Determine the magnitudes of the normal and shear stresses acting on the plane shown in Fig. 2.

Solution—The unit vector perpendicular to the plane is

$$\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3).$$

The traction vector acting on the plane is

$$\begin{aligned} \{t_k\} = [T_{km}]\{n_m\} &= \begin{bmatrix} 100 & -100 & 0 \\ -100 & 100 & 0 \\ 0 & 0 & 300 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 100\sqrt{3} \end{bmatrix} \text{ Pa.} \end{aligned}$$

The normal stress is

$$\sigma = t_1 n_1 + t_2 n_2 + t_3 n_3 = (100\sqrt{3})(1/\sqrt{3}) = 100 \text{ Pa}.$$

The component of **t** parallel to the surface is the shear stress:

$$\{t_k - \sigma n_k\} = \begin{bmatrix} 0 \\ 0 \\ 100\sqrt{3} \end{bmatrix} - 100 \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \frac{100}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

The magnitude of this vector is $\tau = 100\sqrt{2}$ Pa.

EXERCISE 1.27 The Lamé constants of an isotropic material are $\lambda = 1.15(10^{11})$ Pa and $\mu = 0.77(10^{11})$ Pa. The components of the strain tensor at a point P in the material are

$$[E_{km}] = \begin{bmatrix} 0.001 & -0.001 & 0 \\ -0.001 & 0.001 & 0 \\ 0 & 0 & 0.002 \end{bmatrix}.$$

Determine the components of the stress tensor T_{km} at point P.

Solution—From the stress-strain relation, Eq. (1.43), the stress component T_{11} is

$$\begin{aligned} T_{11} = \lambda\delta_{11}(E_{11} + E_{22} + E_{33}) + 2\mu E_{11} &= (\lambda + 2\mu)E_{11} + \lambda(E_{22} + E_{33}) \\ &= [1.15(10^{11}) + 2(0.77)(10^{11})](0.001) \\ &\quad +1.15(10^{11})(0.001 + 0.002) \\ &= 6.14(10^8) \text{ Pa}. \end{aligned}$$

The stress component T_{12} is

$$\begin{aligned} T_{12} = \lambda\delta_{12}(E_{11} + E_{22} + E_{33}) + 2\mu E_{12} &= 2\mu E_{12} \\ &= 2(0.77)(10^{11})(-0.001) \\ &= -1.54(10^8) \text{ Pa}. \end{aligned}$$

Continuing in this way, the stress tensor is

$$[T_{km}] = \begin{bmatrix} 6.14(10^8) & -1.54(10^8) & 0 \\ -1.54(10^8) & 6.14(10^8) & 0 \\ 0 & 0 & 7.68(10^8) \end{bmatrix} \text{ Pa}.$$

EXERCISE 1.28 A bar is 200 mm long and has a square 50 mm × 50 mm cross section in the unloaded state. The bar consists of isotropic material with Lamé constants $\lambda = 4.5(10^{10})$ Pa and $\mu = 3.0(10^{10})$ Pa. The ends of the bar are subjected to a uniform normal traction $\sigma_0 = 2.0(10^8)$ Pa. As a result, the components of the stress tensor at each point of the material are

$$[T_{km}] = \begin{bmatrix} 2.0(10^8) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ Pa.}$$

(a) Determine the length of the bar in the loaded state.
(b) Determine the dimensions of the square cross section of the bar in the loaded state.

Solution—Because the stress components are known, the stress-strain relation, Eq. (1.43), can be solved for the components of the strain. The results are

$$[E_{km}] = \begin{bmatrix} 2.564(10^{-3}) & 0 & 0 \\ 0 & -7.692(10^{-4}) & 0 \\ 0 & 0 & -7.692(10^{-4}) \end{bmatrix}.$$

The length of the loaded bar is

$$(200mm)(1 + E_{11}) = (200mm)[1 + 2.564(10^{-3})] = 200.513\,\text{mm}.$$

The height and width of the loaded bar are

$$(50mm)(1 + E_{22}) = (50mm)[1 - 7.692(10^{-4})] = 4.9962\,\text{mm},$$

so the dimensions of the cross section are 4.9962 mm × 4.9962 mm.

EXERCISE 1.29 The ends of a bar of isotropic material are subjected to a uniform normal traction T_{11}. As a result, the components of the stress tensor at each point of the material are

$$[T_{km}] = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) The ratio E of the stress T_{11} to the longitudinal strain E_{11} in the x_1 direction,

$$E = \frac{T_{11}}{E_{11}},$$

is called the *Young's modulus*, or *modulus of elasticity* of the material. Show that the Young's modulus is related to the Lamé constants by

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

(b) The ratio

$$\nu = -\frac{E_{22}}{E_{11}}$$

is called the *Poisson's ratio* of the material. Show that the Poisson's ratio is related to the Lamé constants by

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

Solution—From the stress-strain relation, Eq. (1.43),

$$T_{11} = \lambda(E_{11} + E_{22} + E_{33}) + 2\lambda E_{11} = (\lambda + 2\mu)E_{11} + 2\lambda E_{22}, \quad (*)$$

where we have used the fact that $E_{22} = E_{33}$, and

$$T_{22} = 0 = \lambda(E_{11} + E_{22} + E_{33}) + 2\lambda E_{22} = \lambda E_{11} + 2(\lambda + \mu)E_{22}.$$

From the latter equation,

$$E_{22} = \frac{-\lambda}{2(\lambda + \mu)}E_{11}. \quad (**)$$

Substituting this relation into Eq. (*) yields

$$T_{11} = (\lambda + 2\mu)E_{11} - \frac{2\lambda^2}{2(\lambda + \mu)}E_{11},$$

So Young's modulus is

$$E = \frac{T_{11}}{E_{11}} = \lambda + 2\mu - \frac{\lambda^2}{\lambda + \mu} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu},$$

and from Eq. (**), Poisson's ratio is

$$\nu = -\frac{E_{22}}{E_{11}} = \frac{\lambda}{2(\lambda + \mu)}.$$

Exercise 1.30 The Young's modulus and Poisson's ratio of an elastic material are defined in Exercise 1.29. Show that the Lamé constants of an isotropic material are given in terms of the Young's modulus and Poisson's ratio of the material by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

Solution—The relations derived in Exercise 1.29 are

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad (*)$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)}. \qquad (**)$$

Solving Eq. (**) for μ,

$$\mu = \frac{(1-2\nu)}{2\nu}\lambda. \quad (***)$$

Substituting this expression into Eq. (*) and simplifying yields

$$E = \frac{(1+\nu)(1-2\nu)}{\nu}\lambda,$$

so

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}.$$

Then substituting this result into Eq. (***) gives the result

$$\mu = \frac{E}{2(1+\nu)}.$$

Exercise 1.31 Show that the strain components of an isotropic linear elastic material are given in terms of the stress components by

$$E_{km} = \frac{1}{2\mu}T_{km} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\delta_{km}T_{jj}.$$

Solution—The stress-strain relation is

$$T_{km} = \lambda\delta_{km}E_{jj} + 2\mu E_{km}. \quad (*)$$

Setting $m = k$,

$$T_{kk} = 3\lambda E_{jj} + 2\mu E_{kk} = (3\lambda + 2\mu)E_{kk},$$

and solving for E_{kk},

$$E_{kk} = \frac{1}{3\lambda + 2\mu} T_{kk}. \ \ (**)$$

Solving Eq. (*) for E_{km},

$$E_{km} = \frac{1}{2\mu} T_{km} - \frac{\lambda}{2\mu} \delta_{km} E_{jj}.$$

and substituting the expression (**) for E_{jj} yields

$$E_{km} = \frac{1}{2\mu} T_{km} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \delta_{km} T_{jj}.$$

Exercise 1.32 By using the Gauss theorem, Eq. (1.3), the transport theorem, Eq. (1.45), and the equation of conservation of mass, Eq. (1.46), show that Eq. (1.48),

$$\frac{d}{dt}\int_V \rho v_m \, dV = \int_S T_{mk} n_k \, dS + \int_V b_m \, dV,$$

can be expressed in the form

$$\int_V \left(\rho a_m - \frac{\partial T_{mk}}{\partial x_k} - b_m \right) dV = 0,$$

where a_m is the acceleration of the material.

Solution—Applying the transport theorem to the term on the left,

$$\begin{aligned} \frac{d}{dt}\int_V \rho v_m \, dV &= \int_V \left[\frac{\partial}{\partial t}(\rho v_m) + \frac{\partial}{\partial x_r}(\rho v_m v_r) \right] dV \\ &= \int_V \left\{ v_m \underbrace{\left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_r)}{\partial x_r} \right]}_{=0} + \rho \underbrace{\left[\frac{\partial v_m}{\partial t} + \frac{\partial v_m}{\partial x_r} v_r \right]}_{=a_m} \right\} dV \\ &= \int_V \rho a_m \, dV. \end{aligned}$$

Applying the Gauss theorem to the first term on the right,

$$\int_S T_{mk} n_k \, dS = \int_V \frac{\partial T_{mk}}{\partial x_k} \, dV.$$

Substituting these two expressions into the equation gives the desired equation.

EXERCISE 1.33 Show that the postulate of balance of angular momentum for a material

$$\frac{d}{dt}\int_V \mathbf{x}\times\rho\mathbf{v}\,dV = \int_S \mathbf{x}\times\mathbf{t}\,dS + \int_V \mathbf{x}\times\mathbf{b}\,dV$$

implies that the stress tensor is symmetric:

$$T_{km} = T_{mk}.$$

Solution—In terms of indices, the postulate is

$$\frac{d}{dt}\int_V e_{kmn}x_k\rho v_m\,dV = \int_S e_{kmn}x_k t_m\,dS + \int_V e_{kmn}x_k b_m\,dV. \quad (*)$$

Using the transport theorem, the left side of this equation is

$$\begin{aligned}
\frac{d}{dt}\int_V e_{kmn}x_k\rho v_m\,dV &= \int_V \left[\frac{\partial}{\partial t}(e_{kmn}x_k\rho v_m) + \frac{\partial}{\partial x_r}(e_{kmn}x_k\rho v_m v_r)\right] dV \\
&= \int_V \left[e_{kmn}x_k\left(\frac{\partial\rho}{\partial t}v_m + \rho\frac{\partial v_m}{\partial t}\right) + e_{kmn}\delta_{kr}\rho v_m v_r \right. \\
&\quad \left. + e_{kmn}x_k\frac{\partial(\rho v_r)}{\partial x_r}v_m + e_{kmn}x_k\rho v_r\frac{\partial v_m}{\partial x_r}\right] dV \\
&= \int_V \left\{e_{kmn}x_k v_m\left[\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_r)}{\partial x_r}\right] + e_{kmn}x_k\rho\left[\frac{\partial v_m}{\partial t} + \frac{\partial v_m}{\partial x_r}v_r\right]\right\} dV \\
&= \int_V e_{kmn}x_k\rho a_m\,dV,
\end{aligned}$$

where we have observed that $e_{kmn}\delta_{kr}\rho v_m v_r = e_{kmn}\rho v_m v_k = 0$.

The first term on the right side of Eq. $(*)$ is

$$\begin{aligned}
\int_S e_{kmn}x_k t_m\,dS &= \int_S e_{kmn}x_k T_{mr}n_r\,dS \\
&= \int_V e_{kmn}\frac{\partial}{\partial x_r}(x_k T_{mr})\,dV \\
&= \int_V \left(e_{kmn}\delta_{kr}T_{mr} + e_{kmn}x_k\frac{\partial T_{mr}}{\partial x_r}\right) dV.
\end{aligned}$$

Therefore we can write Eq. (∗) as

$$\int_V \left[e_{kmn} x_k \left(\rho a_m - \frac{\partial T_{mr}}{\partial x_r} - b_m \right) - e_{kmn} T_{mk} \right] dV = 0.$$

The term in parentheses vanishes due to the balance of linear momentum, resulting in the equation

$$\int_V e_{kmn} T_{mk}\, dV = 0.$$

Thus $e_{kmn} T_{mk} = 0$ at each point. This equation must hold for each value of n. For $n = 1$, it states that

$$e_{231} T_{32} + e_{321} T_{23} = T_{32} - T_{23} = 0.$$

Therefore $T_{32} = T_{23}$. By letting $n = 2$ and $n = 3$, we find that $T_{km} = T_{mk}$.

Exercise 1.34 For a stationary material, the postulate of balance of energy is

$$\frac{d}{dt} \int_V \rho e\, dV = - \int_S q_j n_j\, dS,$$

where V is a material volume with surface S. The term e is the *internal energy* and q_j is the *heat flux vector*. Suppose that the internal energy and the heat flux vector are related to the absolute temperature T of the material by the equations

$$e = cT, \qquad q_j = -k \frac{\partial T}{\partial x_j},$$

where the specific heat c and the thermal conductivity k are constants. Show that the absolute temperature is governed by the heat transfer equation

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c} \frac{\partial^2 T}{\partial x_j \partial x_j}.$$

Solution—Because the material is stationary, $v_m = 0$ and $\rho = \rho_0$. Therefore, applying the transport theorem to the term on the left gives

$$\frac{d}{dt} \int_V \rho e\, dV = \int_V \rho_0 \frac{\partial e}{\partial t}\, dV = \int_V \rho_0 c \frac{\partial T}{\partial t}\, dV.$$

Using the Gauss theorem, the term on the right is

$$- \int_S q_j n_j\, dS = - \int_V \frac{\partial q_j}{\partial x_j}\, dV = \int_V k \frac{\partial^2 T}{\partial x_j \partial x_j}\, dV.$$

The balance of energy postulate becomes

$$\int_V \left[\rho_0 c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x_j \partial x_j}\right] dV,$$

which yields the heat transfer equation.

EXERCISE 1.35 By substituting the Helmholtz decomposition $\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi}$ into the equation of balance of linear momentum (1.56), show that the equation of balance of linear momentum can be written in the form

$$\nabla \left[\rho_0 \frac{\partial^2 \phi}{\partial t^2} - (\lambda + 2\mu)\nabla^2 \phi\right] + \nabla \times \left[\rho_0 \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} - \mu \nabla^2 \boldsymbol{\psi}\right] = 0.$$

Solution—Substituting the Helmholtz decomposition into Eq. (1.56), it becomes

$$\rho_0 \frac{\partial^2}{\partial t^2}(\nabla\phi + \nabla \times \boldsymbol{\psi}) = (\lambda + 2\mu)\nabla(\nabla^2 \phi) - \mu \nabla \times [\nabla \times (\nabla \times \boldsymbol{\psi})],$$

where we have used the identies $\nabla \cdot (\nabla \times \boldsymbol{\psi}) = 0$ and $\nabla \times \nabla\phi = 0$. This equation can be written

$$\nabla \left[\rho_0 \frac{\partial^2 \phi}{\partial t^2} - (\lambda + 2\mu)\nabla^2 \phi\right] + \nabla \times \left[\rho_0 \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} + \mu \nabla \times (\nabla \times \boldsymbol{\psi})\right] = 0. \quad (*)$$

Taking the curl of the definition

$$\nabla^2 \boldsymbol{\psi} = \nabla(\nabla \cdot \boldsymbol{\psi}) - \nabla \times (\nabla \times \boldsymbol{\psi})$$

gives

$$\nabla \times \nabla^2 \boldsymbol{\psi} = 0 - \nabla \times [\nabla \times (\nabla \times \boldsymbol{\psi})].$$

Using this expression in Eq. (*) gives the desired result.

Exercise 1.36 An acoustic medium is defined in Section 1.7. Show that the density ρ of an acoustic medium is governed by the wave equation

$$\frac{\partial^2 \rho}{\partial t^2} = \alpha^2 \nabla^2 \rho.$$

Solution—The motion of an acoustic medium is governed by the equation

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \alpha^2 \nabla(\nabla \cdot \mathbf{u}),$$

or in terms of indices,

$$\frac{\partial^2 u_k}{\partial t^2} = \alpha^2 \frac{\partial^2 u_m}{\partial x_m \partial x_k} = \alpha^2 \frac{\partial E_{mm}}{\partial x_k}.$$

Let us take the divergence of this vector equation by taking the derivative with respect to x_k:

$$\frac{\partial^2 E_{kk}}{\partial t^2} = \alpha^2 \frac{\partial^2 E_{kk}}{\partial x_m \partial x_m}. \quad (*)$$

Notice that we interchanged the indices on the right side of the equation.

The density ρ is given in terms of the strain by

$$\rho = \rho_0 (1 - E_{kk}).$$

From this equation we see that

$$\frac{\partial^2 \rho}{\partial t^2} = -\rho_0 \frac{\partial^2 E_{kk}}{\partial t^2}, \qquad \frac{\partial^2 \rho}{\partial x_m \partial x_m} = -\rho_0 \frac{\partial^2 E_{kk}}{\partial x_m \partial x_m}.$$

With these results we can write Eq. (*) in the form

$$\frac{\partial^2 \rho}{\partial t^2} = \alpha^2 \frac{\partial^2 \rho}{\partial x_m \partial x_m},$$

or

$$\frac{\partial^2 \rho}{\partial t^2} = \alpha^2 \nabla^2 \rho.$$

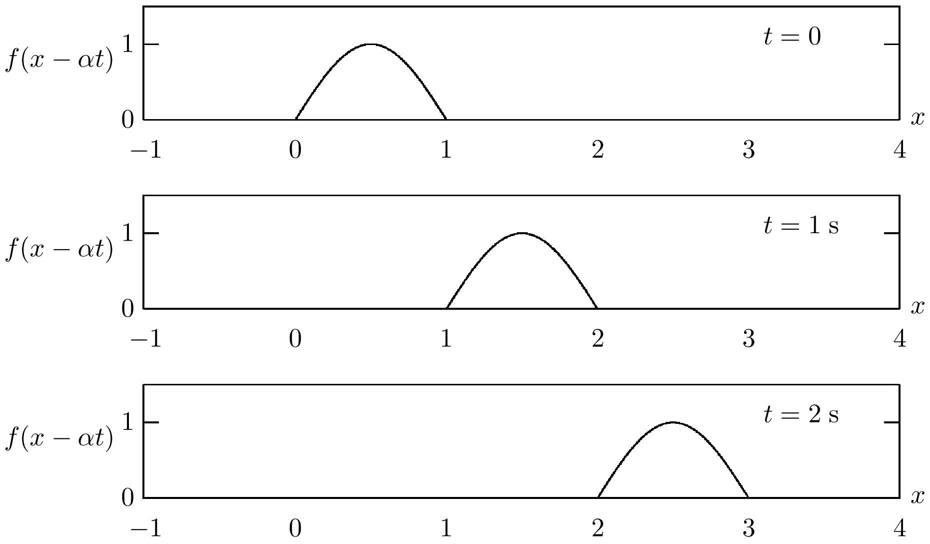
Chapter 2. One-Dimensional Waves

Exercise 2.1 A function $f(x)$ is defined by

$$\begin{aligned} x < 0 &\quad f(x) = 0, \\ 0 \le x \le 1 &\quad f(x) = \sin(\pi x), \\ x > 1 &\quad f(x) = 0. \end{aligned}$$

Plot the function $f(\xi) = f(x - \alpha t)$ as a function of x for $t = 0$, $t = 1$ s, and $t = 2$ s if $\alpha = 1$.

Solution—



Exercise 2.2 Show that as time increases, the graph of the function $g(\eta) = g(x + \alpha t)$ as a function of x translates in the negative x direction with constant velocity α.

Solution—Modify the argument used on page 50 to show that the graph of $f(\xi)$ translates in the positive x direction with constant velocity α.

Exercise 2.3 By using the chain rule, show that the second partial derivative of $u = u(x, t)$ with respect to x can be expressed in terms of partial derivatives of $u = \tilde{u}(\xi, \eta)$ by

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \tilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{u}}{\partial \eta^2}.$$

Solution—The dependent variable u is

$$u = u(x,t) = \tilde{u}(\xi,\eta),$$

where $\xi = x - \alpha t$ and $\eta = x + \alpha t$. Applying the chain rule, the partial derivative of u with respect to x is

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial \tilde{u}}{\partial \eta} \\ &= \frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \tilde{u}}{\partial \eta}.\end{aligned}$$

Then by applying the chain rule again,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \tilde{u}}{\partial \eta}\right) \\ &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}\left(\frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \tilde{u}}{\partial \eta}\right) \\ &\quad + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\left(\frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \tilde{u}}{\partial \eta}\right) \\ &= \left(\frac{\partial^2 \tilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{u}}{\partial \eta^2}\right).\end{aligned}$$

Exercise 2.4 Show that the D'Alembert solution

$$u = f(\xi) + g(\eta)$$

is a solution of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}.$$

Solution—Because the equation is linear, it is sufficient to show that $f(\xi)$ and $g(\eta)$ are each solutions. The partial derivative of $f(\xi)$ with respect to t is

$$\frac{\partial f}{\partial t} = \frac{\partial \xi}{\partial t}\frac{df}{d\xi} = -\alpha \frac{df}{d\xi}.$$

Then by applying the chain rule again,

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(-\alpha\frac{df}{d\xi}\right) = \frac{\partial\xi}{\partial t}\frac{d}{d\xi}\left(-\alpha\frac{df}{d\xi}\right) = \alpha^2\frac{d^2 f}{d\xi^2}.$$

It can be shown in the same way that

$$\frac{\partial^2 f}{\partial x^2} = \frac{d^2 f}{d\xi^2}.$$

Thus $f(\xi)$ is a solution of the one-dimensional wave equation. The same procedure can be used to show that $g(\eta)$ is a solution.

Exercise 2.5 Consider the expression

$$u = Ae^{i(kx - \omega t)}.$$

What conditions must the constants A, k, and ω satisfy in order for this expression to be a solution of the one-dimensional wave equation, Eq. (2.1)?

Solution—Substituting the expression into the one-dimensional wave equation, the result can be written

$$A(\omega^2 - \alpha^2 k^2)e^{i(kx - \omega t)} = 0.$$

Except for the trivial solutions $A = 0$ and $\omega = k = 0$, the expression is a solution only if the constants ω and k satisfy the relation $\omega^2/k^2 = \alpha^2$. In that case, no constraint is placed on the constant A.

Exercise 2.6 Consider the first-order partial differential equation

$$\frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} = 0,$$

where α is a constant. By expressing it in terms of the independent variables $\xi = x - \alpha t$ and $\eta = x + \alpha t$, show that its general solution is

$$u = f(\xi),$$

where f is an arbitrary twice-differentiable function.

Solution—Let $u = u(x,t) = \tilde{u}(\xi,\eta)$. Then

$$\frac{\partial u}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial \tilde{u}}{\partial \eta} = -\alpha\frac{\partial \tilde{u}}{\partial \xi} + \alpha\frac{\partial \tilde{u}}{\partial \eta}$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial \tilde{u}}{\partial \eta} = \frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \tilde{u}}{\partial \eta}.$$

Substituting these expressions into the first-order equation yields the equation

$$\frac{\partial \tilde{u}}{\partial \eta} = 0.$$

Integrating yields the general solution

$$\tilde{u} = f(\xi).$$

Exercise 2.7 A particular type of steel has Lamé constants $\lambda = 1.15 \times 10^{11}$ Pa and $\mu = 0.77 \times 10^{11}$ Pa and density $\rho_0 = 7800$ kg/m^3. Determine (a) the compressional wave velocity α; (b) the shear wave velocity β.

Solution—(a):

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho_0}} = \sqrt{\frac{(1.15 \times 10^{11}) + 2(0.77 \times 10^{11})}{7800}} = 5870 \text{ m/s}.$$

(b):

$$\beta = \sqrt{\frac{\mu}{\rho_0}} = \sqrt{\frac{0.77 \times 10^{11}}{7800}} = 3140 \text{ m/s}.$$

Exercise 2.8 Consider the steel described in Exercise 2.7. A compressional wave propagates through the material. The displacement field is

$$u_1 = 0.001 \sin[2(x_1 - \alpha t)] \text{ m}, \quad u_2 = 0, \quad u_3 = 0.$$

Determine the maximum normal stress T_{11} caused by the wave.

Solution—From the stress-strain relation

$$T_{km} = \lambda \delta_{km} E_{jj} + 2\mu E_{km},$$

The stress component T_{11} is

$$T_{11} = \lambda(E_{11} + E_{22} + E_{33}) + 2\mu E_{11}.$$

For the given displacement field, E_{22} and E_{33} are zero and

$$E_{11} = \frac{\partial u_1}{\partial x_1} = (0.001)(2)\cos[2(x_1 - \alpha t)].$$

The maximum value of E_{11} is 0.002, so the maximum value of T_{11} is

$$T_{11} = (\lambda + 2\mu)E_{11} = [1.15 \times 10^{11} + 2(0.77 \times 10^{11}](0.002) = 538 \text{ MPa},$$

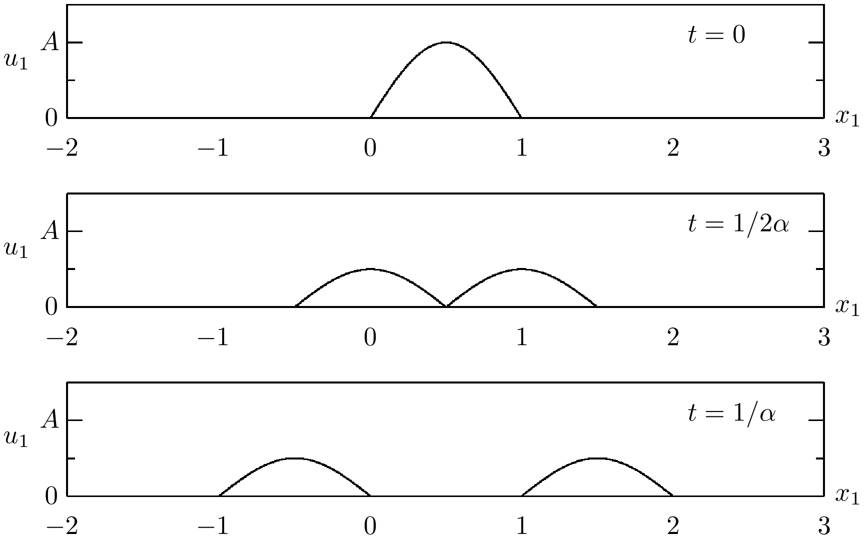
where MPa denotes a megapascal, or 10^6 pascals.

EXERCISE 2.9 Suppose that at $t = 0$ the velocity of an unbounded elastic material is zero and its displacement field is described by $u_1(x_1, 0) = p(x_1)$, where

$$\begin{aligned} x_1 < 0 \quad & p(x_1) = 0, \\ 0 \le x_1 \le 1 \quad & p(x_1) = A\sin(\pi x_1), \\ x_1 > 1 \quad & p(x_1) = 0, \end{aligned}$$

where A is a constant. Plot the displacement field as a function of x_1 when $t = 0$, $t = 1/(2\alpha)$, and $t = 1/\alpha$.

Solution—The displacement field at time t is given by Eq. (2.19):



Exercise 2.10 Suppose that a half space of elastic material is initially undisturbed and at $t = 0$ the boundary is subjected to a uniform normal stress $T_{11} = T\,H(t)$, where T is a constant and the Heaviside, or step function $H(t)$ is defined by

$$H(t) = \begin{cases} 0 & \text{when} \quad t < 0, \\ 1 & \text{when} \quad t \geq 0. \end{cases}$$

(a) Show that the resulting displacement field in the material is

$$u_1 = -\frac{\alpha T}{\lambda + 2\mu}\left(t - \frac{x_1}{\alpha}\right) H\left(t - \frac{x_1}{\alpha}\right).$$

(b) Assume that $T/(\lambda + 2\mu) = 1$. Plot the displacement field as a function of x_1 when $\alpha t = 1$, $\alpha t = 2$, and $\alpha t = 3$.

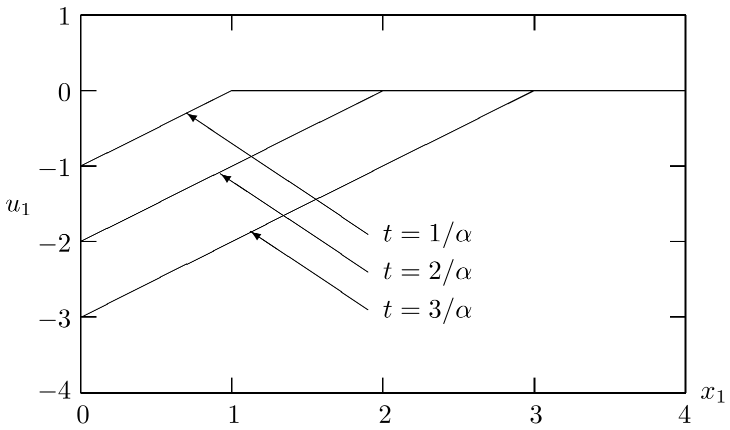
Solution—(a) The solution to this type of problem is discussed in the section beginning on page 66. The displacement field is given by Eq. (2.26), where

$$p(\bar{t}) = \frac{1}{\lambda + 2\mu} T_{11}(0, \bar{t}).$$

In this exercise, $p(\bar{t}) = [T/(\lambda + 2\mu)]H(\bar{t})$, so

$$u_1 = -\frac{\alpha T}{\lambda + 2\mu}\int_0^{t - x_1/\alpha} H(\bar{t})\,d\bar{t} = -\frac{\alpha T}{\lambda + 2\mu}\left(t - \frac{x_1}{\alpha}\right) H\left(t - \frac{x_1}{\alpha}\right).$$

(b)



EXERCISE 2.11 Consider a plate of elastic material of thickness L. Assume that the plate is infinite in extent in the x_2 and x_3 directions. Suppose that the plate is initially undisturbed and at $t = 0$ the left boundary is subjected to a uniform normal stress $T_{11} = T\,H(t)$, where T is a constant and the step function $H(t)$ is defined in Exercise 2.10. This boundary condition will give rise to a wave propagating in the positive x_1 direction. When $t = L/\alpha$, the wave will reach the right boundary and cause a reflected wave. Show that from $t = L/\alpha$ until $t = 2L/\alpha$ the displacement field in the plate is

$$u_1 = -\frac{\alpha T}{\lambda + 2\mu}\left[\left(t - \frac{x_1}{\alpha}\right) H\left(t - \frac{x_1}{\alpha}\right) + \left(t + \frac{x_1}{\alpha} - \frac{2L}{\alpha}\right) H\left(t + \frac{x_1}{\alpha} - \frac{2L}{\alpha}\right)\right].$$

Solution—The solution for the wave propagating in the positive x_1 direction is discussed in the section beginning on page 66. The displacement field is given by Eq. (2.26):

$$u_1(x_1, t) = f(\xi) = -\alpha \int_0^{t - \frac{x_1}{\alpha}} p(\bar{t})\, d\bar{t}, \quad (*) \qquad (2.79)$$

where

$$p(\bar{t}) = \frac{1}{\lambda + 2\mu} T_{11}(0, t).$$

Setting $T_{11}(0, t) = T\,H(t)$, we obtain the displacement field

$$u_1 = -\frac{\alpha T}{\lambda + 2\mu}\left(t - \frac{x_1}{\alpha}\right) H\left(t - \frac{x_1}{\alpha}\right).$$

After the wave reaches the boundary at $x_1 = L$ there will be a reflected wave propagating in the negative x_1 direction. Assume a displacement field

$$u_1(x_1, t) = f(\xi) + g(\eta). \quad (**)$$

The boundary condition at $x_1 = L$ is $T_{11}(L, t) = 0$, which means that

$$\frac{\partial u_1}{\partial x_1}(L, t) = 0.$$

Evaluating the partial derivative,

$$\begin{aligned} \frac{\partial u_1}{\partial x_1} &= \frac{df(\xi)}{d\xi}\frac{\partial \xi}{\partial x_1} + \frac{dg(\eta)}{d\eta}\frac{\partial \eta}{\partial x_1} \\ &= -\frac{1}{\alpha}\frac{df(\xi)}{d\xi} + \frac{1}{\alpha}\frac{dg(\eta)}{d\eta}. \end{aligned}$$

The boundary condition is

$$-\frac{1}{\alpha}\frac{df(t - \dfrac{L}{\alpha})}{d(t - \dfrac{L}{\alpha})} + \frac{1}{\alpha}\frac{dg(t + \dfrac{L}{\alpha})}{d(t + \dfrac{L}{\alpha})} = 0.$$

Defining $\bar{t} = t + L/\alpha$, this is

$$\frac{dg(\bar{t})}{d\bar{t}} = \frac{df(\bar{t} - \dfrac{2L}{\alpha})}{d(\bar{t} - \dfrac{2L}{\alpha})}$$

Integrating this equation yields

$$g(t) = f(t - \frac{2L}{\alpha}) + C,$$

where C is a constant. Using this expression and Eqs. (*) and (*) gives the solution.

Exercise 2.12 Confirm that the solution (2.18) satisfies the initial conditions (2.12).

Solution—It's obvious that the displacement initial condition is satisfied, $u_1(x_1, 0) = p(x_1)$. To confirm that the velocity initial condition is satisfied, we use the following result from calculus: if the limits of an integral are functions of a parameter y,

$$I = \int_{a(y)}^{b(y)} f(x)\,dx,$$

the derivative of the integral with respect to y is

$$\frac{dI}{dy} = f(b(y))\frac{db(y)}{dy} - f(a(y))\frac{da(y)}{dy}.$$

The solution (2.18) is

$$u(x_1,t) = \frac{1}{2}p(\xi) + \frac{1}{2}p(\eta) + \frac{1}{2\alpha}\int_{\xi}^{\eta} q(\bar{x})\,d\bar{x},$$

so

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= \frac{1}{2}\frac{dp(\xi)}{d\xi}\frac{\partial \xi}{\partial t} + \frac{1}{2}\frac{dp(\eta)}{d\eta}\frac{\partial \eta}{\partial t} + \frac{1}{2\alpha}\left[q(\eta)\frac{\partial \eta}{\partial t} - q(\xi)\frac{\partial \xi}{\partial t}\right]\\ &= -\frac{\alpha}{2}\frac{dp(\xi)}{d\xi} + \frac{\alpha}{2}\frac{dp(\eta)}{d\eta} + \frac{1}{2}[q(\eta) + q(\xi)]\end{aligned}$$

Setting $t = 0$ in this expression yields

$$\frac{\partial u_1}{\partial t}(x_1, 0) = q(x_1).$$

EXERCISE 2.13 Confirm that the solution (2.26) satisfies the boundary condition (2.24).

Solution—The solution (2.26) is

$$u_1(x_1,t) = -\alpha\int_0^{\xi} p(\bar{t})\,d\bar{t}.$$

Using the result from calculus quoted in the previous solution,

$$\frac{\partial u_1}{\partial x_1} = -\alpha p(\xi)\frac{\partial \xi}{\partial x_1} = p(\xi).$$

Setting $x_1 = 0$ in this expression yields

$$\frac{\partial u_1}{\partial x_1}(0,t) = p(t).$$

Exercise 2.14 By using Eqs. (2.80) and (2.81), show that the quantity $v+\alpha w$ is constant along any straight line in the x-t plane having slope

$$\frac{dx}{dt} = -\alpha.$$

Solution—The change in the quantity $v + \alpha w$ from the point x, t to the point $x + dx, t + dt$ is

$$\begin{aligned} d(v + \alpha w) &= \frac{\partial}{\partial t}(v + \alpha w)\, dt + \frac{\partial}{\partial x}(v + \alpha w)\, dx \\ &= \left(\frac{\partial v}{\partial t} + \alpha\frac{\partial w}{\partial t}\right) dt + \left(\frac{\partial v}{\partial x} + \alpha\frac{\partial w}{\partial x}\right) dx. \end{aligned}$$

By using Eqs. (2.80) and (2.81), we can write this expression in the form

$$d(v + \alpha w) = \left(\frac{\partial v}{\partial x} + \frac{1}{\alpha}\frac{\partial v}{\partial t}\right)(dx + \alpha\, dt)\,.$$

Thus $d(v + \alpha w)$ is zero if

$$\frac{dx}{dt} = -\alpha.$$

Exercise 2.15 Suppose that a half space of elastic material is initially undisturbed, and at $t = 0$ the boundary is subjected to the stress boundary condition

$$T_{11}(0, t) = p(t),$$

where $p(t)$ is a prescribed function of time that vanishes for $t < 0$. Use the method of characteristics to determine the velocity and strain of the material at an arbitrary point x_1, t in the x_1-t plane.

Answer:

$$\begin{aligned} v(x_1, t) &= -\frac{\alpha}{\lambda + 2\mu} p\left(t - \frac{x_1}{\alpha}\right), \\ w(x_1, t) &= \ \ \frac{1}{\lambda + 2\mu} p\left(t - \frac{x_1}{\alpha}\right). \end{aligned}$$

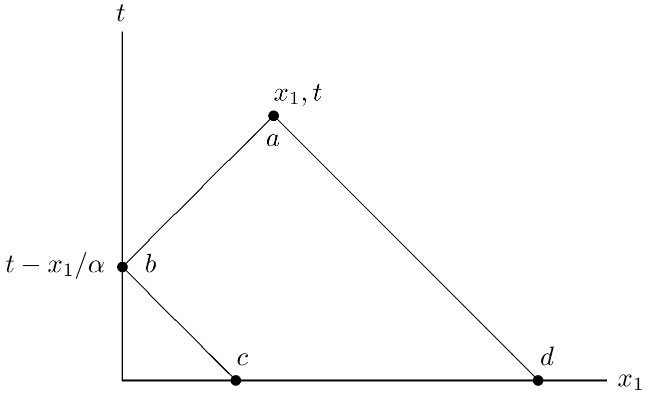
Solution—The boundary condition is

$$T_{11}(0, t) = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1}(0, t) = (\lambda + 2\mu)w(0, t) = p(t),$$

or

$$w(0,t) = \frac{1}{\lambda + 2\mu} p(t).$$

We want to determine the solution at a point x_1, t in the x_1-t plane. Extending left-running and right-running characteristics from this point, we obtain the following diagram:



The initial condition states that v and w equal zero on the x_1 axis. From the characteristic ad, we obtain the equation

$$ad: \quad v(x_1,t) + \alpha w(x_1,t) = 0.$$

From ab, we obtain

$$ab: \quad v(x_1,t) - \alpha w(x_1,t) = v(0, t - \frac{x_1}{\alpha}) - \alpha w(0, t - \frac{x_1}{\alpha}),$$

and from bc, we obtain

$$v(0, t - \frac{x_1}{\alpha}) + \alpha w(0, t - \frac{x_1}{\alpha}) = 0.$$

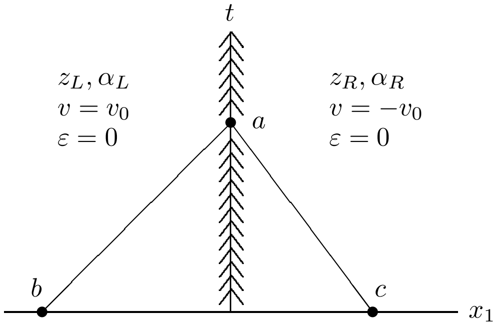
From the boundary condition, the value of w at b is

$$w(0, t - \frac{x_1}{\alpha}) = \frac{1}{\lambda + 2\mu} p(t - \frac{x_1}{\alpha}).$$

We can solve the three equations obtained from the characteristics for the three unknowns $v(0, t - x_1/\alpha)$, $v(x_1,t)$, and $w(x_1,t)$, which yields the solution.

Exercise 2.16 Half spaces of materials with acoustic impedances $z_L = \rho_L \alpha_L$ and $z_R = \rho_R \alpha_R$ approach each other with equal velocities v_0. Use characteristics to show that the velocity of their interface after the collision is $v_0(z_L - z_R)/(z_L + z_R)$.

Solution—The point a in the following x_1-t diagram represents the state of the interface at an arbitrary time after the impact:



Two characteristics extend from a back to the x_1 axis. The right-running characteristic yields the equation

$$ab: \quad v_0 = v(0,t) - \alpha_L \varepsilon_L(0,t),$$

and the left-running characteristic yields the equation

$$ac: \quad -v_0 = v(0,t) + \alpha_R \varepsilon_R(0,t).$$

The normal stresses in the materials have the same value at the boundary. Denoting it by $T_{11}(0,t)$, the stress-strain relations in the two materials are

$$T_{11}(0,t) = (\lambda_L + 2\mu_L)\varepsilon_L(0,t) = z_L \alpha_L \varepsilon_L(0,t),$$

$$T_{11}(0,t) = (\lambda_R + 2\mu_R)\varepsilon_R(0,t) = z_R \alpha_R \varepsilon_R(0,t).$$

Using these two relations, we multiply the right-running characteristic equation by z_L to obtain

$$z_L v_0 = z_L v(0,t) - T_{11}(0,t),$$

and multiply the left-running characteristic equation by z_R to obtain

$$-z_R v_0 = z_R v(0,t) + T_{11}(0,t).$$

Summing these two relations and solving for $v(0,t)$ gives the desired result.

Exercise 2.17 Consider the first-order partial differential equation

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = 0,$$

where α is a constant. Show that its solution $u(x, t)$ is constant along characteristics with slope $dx/dt = \alpha$.

Solution—The change in u from the point x, t to the point $x + dx, t + dt$ is

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial t}\, dt$$

By using the first-order equation, we can write this as

$$du = \frac{\partial u}{\partial x}(dx - \alpha\, dt).$$

Thus $du = 0$, and u is constant, along a line with slope $dx/dt = alpha$.

Exercise 2.18 Use D'Alembert solutions to verify the values shown in Fig. 2.34.

Solution—The D'Alembert solution for the initial wave propagating from the left boundary and the first reflection from the right boundary can be obtained from the solution to Exercise 2.11. That solution gives the displacement of a plate of thickness L subjected to the stress $T_{11} = T_0\, H(t)$ at the left boundary and with the right boundary free of stress. From $t = 0$ until $t = 2L/\alpha$, the displacement is

$$u_1 = \left(t - \frac{x_1}{\alpha}\right) H\left(t - \frac{x_1}{\alpha}\right) + \left(t + \frac{x_1}{\alpha} - \frac{2L}{\alpha}\right) H\left(t + \frac{x_1}{\alpha} - \frac{2L}{\alpha}\right),$$

where we have set $T_0 = -(\lambda + 2\mu)/\alpha$. The velocity field in the plate is

$$v = \frac{\partial u_1}{\partial t} = H\left(t - \frac{x_1}{\alpha}\right) + H\left(t + \frac{x_1}{\alpha} - \frac{2L}{\alpha}\right).$$

The velocity at the left boundary is

$$v(0, t) = H(t) + H\left(t - \frac{2L}{\alpha}\right),$$

which satisfies the velocity condition applied to the left boundary in Fig. 2.34.

Equations (2.89) and (2.90) give

$$t = j\Delta, \quad x_1 = n\alpha\Delta, \quad L = 5\alpha\Delta.$$

Because Δ is a positive constant, substitution of these expressions into the equation for v gives

$$v(n, j) = H(j - n) + H(j + n - 10),$$

which agrees with the values shown in Fig. 2.34.

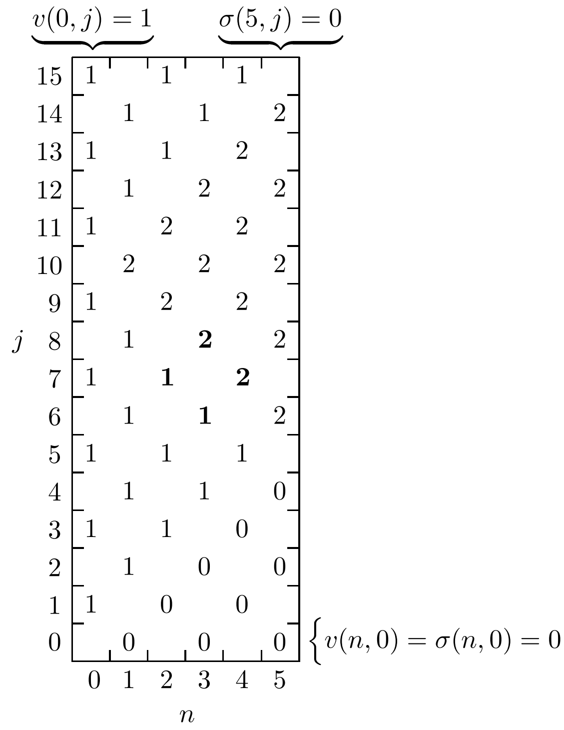
EXERCISE 2.19 (a) Use Eq. (2.98) to verify the values shown in Fig. 2.34.
(b) Extend the calculation shown in Fig. 2.34 to $j = 15$.

Solution—

(a) Use the procedure beginning on page 101 to generate the solution. An example calculation at an interior point is

$$\begin{aligned} v(3, 8) &= v(4, 7) + v(2, 7) - v(3, 6) \\ &= 2 + 1 - 1 \\ &= 2. \end{aligned}$$

This calculation is shown by the bold-face entries in the solution to part (b).
(b) The solution to $j = 15$ is:



	0	1	2	3	4	5
15	1		1		1	
14		1		1		2
13	1		1		2	
12		1		2		2
11	1		2		2	
10		2		2		2
9	1		2		2	
8		1		**2**		2
7	1		**1**		**2**	
6		1		**1**		2
5	1		1		1	
4		1		1		0
3	1		1		0	
2		1		0		0
1	1		0		0	
0		0		0		0

EXERCISE 2.20 Extend the calculations used to obtain Fig. 2.34 to $j = 15$, assuming that at $t = 0$ the left boundary is subjected to a unit step in stress: $\sigma(0,t) = 1$.

Solution—The initial conditions are $v(n,0) = 0$ and $\sigma(n,0) = 0$. The velocity at the left boundary $v(0,1)$ is computed from Eq. (2.93):

$$zv(0,1) + \sigma(0,1) = zv(1,0) + \sigma(1,0) = 0.$$

This yields $v(0,1) = -a$, where z is the acoustic impedance of the layer and the parameter $a = 1/z$. The values of velocity for the odd solution in row $j = 1$ are computed from Eq. (2.94), which reduces to

$$2v(n,1) = v(n+1,0) + v(n-1,0).$$

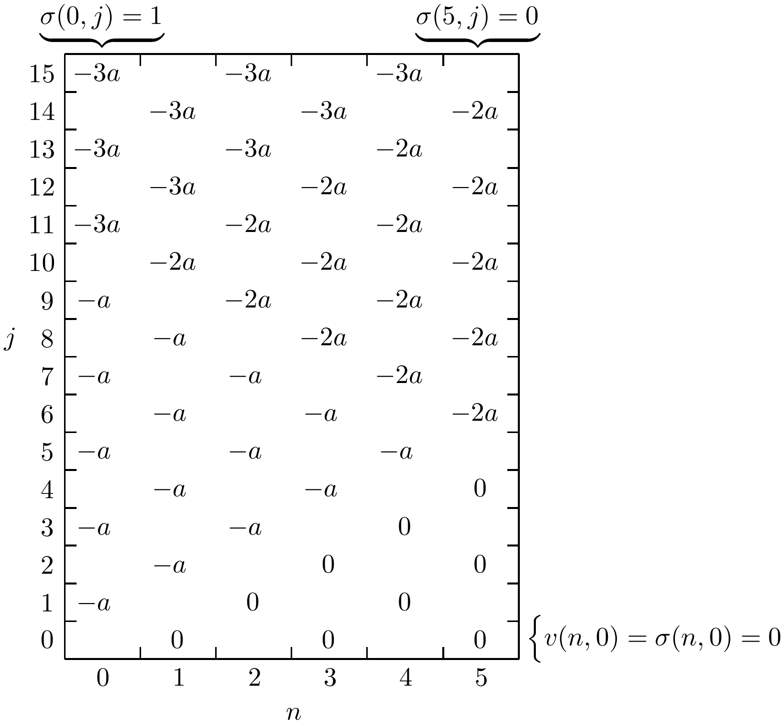
As the solution progresses, the values of $v(0,j)$ are computed from Eq. (2.99), which (because of the constant applied stress) reduces to

$$v(0,j+1) = 2v(1,j) - v(0,j-1).$$

The values $v(5, j)$ at the stress-free right boundary are computed from Eq. (2.100):

$$v(5, j+1) = 2v(4, j) - v(5, j-1).$$

The values $v(n, j)$ at the interior points are computed by application of Eq. (2.98). The results are:



$\underbrace{\sigma(0,j) = 1}$ $\underbrace{\sigma(5,j) = 0}$

15	$-3a$		$-3a$		$-3a$	
14		$-3a$		$-3a$		$-2a$
13	$-3a$		$-3a$		$-2a$	
12		$-3a$		$-2a$		$-2a$
11	$-3a$		$-2a$		$-2a$	
10		$-2a$		$-2a$		$-2a$
9	$-a$		$-2a$		$-2a$	
j 8		$-a$		$-2a$		$-2a$
7	$-a$		$-a$		$-2a$	
6		$-a$		$-a$		$-2a$
5	$-a$		$-a$		$-a$	
4		$-a$		$-a$		0
3	$-a$		$-a$		0	
2		$-a$		0		0
1	$-a$		0		0	
0		0		0		0
	0	1	2	3	4	5

n

$\left\{ v(n,0) = \sigma(n,0) = 0 \right.$

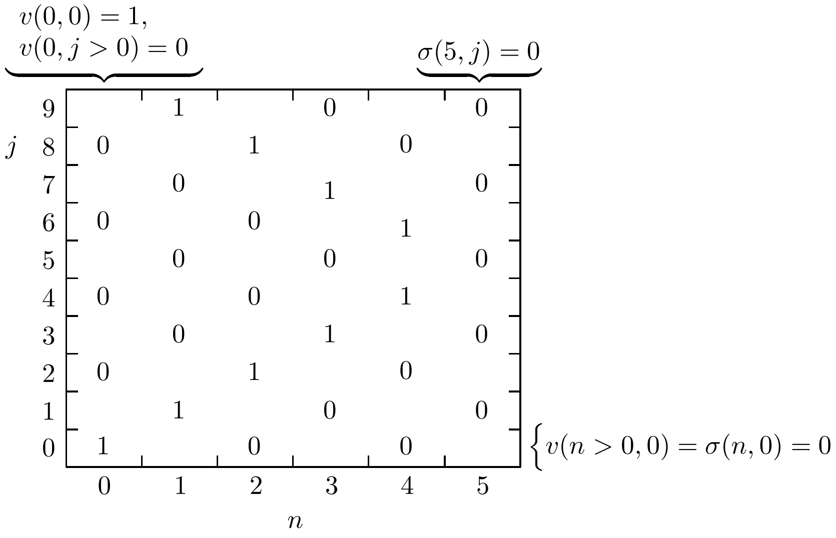
Exercise 2.21 Repeat the calculations used to obtain Fig. 2.33, replacing the unit step in velocity at the left boundary by the pulse

$$v(0, j) = \begin{cases} 1 & j = 0, \\ 0 & j > 0. \end{cases}$$

Solution—Because of the boundary condition, the odd solution (points at which $n + j$ is odd) is zero everywhere, and the even solution is computed instead. The initial conditions are $v(n, 0) = 0$ for $n > 0$ and $\sigma(n, 0) = 0$ for all n. The values of the velocity for the even solution in row $j = 1$ are computed from Eq. (2.94), which reduces to

$$2v(n,1) = v(n+1,0) + v(n-1,0).$$

The boundary conditions for $j > 0$ are $v(0,j) = v(5,j) = 0$. The values $v(n,j)$ at the interior points are computed by application of Eq. (2.98). The results are:



Exercise 2.22 Calculate the results shown in Fig. 2.37.

Solution—Write a program to carry out the algorithm described in example 4 beginning on page 106.

Solutions, Chapter 3. Steady-State Waves

Exercise 3.1 Show that

$$u = Ae^{i(kx - \omega t)}$$

is a solution of Eq. (3.1) if $k = \omega/\alpha$.

Solution—The second partial derivative of u with respect to t is

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2}{\partial t^2}\left[Ae^{i(kx - \omega t)}\right]\\ &= -\omega^2 Ae^{i(kx - \omega t)}.\end{aligned}$$

The second partial derivative of u with respect to x is

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2}{\partial x^2}\left[Ae^{i(kx - \omega t)}\right]\\ &= -k^2 Ae^{i(kx - \omega t)}.\end{aligned}$$

Substituting these expressions into Eq. (3.1), the equation is satisfied if $k = \omega/\alpha$.

Exercise 3.2 A half space of elastic material is subjected to the stress boundary condition

$$T_{11}(0, t) = T_0 e^{-i\omega t},$$

where T_0 is a constant. Determine the resulting steady-state displacement field in the material.

Solution—The displacement field resulting from this boundary condition will be of the form $u_1 = u_1(x_1, t)$, $u_2 = u_3 = 0$. For this displacement field, the equations of motion (1.60) reduce to

$$\frac{\partial^2 u_1}{\partial t^2} = \alpha^2 \frac{\partial^2 u_1}{\partial x_1^2}.$$

Assume a solution

$$u_1 = Ue^{i(kx_1 - \omega t)}, \qquad (*)$$

which represents a right-running wave of undetermined constant amplitude U. Substituting this solution into the equation of motion yields

$$\omega^2 u_1 = \alpha^2 k^2 u_1,$$

which is satisfied if $k = \omega/\alpha$. From Eq. (1.66) and the stress-strain relation (1.68),

$$\begin{aligned}T_{11} &= (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1}\\ &= ik(\lambda + 2\mu)Ue^{i(kx_1 - \omega t)}.\end{aligned}$$

At the boundary $x_1 = 0$, this is

$$T_{11}(0,t) = ik(\lambda + 2\mu)Ue^{-i\omega t}.$$

This expression matches the given boundary condition if the constant U is

$$U = \frac{T_0}{ik(\lambda + 2\mu)}.$$

Substituting this expression into $(*)$, the displacement field is

$$u_1 = \frac{T_0}{ik(\lambda + 2\mu)} e^{i(kx_1 - \omega t)}.$$

EXERCISE 3.3 A plate of elastic material of thickness L is subjected to the steady-state displacement boundary condition

$$u_1(0,t) = Ue^{-i\omega t}$$

at the left boundary. The plate is free at the right boundary. Determine the resulting steady-state displacement field in the material.

Solution—Assume a solution consisting of right-running and left-running waves:

$$u_1 = Te^{i(kx_1 - \omega t)} + Re^{i(-kx_1 - \omega t)},$$

where T and R are constants and $k = \omega/\alpha$. By evaluating this solution at $x_1 = 0$ and equating the result to the displacement applied to the left boundary, we obtain

$$T + R = U. \qquad (*)$$

Because the right boundary is stress free, the stress-strain relation gives

$$T_{11}(L,t) = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1}(L,t) = 0.$$

Therefore

$$\frac{\partial u_1}{\partial x_1}(L,t) = ikTe^{i(kL - \omega t)} - ikRe^{i(-kL - \omega t)} = 0,$$

which yields

$$Te^{ikL} - Re^{-ikL} = 0. \qquad (**)$$

Solving Eqs. (∗) and (∗∗) for T and R, we obtain

$$T = \frac{Ue^{-ikL}}{2\cos kL}, \quad R = \frac{Ue^{ikL}}{2\cos kL}.$$

Substituting these expressions into the assumed solution for u_1 gives

$$u_1 = \frac{U(e^{ikx_1}e^{-ikL} + e^{-ikx_1}e^{ikL})}{2\cos kL}e^{-i\omega t},$$

which can be written as

$$u_1 = \frac{U\cos k(x_1 - L)}{\cos kL}e^{-i\omega t}.$$

Exercise 3.4 A plate of elastic material of thickness L is subjected to the steady-state shear stress boundary condition

$$T_{12}(0,t) = T_0 e^{-i\omega t}$$

at the left boundary. The plate is bonded to a rigid material at the right boundary. Determine the resulting steady-state displacement field in the material.

Solution—The displacement field resulting from this boundary condition will be of the form $u_2 = u_2(x_1, t)$, $u_1 = u_3 = 0$. For this displacement field, the equations of motion (1.60) reduce to

$$\frac{\partial^2 u_2}{\partial t^2} = \beta^2 \frac{\partial^2 u_2}{\partial x_1^2}.$$

Assume a solution for the shear displacement consisting of right-running and left-running waves:

$$u_2 = Te^{i(kx_1 - \omega t)} + Re^{i(-kx_1 - \omega t)}.$$

This solution satisfies the equation of motion if $k = \omega/\beta$. From Eq. (1.66) and the stress-strain relation (1.68), the stress resulting from the assumed solution is

$$\begin{aligned} T_{12} &= \mu \frac{\partial u_2}{\partial x_1} \\ &= ik\mu Te^{i(kx_1 - \omega t)} - ik\mu Re^{i(-kx_1 - \omega t)}. \end{aligned}$$

The boundary condition at $x_1 = 0$ is

$$T_{12}(0,t) = ik\mu Te^{-i\omega t} - ik\mu Re^{-i\omega t} = T_0 e^{-i\omega t},$$

which yields the equation

$$T - R = \frac{-iT_0}{k\mu}. \qquad (*)$$

The boundary condition at $x_1 = L$ is

$$u_2(L,t) = Te^{i(kL - \omega t)} + Re^{i(-kL - \omega t)} = 0,$$

which yields

$$Te^{ikL} + Re^{-ikL} = 0. \qquad (**)$$

Solving Eqs. $(*)$ and $(**)$ for T and!R, we obtain

$$T = \frac{-iT_0 e^{-ikL}}{2k\mu \cos kL}, \quad R = \frac{iT_0 e^{ikL}}{2k\mu \cos kL}.$$

Substituting these expressions into the assumed solution for u_2 gives

$$u_2 = \frac{-iT_0(e^{ikx_1} e^{-ikL} - e^{-ikx_1} e^{ikL})}{2k\mu \cos kL} e^{-i\omega t},$$

which can be written as

$$u_2 = \frac{T_0 \sin k(x_1 - L)}{k\mu \cos kL} e^{-i\omega t}.$$

Exercise 3.5 A compressional wave is described by the potential

$$\phi = Ae^{i(kx_1 \cos\theta + kx_3 \sin\theta - \omega t)}.$$

(a) The Lamé constants of the material are λ and μ. Determine the stress component T_{33} as a function of position and time.
(b) The density of the material in the reference state is ρ_0. Determine the density ρ of the material as a function of position and time.

Solution—
(a) The components of the displacement are

$$\begin{aligned} u_1 &= \frac{\partial \phi}{\partial x_1} = ik\phi\cos\theta, \\ u_2 &= \frac{\partial \phi}{\partial x_2} = 0, \\ u_3 &= \frac{\partial \phi}{\partial x_3} = ik\phi\sin\theta. \end{aligned}$$

From the stress-strain relationship (1.68),

$$T_{33} = \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) + 2\mu\frac{\partial u_1}{\partial x_1}.$$

Substituting the components of the displacement into this expression, we obtain

$$T_{33} = -k^2\phi(\lambda + 2\mu\sin^2\theta).$$

(b) The relationship between the density and the displacements is given by Eq. (1.67):

$$\rho = \rho_0\left(1 - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3}\right).$$

Substitution of the components of the displacement yields

$$\rho = \rho_0(1 + k^2\phi).$$

Exercise 3.6 The motion of an elastic material is described by the displacement field

$$\begin{aligned} u_1 &= 0, \\ u_2 &= u_2(x_1, x_3, t), \\ u_3 &= 0. \end{aligned}$$

(a) Show that u_2 satisfies the wave equation

$$\frac{\partial^2 u_2}{\partial t^2} = \beta^2\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_3^2}\right).$$

For the half-space shown, suppose that the displacement field consists of the incident and reflected waves

$$\begin{aligned} u_2 \quad = \quad & Ie^{i(kx_1\cos\theta - kx_3\sin\theta - \omega t)} \\ & + Re^{i(kx_1\cos\theta_R + kx_3\sin\theta_R - \omega t)}. \end{aligned}$$

(b) Show that $k = \omega/\beta$.
(c) Show that $\theta_R = \theta$ and $R = I$.

Solution—
(a) Note that, for the given displacement field,

$$\frac{\partial^2 u_k}{\partial x_k \partial x_m} = \frac{\partial}{\partial x_m}\left(\frac{\partial u_k}{\partial x_k}\right) = 0,$$

so the equation of motion (1.53) reduces to

$$\frac{\partial^2 u_m}{\partial t^2} = \beta^2 \frac{\partial^2 u_m}{\partial x_k \partial x_k},$$

where $\beta^2 = \mu/\rho_0$. For the given displacement field, the equation of motion is identically zero for $m = 1$ and $m = 3$. The desired wave equation is obtained from $m = 2$.
(b) The second partial derivatives of u_2 are

$$\begin{aligned}
\frac{\partial^2 u_2}{\partial x_1^2} &= -k^2 \cos^2 \theta I e^{i(kx_1 \cos\theta - kx_3 \sin\theta - \omega t)} \\
&\quad -k^2 \cos^2 \theta_R R e^{i(kx_1 \cos\theta_R + kx_3 \sin\theta_R - \omega t)}, \\
\frac{\partial^2 u_2}{\partial x_3^2} &= -k^2 \sin^2 \theta I e^{i(kx_1 \cos\theta - kx_3 \sin\theta - \omega t)} \\
&\quad -k^2 \sin^2 \theta_R R e^{i(kx_1 \cos\theta_R + kx_3 \sin\theta_R - \omega t)}, \\
\frac{\partial^2 u_2}{\partial x_1^2} &= -\omega^2 u_2. \qquad (*)
\end{aligned}$$

Using these results and the identity

$$\sin^2\theta + \cos^2\theta = 1,$$

we obtain

$$\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_3^2} = -k^2 u_2. \qquad (**)$$

Substituting the expressions $(*)$ and $(**)$ into the wave equation for u_2 gives $k = \omega/\beta$.
(c) The surface of the half space is stress free. The components of the unit vector normal to the surface are $n_1 = 0$, $n_2 = 0$, $n_3 = -1$, so from Eq. (ref) the components of the traction vector on the surface are

$$t_k(x_1, 0, t) = -T_{k3}(x_1, 0, t) = 0.$$

Therefore $t_1(x_1, 0, t) = -T_{13}(x_1, 0, t) = 0$, $t_2(x_1, 0, t) = -T_{23}(x_1, 0, t) = 0$, and $t_3(x_1, 0, t) = -T_{33}(x_1, 0, t) = 0$. From Eq. (1.66) and the stress-strain relation

(1.68), the stress components T_{33} and T_{13} are identically zero and the component T_{23} is

$$T_{23} = \mu \frac{\partial u_2}{\partial x_3} = 0.$$

Thus the boundary condition at the surface is

$$\frac{\partial u_2}{\partial x_3}(x_1, 0, t) = 0,$$

which yields the equation

$$I \sin\theta e^{ikx_1 \cos\theta} = R \sin\theta_R e^{ikx_1 \cos\theta_R}.$$

This equation must hold for all values of x_1, which implies that $\theta_R = \theta$ and $R = I$.

EXERCISE 3.7 Show that in terms of Poisson's ratio ν, the ratio of the plate velocity c_p to the shear wave velocity β is

$$\frac{c_p}{\beta} = \left(\frac{2}{1-\nu}\right)^{1/2}.$$

Solution—Using Eqs. (2.8), (3.60) and (3.62),

$$\frac{c_p^2}{\beta^2} = \frac{\left[\dfrac{4\mu(\lambda+\mu)}{\rho_0(\lambda+2\mu)}\right]}{\dfrac{\mu}{\rho_0}} = \frac{4(\lambda+\mu)}{\lambda+2\mu}. \qquad (*)$$

From Appendix B, the term μ is given in terms of λ and the Poisson's ratio by

$$\mu = \frac{\lambda(1-2\nu)}{2\nu}.$$

Substituting this result into Eq. (*) and simplifying yields the desired result.

Exercise 3.8 Show that in the limit $\omega \to \infty$, Eq. (3.58) for the velocities of the Rayleigh-Lamb modes becomes identical to Eq. (3.50) for the velocity of a Rayleigh wave.

Solution—By reversing the order of the terms raised to the one-half power, we can write Eq. (3.58) as

$$\left(2-\frac{c_1^2}{\beta^2}\right)^2 \frac{\tan\left[i\left(\dfrac{\beta^2}{c_1^2}-1\right)^{1/2}\dfrac{\omega h}{\beta}\right]}{\tan\left[i\left(\dfrac{\beta^2}{c_1^2}-\dfrac{\beta^2}{\alpha^2}\right)^{1/2}\dfrac{\omega h}{\beta}\right]} - 4\left(1-\frac{c_1^2}{\beta^2}\frac{\beta^2}{\alpha^2}\right)^{1/2}\left(1-\frac{c_1^2}{\beta^2}\right)^{1/2} = 0.$$

By using the identity

$$\tan iy = \frac{e^{-y}-e^{y}}{i(e^{-y}+e^{y})},$$

we can see that

$$\lim_{\omega\to\infty} \frac{\tan\left[i\left(\dfrac{\beta^2}{c_1^2}-1\right)^{1/2}\dfrac{\omega h}{\beta}\right]}{\tan\left[i\left(\dfrac{\beta^2}{c_1^2}-\dfrac{\beta^2}{\alpha^2}\right)^{1/2}\dfrac{\omega h}{\beta}\right]} = 1,$$

so that we obtain Eq. (3.50).

Exercise 3.9 Show that in the limit as $\omega \to 0$, the solution of Eq. (3.58) for the velocities of the Rayleigh-Lamb modes is the plate velocity c_p.

Solution—We write Eq. (3.58) in the form

$$\begin{aligned}&\left(2-\frac{c_1^2}{\beta^2}\right)^2 \tan\left[\left(1-\frac{\beta^2}{c_1^2}\right)^{1/2}\frac{\omega h}{\beta}\right]\\ &+4\left(\frac{c_1^2}{\beta^2}\frac{\beta^2}{\alpha^2}-1\right)^{1/2}\left(\frac{c_1^2}{\beta^2}-1\right)^{1/2}\tan\left[\left(\frac{\beta^2}{\alpha^2}-\frac{\beta^2}{c_1^2}\right)^{1/2}\frac{\omega h}{\beta}\right]=0.\end{aligned}$$

By expanding the tangents in Taylor series in terms of their arguments, dividing the equation by ω and then letting $\omega \to 0$, we obtain

$$\left(2-\frac{c_1^2}{\beta^2}\right)^2\left(1-\frac{\beta^2}{c_1^2}\right)^{1/2}+4\left(\frac{c_1^2}{\alpha^2}-1\right)^{1/2}\left(\frac{c_1^2}{\beta^2}-1\right)^{1/2}\left(\frac{\beta^2}{\alpha^2}-\frac{\beta^2}{c_1^2}\right)^{1/2}=0.$$

With some rearrangement and cancellation, this equation reduces to

$$\left(2 - \frac{c_1^2}{\beta^2}\right)^2 + 4\left(\frac{c_1^2}{\alpha^2} - 1\right) = 0.$$

Solving for c_1, we obtain

$$c_1^2 = 4\left(\beta^2 - \frac{\beta^4}{\alpha^2}\right).$$

Setting $\alpha^2 = (\lambda + 2\mu)/\rho_0$ and $\beta^2 = \mu/\rho_0$, we obtain

$$c_1^2 = \frac{4\mu(\lambda + \mu)}{\rho_0(\lambda + 2\mu)} = \frac{E_p}{\rho_0} = c_p^2.$$

Exercise 3.10 A wave analogous to a Rayleigh wave can exist at a plane, bonded interface between two different elastic materials. This wave, called a Stoneley wave, attenuates exponentially with distance away from the interface in each material. Derive the characteristic equation for a Stoneley wave.

Solution—Potentials are required for both the upper and lower half spaces. Based on the solution for the phase velocity of Rayleigh waves in Section 3.4, let the potentials in the lower half space be

$$\phi = Ae^{-hx_3}e^{i(k_1x_1 - \omega t)},$$

$$\psi = Ce^{-h_s x_3}e^{i(k_1x_1 - \omega t)},$$

where A and C are constants and

$$h = \left(k_1^2 - \frac{\omega^2}{\alpha^2}\right)^{1/2}, \quad h_s = \left(k_1^2 - \frac{\omega^2}{\beta^2}\right)^{1/2}.$$

In the upper half space let the potentials be

$$\bar{\phi} = \bar{A}e^{-\bar{h}x_3}e^{i(k_1x_1 - \omega t)},$$

$$\bar{\psi} = \bar{C}e^{-\bar{h}_s x_3}e^{i(k_1x_1 - \omega t)},$$

where $\bar{A}$ and $\bar{C}$ are constants and

$$\bar{h} = -\left(k_1^2 - \frac{\omega^2}{\bar{\alpha}^2}\right)^{1/2}, \quad \bar{h}_s = -\left(k_1^2 - \frac{\omega^2}{\bar{\beta}^2}\right)^{1/2}.$$

The assumed potentials have the same wave number, which is necessary to satisfy the boundary conditions at the interface. Also, the signs of h, h_s, $\bar{h}$, and $\bar{h}_s$ have been chosen to yield solutions in both half spaces that decay with distance from the interface.

The components of displacement in the lower half space are

$$\begin{aligned} u_1 &= \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3} \\ &= (ik_1 A e^{-hx_3} + h_s C e^{-h_s x_3}) e^{i(k_1 x_1 - \omega t)}, \\ u_3 &= \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi}{\partial x_1} \\ &= (-hA e^{-hx_3} + ik_1 C e^{-h_s x_3}) e^{i(k_1 x_1 - \omega t)}, \end{aligned}$$

with comparable expressions for the components of displacement $\bar{u}_1$ and $\bar{u}_3$ in the lower half space. The components of displacement in the upper and lower half spaces are equal at $x_3 = 0$:

$$\begin{aligned} (\bar{u}_1)_{x_3=0} &= (u_1)_{x_3=0}, \\ (\bar{u}_3)_{x_3=0} &= (u_3)_{x_3=0}. \end{aligned}$$

Substituting the expressions for the displacement components in terms of the potentials, these two boundary conditions result in the equations

$$\left. \begin{aligned} ik_1 \bar{A} + \bar{h}_s \bar{C} &= ik_1 A + h_s C, \\ -\bar{h} \bar{A} + ik_1 \bar{C} &= -hA + ik_1 C. \end{aligned} \right\} \quad (*)$$

The assumed solutions must also satisfy the stress boundary conditions

$$\begin{aligned} (\bar{T}_{13})_{x_3=0} &= (T_{13})_{x_3=0}, \\ (\bar{T}_{33})_{x_3=0} &= (T_{33})_{x_3=0}. \end{aligned}$$

From Eq. (1.66) and the stress-strain relations (1.68),

$$\begin{aligned} T_{13} &= \mu \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right), \\ T_{33} &= \lambda \frac{\partial u_1}{\partial x_1} + (\lambda + 2\mu) \frac{\partial u_3}{\partial x_3}. \end{aligned}$$

Substituting these relations into the stress boundary conditions and using the expressions for the displacement components in terms of the potentials together with the relations

$$\lambda + 2\mu = \rho_0 \alpha^2 = \mu \frac{\alpha^2}{\beta^2}, \quad \lambda = \mu \left(\frac{\alpha^2}{\beta^2} - 2 \right),$$

the stress boundary conditions yield the equations

$$\left.\begin{aligned}
-2ik_1\bar{h}\bar{A} - (\bar{h}_s^2 + k_1^2)\bar{C} &= -2ik_1 hA - (h_s^2 + k_1^2)C,\\
i\bar{\mu}\left[\frac{\bar{\alpha}^2}{\bar{\beta}^2}(k_1^2 - \bar{h}^2) - 2k_1^2\right]\bar{A} - 2\bar{\mu}k_1\bar{h}_s\bar{C} &= i\mu\left[\frac{\alpha^2}{\beta^2}(k_1^2 - h^2) - 2k_1^2\right]A - 2\mu k_1 h_s C.
\end{aligned}\right\} \quad (**)$$

The four equations $(*)$ and $(**)$ can be written in the matrix form

$$[M]\begin{bmatrix} \bar{A} \\ \bar{C} \\ A \\ C \end{bmatrix} = [0],$$

which has a nontrivial solution only if the determinate $|M| = 0$. In terms of the wave number k_1 and frequency ω, the Stonely wave phase is $c_{st} = \omega/k_1$. Note from the definition of h that

$$\frac{\alpha^2}{\beta^2}(k_1^2 - h^2) = \frac{\omega^2}{\beta^2}, \qquad \frac{h}{k_1} = \left(1 - \frac{c_{st}^2}{\alpha^2}\right)^{1/2}, \qquad \frac{h_s}{k_1} = \left(1 - \frac{c_{st}^2}{\beta^2}\right)^{1/2}.$$

Using these relations, the determinate $|M|$ can be written

$$\begin{vmatrix}
1 & -\left(1 - \frac{c_{st}^2}{\bar{\beta}^2}\right)^{1/2} & -1 & -\left(1 - \frac{c_{st}^2}{\beta^2}\right)^{1/2} \\
\left(1 - \frac{c_{st}^2}{\bar{\alpha}^2}\right)^{1/2} & -1 & \left(1 - \frac{c_{st}^2}{\alpha^2}\right)^{1/2} & 1 \\
2\left(1 - \frac{c_{st}^2}{\bar{\alpha}^2}\right)^{1/2} & -\left(2 - \frac{c_{st}^2}{\bar{\beta}^2}\right) & 2\frac{\mu}{\bar{\mu}}\left(1 - \frac{c_{st}^2}{\alpha^2}\right)^{1/2} & \frac{\mu}{\bar{\mu}}\left(2 - \frac{c_{st}^2}{\beta^2}\right) \\
\left(2 - \frac{c_{st}^2}{\bar{\beta}^2}\right) & -2\left(1 - \frac{c_{st}^2}{\bar{\beta}^2}\right)^{1/2} & -\frac{\mu}{\bar{\mu}}\left(2 - \frac{c_{st}^2}{\beta^2}\right) & -2\frac{\mu}{\bar{\mu}}\left(1 - \frac{c_{st}^2}{\beta^2}\right)^{1/2}
\end{vmatrix} = 0.$$

This equation, which can be solved for the phase velocity c_{st}, is the characteristic equation for a Stonely wave. Notice that the phase velocity does not depend on the frequency or wave number, but is a function solely of the properties of the two materials.

EXERCISE 3.11 Show that when $z_a = z_b$, Eq. (3.73) yields the expression given in Eq. (3.74) for the phase velocity $c_1 = \omega/k$ in a layered material.

Solution—Setting $z_a = z_b$ in Eq. (3.73) yields

$$\cos kd = \cos\left(\frac{\omega d_a}{\alpha_a}\right)\cos\left(\frac{\omega d_b}{\alpha_b}\right) - \sin\left(\frac{\omega d_a}{\alpha_a}\right)\sin\left(\frac{\omega d_b}{\alpha_b}\right).$$

By using the identity $\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$, this equation can be written

$$\cos kd = \cos\left(\frac{\omega d_a}{\alpha_a} + \frac{\omega d_b}{\alpha_b}\right),$$

which has solutions

$$kd = \left(\frac{d_a}{\alpha_a} + \frac{d_b}{\alpha_b}\right)\omega + 2\pi m, \quad m = 0, 1, \ldots$$

The solution for $m = 0$ is the desired result.

Exercise 3.12 Show that in the limit $\omega \to 0$, the solution of Eq. (3.73) for the phase velocity of a steady-state wave in a layered material is given by Eq. (3.75).

Solution—Expressing the sine and cosine terms in Eq. (3.73) as Taylor series of their arguments and dividing the equation by k^2, it can be written

$$d^2 + O(k^2) = [\Delta_d^2 + O(\omega^2)]c_1^2, \quad (*)$$

where $c_1 = \omega/k$ is the phase velocity and

$$\Delta_d^2 = \left(\frac{d_a}{\alpha_a}\right)^2 + \left(\frac{z_a}{z_b} + \frac{z_b}{z_a}\right)\frac{d_a d_b}{\alpha_a \alpha_b} + \left(\frac{d_b}{\alpha_b}\right)^2.$$

The notation $O(k^2)$ means "terms of order 2 or greater in k." Solving Eq. $(*)$ for the phase velocity yields

$$c_1 = \left[\frac{d^2 + O(k^2)}{\Delta_d^2 + O(\omega^2)}\right]^{1/2}. \quad (**)$$

A non-zero limit of $c_1 = \omega/k$ as $\omega \to 0$ exists only if $k \to 0$ as $\omega \to 0$. Therefore we see from Eq. $(**)$ that

$$\lim_{\omega\to 0} c_1 = \frac{d}{\Delta_d},$$

which is Eq. (3.75).

Exercise 3.13 For the layered material discussed in Section 3.6, the fraction of the volume of the material occupied by material a is $\phi = d_a/(d_a + d_b) = d_a/d$. Derive an equation for the low-frequency limit of the phase velocity as a function of ϕ. Using the properties of tungsten for material a and aluminum for material b (see Table B.2 in Appendix B), plot your equation for values of ϕ from zero to one.

Solution—Noting that $\phi = d_a/d$ and $1 - \phi = d_b/d$, we can write Eq. (3.76) as

$$\frac{1}{c_1^2} = \frac{\phi^2}{\alpha_a^2} + \left(\frac{z_a}{z_b} + \frac{z_b}{z_a}\right)\frac{\phi(1-\phi)}{\alpha_a\alpha_b} + \frac{(1-\phi)^2}{\alpha_b^2}. \qquad (*)$$

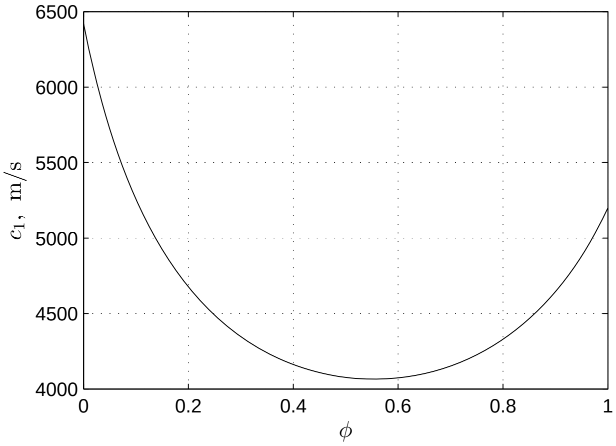
From Table B.2 in Appendix B, the density of tungsten is $\rho_a = 19.4 \times 10^3$ kg/m^3 and its compressional wave velocity is $\alpha_a = 5.20 \times 10^3$ m/s. Its acoustic impedance is

$$\begin{aligned} z_a &= \rho_a\alpha_a \\ &= (19.4 \times 10^3 \text{ kg/m}^3)(5.20 \times 10^3 \text{ m/s}) \\ &= 1.01 \times 10^8 \text{ kg/m}^2\text{s}. \end{aligned}$$

The density of aluminum is $\rho_b = 2.70 \times 10^3$ kg/m^3 and its compressional wave velocity is $\alpha_b = 6.42 \times 10^3$ m/s. Its acoustic impedance is

$$\begin{aligned} z_b &= \rho_b\alpha_b \\ &= (2.70 \times 10^3 \text{ kg/m}^3)(6.42 \times 10^3 \text{ m/s}) \\ &= 1.73 \times 10^7 \text{ kg/m}^2\text{s}. \end{aligned}$$

Using this data, we solve Eq. $(*)$ for c_1 as a function of ϕ:



Notice that there is a range of values of ϕ for which c_1 is less than the compressional wave velocity in either material.

Chapter 4. Transient Waves

EXERCISE 4.1 Suppose that an elastic half space is initially undisturbed and is subjected to the velocity boundary condition

$$\frac{\partial u_1}{\partial t}(0,t) = H(t)e^{-bt},$$

where $H(t)$ is the step function and b is a positive real number.
(a) By using a Laplace transform with respect to time, show that the solution for the Laplace transform of the displacement is

$$u_1^L = \frac{e^{-sx_1/\alpha}}{s(s+b)}.$$

(b) By inverting the Laplace transform obtained in Part (a), show that the solution for the displacement is

$$\begin{aligned} u_1 &= \frac{1}{b}\left[1 - e^{-b(t - x_1/\alpha)}\right] && \text{when} \quad t > \frac{x_1}{\alpha}, \\ u_1 &= 0 && \text{when} \quad t < \frac{x_1}{\alpha}. \end{aligned}$$

Solution—

(a) The Laplace transform of the wave equation for an initially undisturbed material is given by Eq. (4.6):

$$\frac{d^2u_1^L}{dx_1^2} - \frac{s^2}{\alpha^2}u_1^L = 0.$$

The solution for a wave propagating in the positive x_1 direction is

$$u_1^L = Be^{-sx_1/\alpha}. \quad (*)$$

To determine the constant B in this expression, we evaluate the Laplace transform of the boundary condition:

$$\begin{aligned}\int_0^\infty \frac{\partial u_1}{\partial t}(0,t)e^{-st}\,dt &= \int_0^\infty H(t)e^{-(s+b)t}\,dt\\ &= -\left.\frac{e^{-(s+b)t}}{s+b}\right|_0^\infty\\ &= \frac{1}{s+b}.\end{aligned}$$

Integrating the left side of this equation by parts,

$$u_1(0,t)e^{-st}\Big|_0^\infty + s\int_0^\infty u_1(0,t)e^{-st}\,dt = \frac{1}{s+b},$$

we obtain

$$(u_1^L)_{x_1=0} = \frac{1}{s(s+b)}.$$

Using this expression to evaluate the constant B in Eq. $(*)$ gives the desired result.

(b) The inversion integral (4.1) gives

$$u_1(x_1,t) = \frac{1}{2\pi i}\int_{C_\infty} \frac{e^{s(t-x_1/\alpha)}}{s(s+b)}\,ds.$$

The integrand

$$g(s) = \frac{1}{2\pi is\,(s+b)}e^{s(t-x_1/\alpha)}$$

has first-order poles at $s = 0$ and $s = -b$. By using the closed contour in Fig. 4.3.a and evaluating the residues, we obtain

$$\begin{aligned}\lim_{R\to\infty}\sum_{k=1}^{4}\int_{C_k} g(s)\,ds &= \lim_{s\to 0}\frac{1}{s+b}e^{s(t-x_1/\alpha)} + \lim_{s\to -b}\frac{1}{s}e^{s(t-x_1/\alpha)}\\ &= \frac{1}{b}\left[1 - e^{-b(t-x_1/\alpha)}\right].\end{aligned}$$

Using the arguments on page 163, the integrals along C_2 and C_4 approach zero. Jordan's lemma shows that the integral along C_3 also approaches zero provided that $t > x_1/\alpha$. Therefore

$$u_1(x_1, t) = \frac{1}{b}\left[1 - e^{-b(t - x_1/\alpha)}\right] \quad \text{when} \quad t > x_1/\alpha.$$

When $t < x_1/\alpha$, we can evaluate the displacement using the closed contour shown in Fig. 4.3.b. In this case the contour contains no poles, so the integral over the closed contour is zero and the resulting displacement is zero.

EXERCISE 4.2 Use Jordan's lemma to show that

$$\lim_{R\to\infty} \int_{C_S} \frac{1}{2\pi(b + i\omega)^2} e^{i\omega(t - x_1/\alpha)}\, d\omega = 0 \quad \text{when} \quad t > \frac{x_1}{\alpha},$$

where C_S is the semicircular contour shown in Fig. 4.4.b.

Solution—If we let

$$s = i\omega, \quad a = t - \frac{x_1}{\alpha}, \quad h(s) = \frac{1}{2\pi i(b + s)^2},$$

the integral becomes

$$I = \int_{C_S} h(s) e^{as}\, ds.$$

Along the contour C_S, $\omega = d + ie$ and $s = -e + id$, where e and d are real and $e \geq 0$. Thus $\mathrm{Re}(as) = -e(t - x_1/\alpha) \leq 0$ when $t > x_1/\alpha$. Also, as $R \to \infty$, $h(s) \to 1/2\pi i s^2$, so $|h(s)| \to 0$. Therefore Jordan's lemma states that $I \to 0$ as $R \to 0$.

EXERCISE 4.3 Show that when $t < x_1/\alpha$, the solution of Eq. (4.26) is

$$u_1 = \lim_{R\to\infty} \int_{C_R} g(\omega)\, d\omega = 0.$$

Solution—From Eq. (4.25),

$$g(\omega) = h(\omega)e^{a\omega}$$

where

$$h(\omega) = \frac{1}{2\pi(b+i\omega)^2}, \quad a = i\omega\left(t - \frac{x_1}{\alpha}\right).$$

The function $g(\omega)$ has a second-order pole at $\omega = bi$.

Consider the contour composed of segments C_R and C'_S shown in this exercise. It contains no poles, so

$$\lim_{R\to\infty}\int_{C_R} g(\omega)\,d\omega + \lim_{R\to\infty}\int_{C'_S} g(\omega)\,d\omega = 0.$$

On C'_S, $\omega = d - ie$, where d and e are real and $e \geq 0$. Also on C'_S, $|h(\omega)| \to 0$ as $R \to \infty$, and $\mathrm{Re}(a\omega) = e(t - x_1/\alpha) \leq 0$ provided that $t < x_1/\alpha$. Therefore Jordan's lemma states that the second integral in the above equation approaches zero as $R \to 0$, giving the desired result.

Exercise 4.4 Suppose that an unbounded elastic material is subjected to the initial displacement and velocity fields

$$\begin{aligned} u_1(x_1, 0) &= p(x_1), \\ \frac{\partial u_1}{\partial t}(x_1, 0) &= \quad 0, \end{aligned}$$

where $p(x_1)$ is a prescribed function. By taking a Fourier transform with respect to x_1 with the transform variable denoted by k, show that the solution for the Fourier transform of the displacement is

$$u_1^F = \tfrac{1}{2}p^F\left(e^{-i\alpha kt} + e^{i\alpha kt}\right),$$

where p^F is the Fourier transform of the function $p(x_1)$.

Solution—We take the Fourier transform of the wave equation with respect to x_1 with the transform variable denoted by k:

$$\int_{-\infty}^{\infty} \frac{\partial^2 u_1}{\partial t^2} e^{-ikx_1}\,dx_1 = \int_{-\infty}^{\infty} \alpha^2 \frac{\partial^2 u_1}{\partial x_1^2} e^{-ikx_1}\,dx_1.$$

Integrating the left side by parts twice and using the conditions that u_1 and $\partial u_1/\partial x_1$ are zero at $x_1 = \pm\infty$, we obtain

$$\frac{d^2 u_1^F}{dt^2} + k^2 \alpha^2 u_1^F = 0.$$

The solution of this ordinary differential equation is

$$u_1^F(k,t) = Ae^{-i\alpha kt} + Be^{\alpha kt}.$$

The initial conditions give

$$u_1^F\big|_{t=0} = p^F = A + B$$

and

$$\left.\frac{du_1^F}{dt}\right|_{t=0} = 0 = -ik\alpha A + ik\alpha B.$$

Solving these two equations, we obtain $A = B = p^F/2$, which yields the desired solution.

EXERCISE 4.5 Consider the function of time $f(t) = t$.
(a) If you represent this function as a discrete Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} A_n e^{i\omega_n t}$$

over the interval $0 \le t \le 1$, show that the coefficients are

$$A_n = \frac{e^{-i\omega_n}(i\omega_n + 1)}{\omega_n^2} - \frac{1}{\omega_n^2}. \tag{4.80}$$

(b) Show that $A_0 = 1/2$.

Solution—
(a) The coefficients are

$$\begin{aligned} A_n &= \int_0^1 f(t)e^{-i\omega_n t}\,dt \\ &= \int_0^1 te^{-i\omega_n t}\,dt \\ &= \left[\frac{e^{-i\omega_n t}}{(-i\omega_n)^2}(-i\omega_n - 1)\right]_0^1 \\ &= \frac{e^{-i\omega_n}(i\omega_n + 1)}{\omega_n^2} - \frac{1}{\omega_n^2}. \end{aligned}$$

(b) The Taylor series of the term $e^{-i\omega_n}$ is

$$e^{-i\omega_n} = 1 - i\omega_n - \frac{1}{2}\omega_n^2 + \frac{1}{6}i\omega_n^3 + O(\omega_n^4),$$

where $O(\omega_n^4)$ means “terms of order 4 or greater in ω_n.” Dividing this expression by ω_n^2 gives

$$\frac{e^{-i\omega_n}}{\omega_n^2} = \frac{1}{\omega_n^2} - \frac{i}{\omega_n} - \frac{1}{2} + \frac{1}{6}i\omega_n + O(\omega_n^2).$$

Substituting this series into the expression for A_n gives

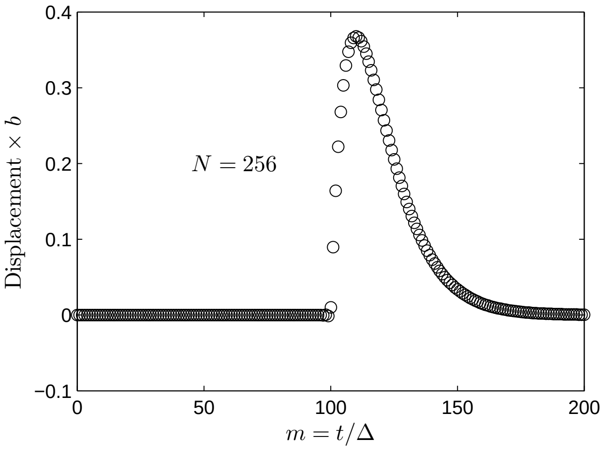
$$A_n = \frac{1}{2} + O(\omega_n),$$

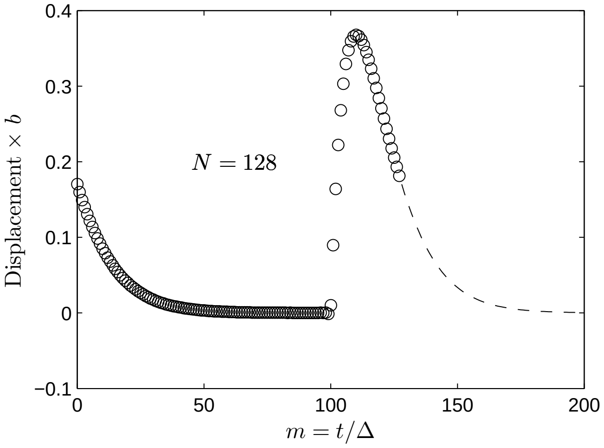
so that $A_0 = 1/2$.

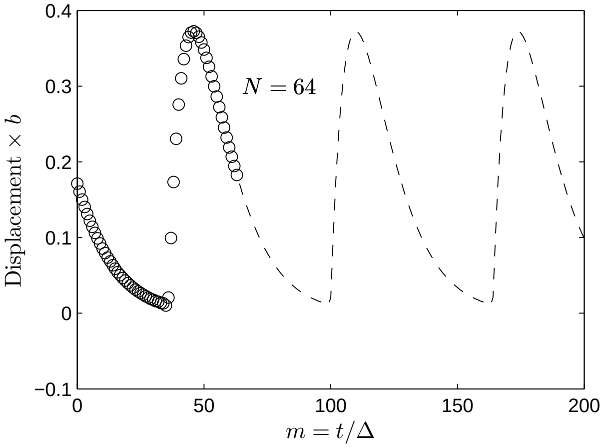
Exercise 4.6 Calculate the results shown in Fig. 4.6. Use $T = 100$ and $\alpha = 1$. Repeat the calculation using N =256, 128, and 64. What causes the solution to change?

Discussion—In carrying out these computations, you need to be aware that implementations of the FFT vary. Some are specialized to real values of f_n, but some accept complex values. The particular implementation used influences your choice of N. Another variation is the placement of the normalization factor $1/N$; it can appear in Eq. (4.36) or in Eq. (4.41), with a corresponding affect on Eq. (4.43).

Solution—The FFT solution is a periodic function over the interval $0 \leq m \leq N$. The dashed lines show the periodic continuations of the solutions.


0.4
0.3
0.2
0.1
0
−0.1
Displacement × b
N = 256
0
50
100
150
200
m = t/Δ


0.4
0.3
0.2
0.1
0
−0.1
Displacement × b
N = 128
0
50
100
150
200
m = t/Δ



EXERCISE 4.7 In Exercise 4.6, assume that there is a second boundary at a distance $x_1 = L = 200\alpha\Delta$ from the existing boundary. Assuming the material is fixed at the new boundary, derive the finite Fourier transform of the displacement field.

Solution—We substitute the conditions at the two boundaries into Eq. (4.21), obtaining

$$\begin{aligned} u_1^F(0,\omega) &= \frac{1}{(b+i\omega)^2} = A + B, \\ u_1^F(L,\omega) &= \quad 0 \quad = Ae^{i\omega L/\alpha} + Be^{-i\omega L/\alpha}. \end{aligned}$$

Solving these equations for A and B and substituting the results into Eq. (4.21), the resulting expression for the Fourier transform of the displacement field is

$$u_1^F = \frac{\sin[\omega(x_1 - L)/\alpha]}{(b+i\omega)^2 \sin(-\omega L/\alpha)}.$$

To obtain the digital Fourier transform, we apply Eq. (4.38) with $\omega_m = 2\pi m/N\Delta$, obtaining

$$u_1^{DF}(n,\omega_m) = \frac{\sin[2\pi m(n-200)/N]}{b^2T(1+2\pi imr/N)^2 \sin[-2\pi m(200)/N]},$$

where we have set $L = 200\alpha\Delta$ and introduced the parameters $r = 1/b\Delta$ and $n = x_1/\alpha\Delta$.

Exercise 4.8 The discrete Fourier transform converts N real numbers into N complex numbers. Use the properties of periodicity and symmetry of f_n^{DF} to show that half of the values of the f_n^{DF} are sufficient to determine the other half.

Solution—From the definition of the discrete Fourier transform, Eq. (4.36), we obtain

$$f_{-m}^{DF} = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{2\pi imn/N}.$$

Because f_n is real, the complex conjugate of f_{-m}^{DF} is

$$\bar{f}_{-m}^{DF} = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-2\pi imn/N} = f_{-m}^{DF}. \quad (*)$$

Application of the periodicity condition (4.37) gives

$$\bar{f}_{N-m}^{DF} = f_m^{DF}.$$

For $m = 1, 2, \ldots, N-1$, half of the values of f_m^{DF} are equal to the complex conjugates of the other half. For the case $m = 0$, Eq. (*) gives $\bar{f}_0^{DF} = f_0^{DF}$. Thus

$$\mathrm{Im}(f_0^{DF}) = 0.$$

Consequently, of the $2N$ real numbers representing the N complex values of f_m^{DF}, one of them is zero and $N-1$ other values are negatives of the remaining numbers.

Exercise 4.9 Show that the group velocity $c_g = d\omega/dk$ is related to the phase velocity c_1 by

$$c_g = c_1 + k_1\frac{dc_1}{dk_1} = \frac{c_1}{1 - \dfrac{\omega}{c_1}\dfrac{dc_1}{d\omega}}.$$

Solution—Beginning with the definition $c_1 = \omega/k_1$, we obtain

$$\frac{dc_1}{dk_1} = \frac{1}{k_1}\frac{d\omega}{dk_1} - \frac{\omega}{k_1^2} = \frac{1}{k_1}(c_g - c_1),$$

which yields the first result. Similarly,

$$\frac{dc_1}{d\omega} = \frac{1}{k_1} - \frac{\omega}{k_1^2}\frac{dk_1}{d\omega} = \frac{1}{k_1}\left(1 - \frac{c_1}{c_g}\right),$$

which yields the second result.

Exercise 4.10 In Section 3.5, we analyzed acoustic waves in a channel.
(a) Show that the group velocity of such waves is related to the phase velocity by $c_g c_1 = \alpha^2$.
(b) What is the maximum value of the group velocity?

Solution—
(a) We write the dispersion relation, Eq. (3.54), in the form

$$k_1^2\alpha^2 = \omega^2 - \left(\frac{\alpha n\pi}{h}\right)^2.$$

The derivative of this expression with respect to k_1 yields

$$k_1\alpha^2 = \omega\frac{d\omega}{dk_1},$$

or

$$\alpha^2 = c_1 c_g. \quad (*)$$

(b) By substituting Eq. (3.54) into the relation $(*)$, we obtain

$$c_g = \frac{\alpha^2}{c_1} = \alpha\left[1 - \left(\frac{\alpha n\pi}{\omega h}\right)^2\right]^{1/2}.$$

For frequencies above the cutoff frequency, $1 \geq 1 - (\alpha n\pi/\omega h)^2 \geq 0$. Therefore

$$c_g \leq \alpha.$$

For frequencies below the cutoff frequency, c_1 and c_g are imaginary and the solution does not describe a propagating wave.

Exercise 4.11 Use Eq. (3.73) to derive an expression for the group velocity of a layered medium with periodic layers.

Solution—Differentiation of Eq. (3.73) with respect to k yields

$$\begin{aligned} d\sin(kd) &= \frac{d\omega}{dk}\Bigg\{\frac{d_a}{\alpha_a}\sin\left(\frac{\omega d_a}{\alpha_a}\right)\cos\left(\frac{\omega d_b}{\alpha_b}\right)+\frac{d_b}{\alpha_b}\cos\left(\frac{\omega d_a}{\alpha_a}\right)\sin\left(\frac{\omega d_b}{\alpha_b}\right) \\ &\qquad +\frac{1}{2}\left(\frac{z_a}{z_b}+\frac{z_b}{z_a}\right)\left[\frac{d_a}{\alpha_a}\cos\left(\frac{\omega d_a}{\alpha_a}\right)\sin\left(\frac{\omega d_b}{\alpha_b}\right)\right. \\ &\qquad\qquad \left.+\frac{d_b}{\alpha_b}\sin\left(\frac{\omega d_a}{\alpha_a}\right)\cos\left(\frac{\omega d_b}{\alpha_b}\right)\right]\Bigg\}. \end{aligned}$$

Solving for $d\omega/dk = c_g$ and rearranging terms yields

$$\begin{aligned} c_g &= d\sin(kd)\Bigg/\Bigg\{\left[\frac{d_a}{\alpha_a}+\frac{1}{2}\left(\frac{z_a}{z_b}+\frac{z_b}{z_a}\right)\frac{d_b}{\alpha_b}\right]\sin\left(\frac{\omega d_a}{\alpha_a}\right)\cos\left(\frac{\omega d_b}{\alpha_b}\right) \\ &\qquad +\left[\frac{d_b}{\alpha_b}+\frac{1}{2}\left(\frac{z_a}{z_b}+\frac{z_b}{z_a}\right)\frac{d_a}{\alpha_a}\right]\cos\left(\frac{\omega d_a}{\alpha_a}\right)\sin\left(\frac{\omega d_b}{\alpha_b}\right)\Bigg\}. \end{aligned}$$

Exercise 4.12 Use the result of Exercise 4.11 to show that the group velocity is zero at the edges of a pass band.

Solution—In Fig. 3.32, the edges of the pass bands separate intervals where k is real from intervals where k is complex. From the dispersion relation (3.73), the boundaries between real and complex solutions are where $\cos(kd) = \pm 1$. At these values of k, $\sin(kd) = 0$. Provided the denominator of the result for c_g from Exercise 4.11 is not zero, c_g is zero at the edges of the pass bands.

Exercise 4.13 Consider Lamb's problem for an acoustic medium. (See the discussion of an acoustic medium on page 44.) Use the Cagniard-de Hoop method to show that the solution for the stress component T_{33} is

$$T_{33} = -\frac{T_0}{\pi}H\left(t-\frac{r}{\alpha}\right)\frac{\sin\theta}{r\left(1-\dfrac{r^2}{t^2\alpha^2}\right)^{1/2}}.$$

Solution—An acoustic medium is a linear elastic material with zero shear modulus ($\mu = 0$). This exercise requires modifying the solution of Lamb's problem presented in Section 4.5 to the case of an acoustic medium.

Because $\mu = 0$, the Helmholtz decomposition is

$$\mathbf{u} = \nabla\phi,$$

and, from Eqs. (4.53) and (4.55),

$$T_{33} = \lambda\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) = \lambda\left(\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_3^2}\right). \quad (1)$$

From Eq. (4.56), the wave equation satisfied by ϕ is

$$\frac{\partial^2\phi}{\partial t^2} = \alpha^2\left(\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_3^2}\right). \quad (2)$$

The Laplace transform of the potential ϕ is

$$\phi^L = \int_0^\infty \phi e^{-st}\,dt,$$

and the Fourier transform of ϕ^L with respect to x_3 is defined to be

$$\phi^{LF} = \int_{-\infty}^{\infty} \phi^L e^{-iksx_1}\,dx_1.$$

From Eq. (4.61) and the discussion that follows it, applying the Laplace and Fourier transforms to the wave equation (2) leads to the solution

$$\phi^{LF} = Ae^{-(k^2 + v_P^2)^{1/2} s x_3},$$

where A is a constant and $v_P = 1/\alpha$ is the compressional "showness." From Eq. (4.62), applying the Laplace and Fourier transforms to the normal stress boundary condition leads to the equation

$$\left[\frac{d^2\phi^{LF}}{dx_3^2} - k^2 s^2 \phi^{LF}\right]_{x_3=0} = -\frac{T_0}{\lambda s}.$$

Using this equation to solve for A, we obtain the solution for ϕ^{LF}:

$$\phi^{LF} = -\frac{T_0}{\lambda s^3 v_P^2} e^{-(k^2 + v_P^2)^{1/2} s x_3}.$$

We apply the Fourier inversion integral to this expression:

$$\phi^L = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{T_0}{\lambda s^2 v_P^2} e^{-[(k^2 + v_P^2)^{1/2} x_3 - ikx_1]s}\,dk.$$

We substitute this result into the Laplace transform of Eq. (1), obtaining

$$T_{33}^L = -\frac{T_0}{2\pi}\int_{-\infty}^{\infty} e^{-[(k^2 + v_P^2)^{1/2}x_3 - ikx_1]s}\, dk. \quad (3)$$

To evaluate this integral, we define a real parameter t by

$$t = (k^2 + v_P^2)^{1/2}x_3 - ikx_1.$$

From Eq. (4.66), the solutions for k in terms of t are

$$k_\pm = \frac{it}{r}\cos\theta \pm \left(\frac{t^2}{r^2} - v_P^2\right)^{1/2}\sin\theta. \quad (4)$$

As the value of the parameter t goes from $v_P r$ to ∞, the values of k given by this equation describe the two paths B_- and B_+ in the complex k-plane shown in Fig. 4.27.

The integrand of Eq. (3) has branch points at $k = \pm v_P i$ but no poles. The discussion on page 200 applies to this integral. Consequently, the integration along the path C_1 can be replaced by integration along the paths B_- and B_+:

$$T_{33}^L = -\frac{T_0}{2\pi}\left[\int_{v_P r}^{\infty} e^{-st}\frac{\partial k_+}{\partial t}\, dt - \int_{v_P r}^{\infty} e^{-st}\frac{\partial k_-}{\partial t}\, dt\right].$$

By introducing the step function $H(t - v_P r)$, the lower limits of integration can be changed to zero:

$$T_{33}^L = -\frac{T_0}{2\pi}\int_0^{\infty} H(t - v_P r)\left(\frac{\partial k_+}{\partial t} - \frac{\partial k_-}{\partial t}\right)e^{-st}\, dt.$$

Because the parameter s appears only in the exponential term, this integral matches the definition of the Laplace transform. Therefore

$$T_{33} = -\frac{T_0}{2\pi}H(t - v_P r)\left(\frac{\partial k_+}{\partial t} - \frac{\partial k_-}{\partial t}\right).$$

From Eq. (4),

$$\frac{\partial k_+}{\partial t} - \frac{\partial k_-}{\partial t} = \frac{2t\sin\theta}{r^2\left(\frac{t^2}{r^2} - v_P^2\right)^{1/2}},$$

which completes the solution.

Chapter 5. Nonlinear Wave Propagation

Exercise 5.1 Consider the one-dimensional linear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2\frac{\partial^2 u}{\partial x^2},$$

where α is constant. Introducing the variables $v = \partial u/\partial t$ and $w = \partial u/\partial x$, it can be written as the system of first-order equations

$$\begin{aligned}\frac{\partial v}{\partial t} &= \alpha^2 \frac{\partial w}{\partial x},\\ \frac{\partial v}{\partial x} &= \ \frac{\partial w}{\partial t}.\end{aligned}$$

(a) Use Eq. (5.21) to show that there are two families of right-running and left-running characteristics in the x-t plane and that the characteristics are straight lines.
(b) Show that the system of first-order equations is hyperbolic.

Solution—
(a) Express the system of first-order equations in the form of Eq. (5.17) by letting

$$u_k = \begin{bmatrix} v \\ w \end{bmatrix}, \quad A_{km} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{km} = \begin{bmatrix} 0 & -\alpha^2 \\ -1 & 0 \end{bmatrix}.$$

Equation (5.21) becomes

$$\begin{bmatrix} c & \alpha^2 \\ 1 & c \end{bmatrix} = 0,$$

which has the roots

$$c = \pm\alpha.$$

Because α is a constant, the characteristics are straight lines.
(b) Equation (5.27) becomes

$$\begin{bmatrix} c & 1 \\ \alpha^2 & c \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 0.$$

There are two solutions for the vector $\mathbf{r}$:

$$r_2 = -\alpha r_1 \quad \text{for} \quad c = \alpha$$

and

$$r_2 = \alpha r_1 \quad \text{for} \quad c = -\alpha.$$

The system is hyperbolic because the roots c are real and two linearly independent vectors $\mathbf{r}$ exist.

Exercise 5.2 Consider the system of first-order equations in Exercise 5.1. Use the interior equation, Eq. (5.26), to show that $v - \alpha w$ is constant along a right-running characteristic and $v + \alpha w$ is constant along a left-running characteristic.

Solution—For a right-running characteristic, $c = \alpha$ and $r_2 = -\alpha r_1$. The interior equation for this characteristic is

$$r_1 \begin{bmatrix} 1 & -\alpha \end{bmatrix} \begin{bmatrix} \zeta_t & -\alpha^2 \zeta_x \\ -\zeta_x & \zeta_t \end{bmatrix} \begin{bmatrix} v_\zeta \\ w_\zeta \end{bmatrix},$$

which can be written

$$(v - \alpha w)_\zeta \, d\zeta = 0.$$

Along a right-running characteristic, $d\eta = 0$, and consequently

$$d(v - \alpha w) = (v - \alpha w)_\zeta \, d\zeta = 0.$$

Therefore $v - \alpha w$ is constant along a right-running characteristic. Use the same argument to show that $v + \alpha w$ is constant along a left-running characteristic, for which $c = -\alpha$ and $r_2 = \alpha r_1$.

Exercise 5.3 Consider the simple wave solution beginning on page 224. Suppose that the relationship between the stress and the deformation gradient is $\tilde{T}(F) = E_0 \ln F$, where E_0 is a constant.

(a) Show that within the characteristic fan, the velocity is given in terms of the deformation gradient by

$$v = -2\alpha_0(F^{1/2} - 1),$$

where $\alpha_0 = (E_0/\rho_0)^{1/2}$.

(b) Show that within the characteristic fan, the deformation gradient and velocity are given as functions of X, t by

$$F = \frac{\alpha_0^2 t^2}{X^2}, \qquad v = -2\alpha_0 \left(\frac{\alpha_0 t}{X} - 1) \right).$$

Solution—

(a) The derivative of the stress with respect to the deformation gradient is

$$\tilde{T}_F = \frac{E_0}{F},$$

therefore

$$\alpha = \left(\frac{\tilde{T}_F}{\rho_0}\right)^{1/2} = \frac{\alpha_0}{F^{1/2}}. \quad (*)$$

Substitution of this expression into Eq. (5.30) gives

$$v = -2\alpha_0(F^{1/2} - 1).$$

(b) Because each characteristic within the fan is a straight line through the origin, $\alpha = X/t$. From Eq. $(*)$,

$$F = \frac{\alpha_0^2}{\alpha^2} = \frac{\alpha_0^2 t^2}{X^2}.$$

Substituting this expression into the answer to Part (a) yields the desired result.

EXERCISE 5.4 A half space of elastic material is initially undisturbed. At $t = 0$ the boundary is subjected to a uniform constant stress T_0. The stress is related to the deformation gradient by the logarithmic expression $T = \tilde{T}(F) = E_0 \ln F$, where E_0 is a constant. Show that the resulting velocity of the boundary of the half space toward the left is

$$\left(\frac{E_0}{\rho_0}\right)^{1/2} \left[e^{(T_0/2E_0)} - 1\right].$$

Solution—In Fig. 5.10, the characteristic fan separates region 1 from region 2. The stress T and deformation gradient F in region 2 are determined by the applied stress:

$$T = E_0 \ln F = T_0.$$

We solve for F,

$$F = e^{T_0/E_0},$$

and substitute this result into the expression for v from Part (a) of Exercise 5.3, obtaining

$$v = -\left(\frac{E_0}{\rho_0}\right)^{1/2} \left[e^{(T_0/2E_0)} - 1\right].$$

Exercise 5.5 Consider a weak wave propagating into an undisturbed nonlinear elastic material. Show that the equation governing the rate of change of the discontinuity in $\hat{v}_X$ is

$$\frac{d}{dt}[\![\hat{v}_X]\!] = \left(\frac{\tilde{T}_{FF}}{2\rho_0\alpha_0^2}\right)[\![\hat{v}_X]\!]^2.$$

Solution—Applying the chain rule,

$$\hat{v}_X = \eta_X v_\eta + \zeta_X v_\zeta.$$

Letting $\zeta = t$ and $\eta = t - X/\alpha_0$ in this equation, we obtain

$$v_\eta = -\alpha_0 \hat{v}_X.$$

Substitution of this expression into Eq. (5.36) yields the desired result.

Exercise 5.6 (a) Determine the relation $T = \tilde{T}(F)$ for a linear elastic material. (b) What is the velocity of an acceleration wave in a linear elastic material? (c) What is the velocity of a shock wave in a linear elastic material?

Solution—

(a) In one-dimensional motion of a linear elastic material, the density is related to the longitudinal strain by

$$\frac{\rho_0}{\rho} = 1 + E_{11},$$

so

$$E_{11} = \frac{\rho_0}{\rho} - 1 = F - 1,$$

and the stress-strain relation is

$$T_{11} = (\lambda + 2\mu)E_{11}.$$

Therefore

$$T = \tilde{T}(F) = (\lambda + 2\mu)(F - 1). \quad (*)$$

(b) From Eq. (∗), $\tilde{T}_F = d\tilde{T}/dF = \lambda + 2\mu$. Then from Eq. (5.48), the velocity of an acceleration wave is

$$v_s = \left(\frac{\tilde{T}_F}{\rho_0}\right)^{1/2} = \left(\frac{\lambda + 2\mu}{\rho_0}\right)^{1/2}.$$

(c) From Eq. (∗), $[\![T]\!] = (\lambda + 2\mu)[\![F]\!]$. Then from Eq. (5.56), the velocity of a shock wave is

$$v_s = \left(\frac{[\![T]\!]}{\rho_0[\![F]\!]}\right)^{1/2} = \left(\frac{\lambda + 2\mu}{\rho_0}\right)^{1/2}.$$

Exercise 5.7 The relation between the stress and the deformation gradient in a material is $T = \tilde{T}(F) = E_0 \ln F$, where E_0 is a constant. The density of the undeformed material is ρ_0.

(a) What is the velocity of an acceleration wave in the undeformed material?

(b) If the material is homogeneously compressed so that its density is $2\rho_0$, what is the velocity of an acceleration wave?

Solution—

(a) The velocity of an acceleration wave is

$$v_s = \left(\frac{\tilde{T}_F}{\rho_0}\right)^{1/2} = \left(\frac{E_0}{\rho_0 F}\right)^{1/2}. \quad (*)$$

The deformation gradient in the undisturbed material ahead of the wave is $F = 1$. Consequently, in the undisturbed material

$$v_s = \left(\frac{E_0}{\rho_0}\right)^{1/2}.$$

(b) In a material compressed to $2\rho_0$, $F = 1/2$. From Eq. (∗), the velocity is

$$v_s = \left(\frac{2E_0}{\rho_0}\right)^{1/2}.$$

Exercise 5.8 The relation between the stress and the deformation gradient in a material is $T = \tilde{T}(F) = E_0 \ln F$, where E_0 is a constant. The density of the undeformed material is ρ_0. An acceleration wave propagates into the undeformed material. At $t = 0$, the acceleration of the material just behind the wave is a_0 in the direction of propagation. What is the acceleration of the material just behind the wave at time t?

Solution—Because F is continuous across an acceleration wave, $F = 1$ just behind the wave. The growth of the amplitude of the wave is given by Eq. (5.53):

$$\frac{d}{dt}\left[\!\!\left[\frac{\partial \hat{v}}{\partial t}\right]\!\!\right] = -\frac{\tilde{T}_{FF}}{2\rho_0 v_s^3}\left[\!\!\left[\frac{\partial \hat{v}}{\partial t}\right]\!\!\right]^2 = \frac{1}{2(E_0/\rho_0)^{1/2}}\left[\!\!\left[\frac{\partial \hat{v}}{\partial t}\right]\!\!\right]^2.$$

Integrating this equation with respect to time yields

$$\left[\!\!\left[\frac{\partial \hat{v}}{\partial t}\right]\!\!\right]^{-1} = \frac{-t}{2(E_0/\rho_0)^{1/2}} + \frac{1}{a_0},$$

which after rearrangement yields the result.

Exercise 5.9 If a gas behaves isentropically, the relation between the pressure and the density is

$$\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma},$$

where γ, the ratio of specific heats, is a positive constant. In the one-dimensional problems we are discussing, the stress $T = -p$.

(a) Show that the velocity of an acceleration wave propagating into undisturbed gas with pressure p_0 and density ρ_0 is

$$v_s = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}.$$

(b) If the amplitude $[\![\partial \hat{v}/\partial t]\!]$ of an acceleration wave propagating into undisturbed gas with pressure p_0 and density ρ_0 is positive, show that the amplitude will increase with time.

Solution—(a) Expressing the isentropic relation in terms of the stress $T = -p$ and $F = \rho_0/\rho$, we obtain

$$T = \tilde{T}(F) = -p_0 F^{-\gamma}.$$

Therefore

$$\tilde{T}_F = \frac{d\tilde{T}}{dF} = \gamma p_0 F^{-(\gamma+1)}. \quad (*)$$

At the wavefront, $\rho = \rho_0$, so $F = 1$ and $\tilde{T}_F = \gamma p_0$. Therefore the acceleration wave velocity is

$$v_s = \left(\frac{\tilde{T}_F}{\rho_0}\right)^{1/2} = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}.$$

(b) From Eq. $(*)$,

$$\tilde{T}_{FF} = -(\gamma + 1)\gamma p_0 F^{-(\gamma+2)}.$$

At the wavefront, $\tilde{T}_{FF} = -(\gamma + 1)\gamma p_0$. Because $\tilde{T}_{FF}$ is negative, Eq. (5.53) indicates that the rate of change of $[\![\partial\hat{v}/\partial t]\!]$ with respect to time is positive.

Exercise 5.10 The relation between the stress and the deformation gradient in a material is $T = \tilde{T}(F) = E_0 \ln F$, where E_0 is a constant. The density of the undeformed material is ρ_0. A shock wave propagates into the undeformed material. If the density of the material just behind the wave is $2\rho_0$ at a particular time, show that the velocity of the wave at that time is

$$v_s = \left[\frac{2E_0 \ln 2}{\rho_0}\right]^{1/2}.$$

Solution—The velocity of a shock wave is given by Eq. (5.56):

$$v_s = \left(\frac{[\![T]\!]}{\rho_0[\![F]\!]}\right)^{1/2}.$$

In the undisturbed material, $F = 1$ and $T = 0$. Behind the shock wave, $F = \rho_0/2\rho_0 = 1/2$ and $T = E_0 \ln(1/2)$. Therefore

$$v_s = \left[\frac{E_0 \ln(1/2)}{-\rho_0/2}\right]^{1/2}.$$

Exercise 5.11 The stress-strain relation of an elastic material is given by the solid curve. The point labeled (−) is the state ahead of a shock wave and the point labeled (+) is the state just behind the wave at a time t. Determine whether $[\![F]\!]$ will increase, remain the same, or decrease as a function of time in cases (a), (b), and (c).

Solution—From Eq. (5.56),

$$\rho_0 v_s^2 = \frac{[\![T]\!]}{[\![F]\!]} = \frac{T_- - T_+}{F_- - F_+}.$$

The expression on the right is the slope of the dashed straight line from point (+) to point (−). The term $(\tilde{T}_F)_+$ in Eq. (5.61) is the slope of the solid curve at point (+), which is greater than the slope of the dashed straight line. Therefore the term

$$(\tilde{T}_F)_+ - \rho_0 v_s^2$$

in Eq. (5.61) is positive. In cases (a), (b), and (c), $[\![\partial F/\partial X]\!]$ is positive, zero, and negative, respectively, so Eq. (5.61) indicates that $[\![F]\!]$ will decrease with time in case (a), remain constant in case (b), and increase with time in case c. (Because F_+ is negative, notice that this means that the *magnitude* of the jump in F will *increase* with time in case (a), remain constant in case (b), and *decrease* with time in case (c).)

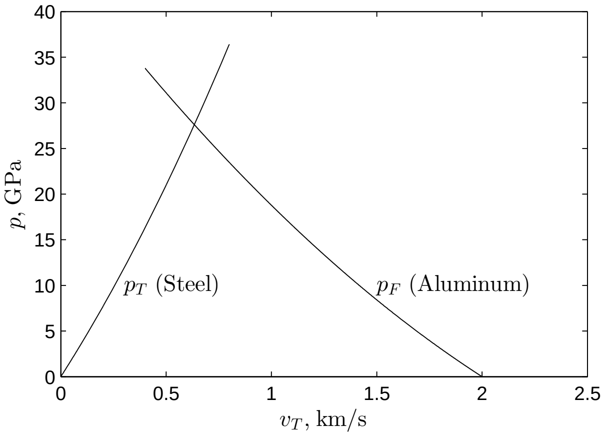
Exercise 5.12 Consider a flyer plate experiment in which the flyer plate and the target are identical materials with mass density 2.79 Mg/m^3, the measured shock wave velocity is 6.66 km/s, and the measured flyer-plate velocity is 2 km/s. What is the pressure in the equilibrium region behind the loading wave?

Solution—From Eqs. (5.72), the pressure in the equilibrium region behind the loading wave can be expressed as $p_e = \rho_0 U v_e$, where U is the shock wave velocity and v_e is the velocity of the material behind the loading wave. Because the flyer plate and the target are identical materials, the Hugoniot curves in Fig. 5.22 intersect at $v_T = v_0/2 = 1$ km/s. Another way to obtain this result is to view the experiment using a reference frame that is moving to the right with velocity $v_0/2$. Relative to this reference, the flyer plate and the target approach each other with the same velocity $v_0/2$. Because the flyer plate and the target consist of the same material, the velocity of the materials behind the loading waves must be zero. Then if we revert to the fixed reference frame, the velocity of the target material behind the loading wave is $v_0/2$. Therefore, the pressure of the material behind the loading wave is

$$\begin{aligned}p_e &= \rho_0 U v_e \\ &= (2.79 \times 10^3\ \mathrm{kg/m^3})(6660\ \mathrm{m/s})(1000\ \mathrm{m/s}) \\ &= 18.6 \times 10^9\ \mathrm{Pa}\ \ (18.6\ \mathrm{GPa}).\end{aligned}$$

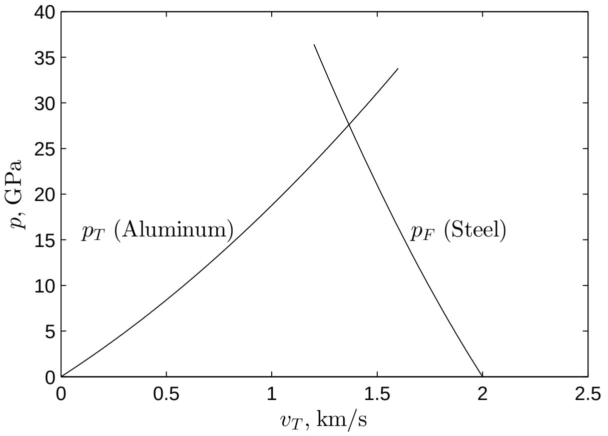
EXERCISE 5.13 (a) Consider a flyer plate experiment in which the flyer plate is aluminum with properties $\rho_0 = 2.79\ \mathrm{Mg/m^2}$, $c = 5.33$ km/s, and $s = 1.40$, the target is stainless steel with properties $\rho_0 = 7.90\ \mathrm{Mg/m^2}$, $c = 4.57$ km/s, and $s = 1.49$, and the velocity of the flyer plate is 2 km/s. What is the pressure in both materials in the equilibrium region behind the loading waves, and what is the velocity of their interface?
(b) If the materials are interchanged so that the flyer plate is stainless steel and the target is aluminum, what is the pressure in both materials in the equilibrium region behind the loading waves and what is the velocity of their interface?

Solution—
(a) We use Eq. (5.74) to obtain the graph of p_T as a function of v_T (the Hugoniot) for the stainless steel target. We use Eqs. (5.75) and (5.77) to obtain the graph of p_F as a function of v_T for the aluminum flyer plate, obtaining



The point where the two curves intersect, $v_T = 0.634$ km/s and $p = 27.6$ GPa, are the velocity and pressure of the materials in the equilibrium region.

(b) Reversing the materials, we obtain the graph

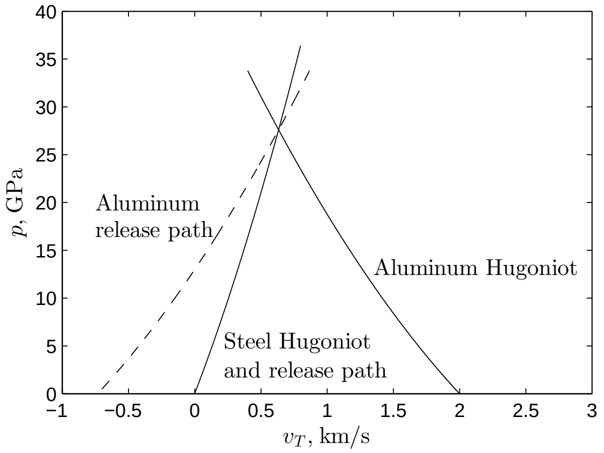


The curves now intersect at $v_T = 1.37$ km/s and $p = 27.6$ GPa

Exercise 5.14 In Exercise 5.13, assume that the target is much thicker than the flyer plate. After the loading wave in the flyer plate reaches the back surface of the plate, it is reflected as a release wave and returns to the interface between the materials. Determine whether the plates separate when the release wave reaches the interface: (a) if the flyer plate is aluminum and the target is stainless steel; (b) if the flyer plate is stainless steel and the target is aluminum.

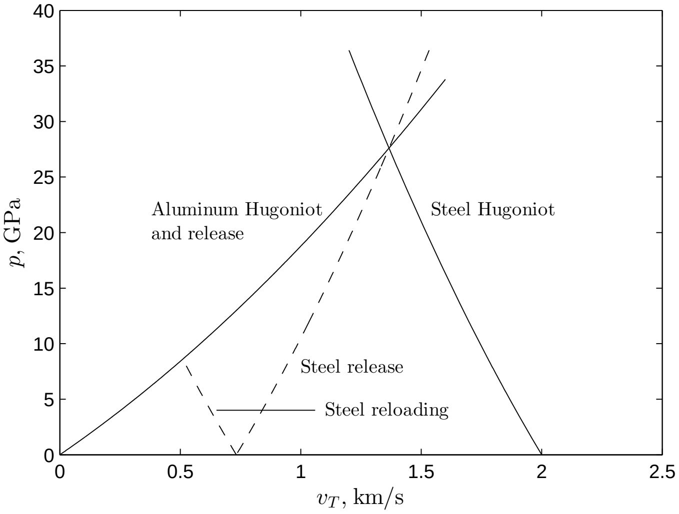
Solution—

(a) We use Eq. (5.74) to obtain the graph of p_T as a function of v_T (the Hugoniot) for the steel target. We use Eqs. (5.75) and (5.77) to obtain the graph of p_F as a function of v_T for the aluminum flyer plate. These Hugoniots are the solid curves shown below. Because the back surface of the flyer plate is free of stress, the loading wave reflects as a release wave. The release path from the equilibrium state is approximated by the aluminum Hugoniot, shown as a dashed line:



Within the release wave in the aluminum, the pressure decreases to zero as the material velocity decreases to −732 m/s. When the release wave reaches the interface between the materials, the steel unloads along its original Hugoniot and the pressure and material velocity decrease to zero. Because the interface cannot support tension, the materials separate when the pressure in the steel reaches zero.

(b) The Hugoniots for the steel flyer plate and the aluminum target are the solid curves shown below. The release wave that reflects from the back surface of the flyer plate causes the pressure in the steel to decrease to zero and the velocity to decrease to 732 m/s. When the release wave reaches the interface between the materials, the aluminum unloads along its Hugoniot. However, because the steel is moving toward the aluminum target at 732 m/s, the pressure in the aluminum does not drop to zero. The target and flyer plate remain in contact, a partial release wave continues into the aluminum, and a reloading wave is formed in the steel. Because the release wave spreads as it travels from the free left surface of the aluminum plate to the interface, the reloading wave in the steel is not a shock wave and approximately follows the steel Hugoniot shown.



Appendix A. Complex Analysis

Exercise A.1 Show that the magnitude of a complex variable z is given by the relation

$$|z| = (z\bar{z})^{1/2}.$$

Solution—

$$\begin{aligned}(z\bar{z})^{1/2} &= [(x+iy)(x-iy)]^{1/2}\\ &= (x^2 - i^2y^2)^{1/2}\\ &= (x^2+y^2)^{1/2}\\ &= |z|.\end{aligned}$$

Exercise A.2 Show that for any two complex numbers z_1 and z_2,

$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Solution—Consider the polar form of a complex variable

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta).$$

Its magnitude is

$$\begin{aligned} |re^{i\theta}| &= |r\cos\theta + ir\sin\theta| \\ &= (r^2\cos^2\theta + r^2\sin^2\theta)^{1/2} \\ &= r. \end{aligned}$$

Using that result, we express z_1 and z_2 in polar form and obtain

$$\begin{aligned} |z_1||z_2| &= |r_1 e^{i\theta_1}||r_2 e^{i\theta_2}| = r_1 r_2, \\ |z_1 z_2| &= |r_1 r_2 e^{i(\theta_1 + \theta_2)}| = r_1 r_2, \end{aligned}$$

and

$$\begin{aligned} \frac{|z_1|}{|z_2|} &= \frac{|r_1 e^{i\theta_1}|}{|r_2 e^{i\theta_2}|} = \frac{r_1}{r_2}, \\ \left|\frac{z_1}{z_2}\right| &= \left|\frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}\right| = \frac{r_1}{r_2}. \end{aligned}$$

Exercise A.3

(a) Show that the exponential function $f(z) = e^z$ is analytic.

(b) Show that the function $f(z) = \bar{z}$ is not analytic anywhere.

Solution—

(a) Letting $z = x + iy$, the exponential function can be written

$$e^z = e^x(\cos y + i\sin y).$$

Its real and imaginary parts are

$$u = e^x \cos y, \quad v = e^x \sin y.$$

Evaluating their partial derivatives,

$$\begin{aligned}\frac{\partial u}{\partial x} &= e^x \cos y, & \frac{\partial u}{\partial y} &= -e^x \sin y,\\ \frac{\partial v}{\partial x} &= e^x \sin y, & \frac{\partial v}{\partial y} &= e^x \cos y,\end{aligned}$$

we see that

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y},\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned}$$

Thus the Cauchy-Riemann equations hold for all values of z, so the exponential function is analytic everywhere.

(b) The real and imaginary parts of the conjugate function $f(z) = \bar{z}$ are

$$u = x, \quad v = -y.$$

Their partial derivatives are

$$\begin{aligned}\frac{\partial u}{\partial x} &= 1, & \frac{\partial u}{\partial y} &= 0,\\ \frac{\partial v}{\partial x} &= 0, & \frac{\partial v}{\partial y} &= -1.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\partial u}{\partial x} &\neq \frac{\partial v}{\partial y},\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned}$$

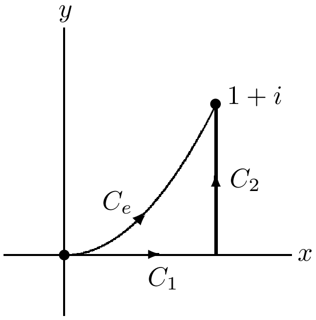
The Cauchy-Riemann equations do not hold for any value of z, so the conjugate function is not analytic everywhere.

EXERCISE A.4 Evaluate the contour integral

$$\int_C (1+z)\, dz$$

along the contour shown from the point $z = 0$ to the point $z = 1 + i$. Compare your result with the result of the example on page 357.

Solution—Consider two contours:



The contour $C_1 + C_2$ is the integration path for this exercise. The contour C_e is the integration path in the example on page 357.

Expressing the integrand in terms of its real and imaginary parts, for any path C,

$$\int_C (1+z)\,dz = \int_C [(1+x)dx - y\,dy] + i\int_C [y\,dx + (1+x)dy].$$

To integrate along the path $C_1 + C_2$, we note that $y = 0$ on C_1 and $x = 1$ on C_2, obtaining

$$\begin{aligned}\int_{C_1+C_2} (1+z)\,dz &= \int_0^1 (1+x)\,dx - \int_0^1 y\,dy + i\int_0^1 2dy \\ &= \left(x + \frac{x}{2}\right)\Big|_0^1 - \frac{y^2}{2}\Big|_0^1 + 2y|_0^1 \\ &= 1 + 2i.\end{aligned}$$

This is the same value we obtained on page 357 for integration along the path C_e. This result is confirmed by the Cauchy integral theorem. Because the integrand $1 + z$ is analytic and C_1, C_2 and C_e form a piecewise-smooth closed contour, the theorem states that

$$\int_{C_1} (1+z)\,dz + \int_{C_2} (1+z)\,dz - \int_{C_e} (1+z)\,dz = 0.$$

Exercise A.5

(a) Show that the function

$$f(z) = \frac{1}{z-1}$$

has a first-order pole at $z = 1$.

(b) Show that the function

$$f(z) = \frac{1}{(z-1)^2}$$

has a second-order pole at $z = 1$.

Solution—
(a) From Eq. (A.8), the function $f(z) = 1/(z-1)$ is analytic except at $z = 1$. Because

$$(z-1)f(z) = 1$$

is analytic, $f(z)$ has a first-order pole at $z = 1$.
(b) From Eq. (A.8), the function $f(z) = 1/(z-1)^2$ is analytic except at $z = 1$. Because

$$(z-1)f(z) = \frac{1}{z-1}$$

has an isolated singularity at $z = 1$ that is not removable and

$$(z-1)^2 f(z) = 1$$

is analytic, $f(z)$ has a second-order pole at $z = 1$.

Exercise A.6 Evaluate the contour integral

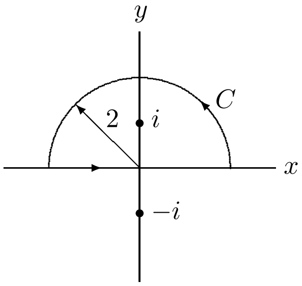
$$\int_C \frac{dz}{z^2+1}$$

for the closed contour shown.

Solution—Write the contour integral as

$$\int_C \frac{dz}{z^2+1} = \int_C \frac{dz}{(z+i)(z-i)}.$$

The integrand has first-order poles at $z = -i$ and $z = i$, shown below.



Only the pole at $z = i$ is within the closed contour. Its residue is

$$\lim_{z\to i}\left(\frac{1}{z+i}\right) = \frac{1}{2i}.$$

The value of the contour integral is $2\pi i$ times the sum of the residues of the poles enclosed by the contour:

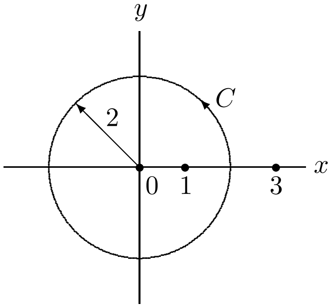
$$\int_C \frac{dz}{z^2+1} = 2\pi i\left(\frac{1}{2i}\right) = \pi.$$

EXERCISE A.7 Evaluate the contour integral

$$\int_C \frac{dz}{z(z-1)(z-3)^2}$$

for the closed circular contour shown.

Solution—The integrand has poles at $z = 0$, $z = 1$, and $z = 3$, shown below. Only the first-order poles at $z = 0$ and $z = 1$ are within the closed contour.



The residue of the pole at $z = 0$ is

$$\lim_{z\to 0}\left[\frac{1}{(z-1)(z-3)^2}\right] = -\frac{1}{9}.$$

The residue of the pole at $z = 1$ is

$$\lim_{z\to 1}\left[\frac{1}{z(z-3)^2}\right] = \frac{1}{4}.$$

The value of the contour integral is $2\pi i$ times the sum of the residues of the poles enclosed by the contour:

$$\int_C \frac{dz}{z(z-1)(z-3)^2} = 2\pi i\left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5}{18}\pi i.$$

Exercise A.8 By using the definition of the square root of the complex variable z given in Eq. (A.15), show that:
(a) the two values of the square root of 1 are $+1$ and -1;
(b) the two values of the square root of i are $(1+i)/\sqrt{2}$ and $-(1+i)/\sqrt{2}$.

Solution—
(a) From Eq. (A.15),

$$\begin{aligned} 1^{1/2} &= e^{i(2\pi m)/2}, \qquad m = 0, 1 \\ &= \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right. . \end{aligned}$$

(b) Similarly,

$$\begin{aligned} i^{1/2} &= e^{i(\pi/2 + 2\pi m)/2}, \qquad m = 0, 1 \\ &= \left\{ \begin{array}{c} e^{\pi i/4} \\ e^{5\pi i/4} \end{array} \right. \\ &= \left\{ \begin{array}{c} e^{\pi i/4} \\ -e^{\pi i/4} \end{array} \right. . \end{aligned}$$

Exercise A.9 By using the definition of the square root of the complex variable z given in Eq. (A.15), show that there are only two values of the square root of z.

Solution— From Eq. (A.15),

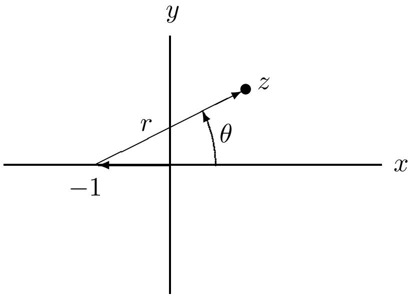
$$\begin{aligned} z^{1/2} &= r^{1/2} e^{i(\theta + 2\pi m)/2}, \qquad m = 0, 1 \\ &= \left\{ \begin{array}{c} r^{1/2} e^{i\theta/2} \\ r^{1/2} e^{i(\theta/2 + \pi)} \end{array} \right. \\ &= \left\{ \begin{array}{c} r^{1/2} e^{i\theta/2} \\ -r^{1/2} e^{i\theta/2} \end{array} \right. . \end{aligned}$$

EXERCISE A.10 Suppose that you make the function $f(z) = (z+1)^{1/2}$ single valued by using the branch cut shown. If $f(i) = 1.099 + 0.455i$, what is the value of $f(-i)$?

Solution—Let the complex variable z be written as

$$z = -1 + re^{i\theta},$$

where the argument θ is measured counterclockwise from the positive x axis:



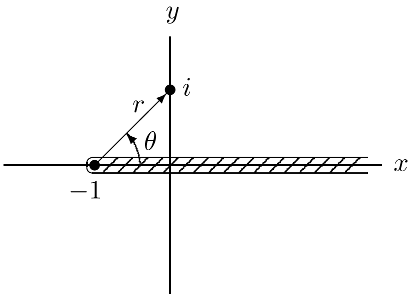
Then

$$z + 1 = re^{i\theta},$$

and the two branches of $(z+1)^{1/2}$ are (see the solution to Exercise A.9)

$$(z+1)^{1/2} = \begin{cases} r^{1/2}e^{i\theta/2} \\ -r^{1/2}e^{i\theta/2} \end{cases}.$$

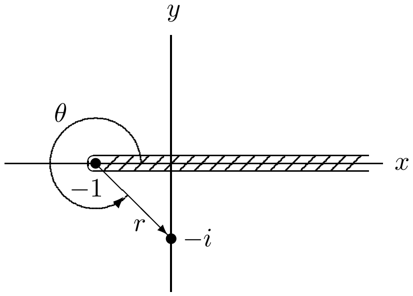
When $z = i$,



so that $r = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$ and $\theta = \pi/4$, the first branch of $(z+1)^{1/2}$ gives

$$\begin{aligned}(z+1)^{1/2} &= (2)^{1/4}(\cos\pi/8 + i\sin\pi/8) \\ &= 1.099 + 0.455i.\end{aligned}$$

To have a single-valued function, we must remain on the first branch. When $z = -i$,



so that $r = \sqrt{2}$ and $\theta = 7\pi/4$, the first branch of $(z+1)^{1/2}$ gives

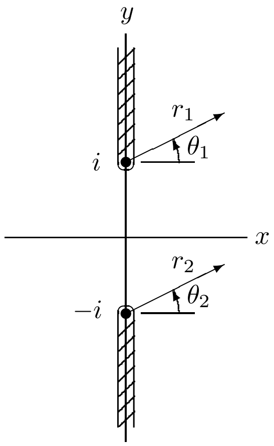
$$\begin{aligned}(z+1)^{1/2} &= (2)^{1/4}(\cos 7\pi/8 + i \sin 7\pi/8)\\ &= -1.099 + 0.455i.\end{aligned}$$

EXERCISE A.11 Suppose that you make the function $f(z) = (z^2+1)^{1/2}$ single valued by using the branch cuts shown. If $f(1+2i) = 1.112 + 1.799i$, what is the value of $f(-1+2i)$?

Solution—Notice that

$$f(z) = (z-i)^{1/2}(z+i)^{1/2},$$

so $f(z)$ is the product of two multivalued functions. We introduce variables r_1, r_2, θ_1, and θ_2 defined as shown below:



If the complex variable z is written as

$$z = i + r_1 e^{i\theta_1},$$

then

$$z - i = r_1 e^{i\theta_1}$$

and the two branches of $(z-i)^{1/2}$ are (see the solution to Exercise A.9)

$$(z-i)^{1/2} = \begin{cases} r_1^{1/2} e^{i\theta_1/2} \\ -r_1^{1/2} e^{i\theta_1/2} \end{cases} .$$

Then if we write the complex variable z as

$$z = -i + r_2 e^{i\theta_2},$$

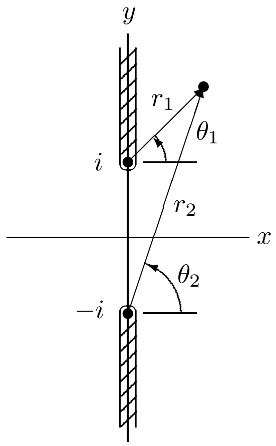
the expression

$$z + i = r_2 e^{i\theta_2}$$

and the two branches of $(z+i)^{1/2}$ are

$$(z+i)^{1/2} = \begin{cases} r_2^{1/2} e^{i\theta_2/2} \\ -r_2^{1/2} e^{i\theta_2/2} \end{cases} .$$

When $z = 1 + 2i$,



the terms

$$r_1 = \sqrt{(1)^2 + (1)^2} = \sqrt{2},$$

$$r_2 = \sqrt{(1)^2 + (3)^2} = \sqrt{10},$$

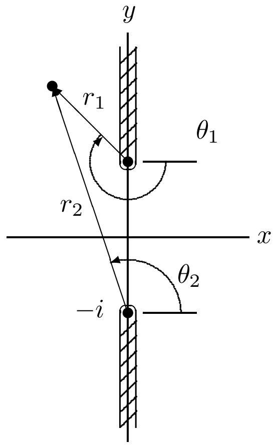
$$\theta_1 = \arctan(1/1) = 45°,$$

$$\theta_2 = \arctan(3/1) = 71.6°.$$

Using the first branches of the two square roots, we obtain

$$\begin{aligned} f(1 + 2i) &= r_1^{1/2} r_2^{1/2} e^{i(\theta_1 + \theta_2)/2} \\ &= 1.112 + 1.799i. \end{aligned}$$

To obtain a single-valued function, we must continue to use the first branches of the square roots. When $z = -1 + 2i$,



the terms

$$\begin{aligned} r_1 &= \sqrt{2}, \\ r_2 &= \sqrt{10}, \\ \theta_1 &= -225^\circ, \\ \theta_2 &= 90^\circ + \arctan(1/3) = 108^\circ. \end{aligned}$$

Using the first branches of the two square roots, we obtain

$$\begin{aligned} f(-1+2i) &= r_1^{1/2} r_2^{1/2} e^{i(\theta_1 + \theta_2)/2} \\ &= 1.112 - 1.799i. \end{aligned}$$

Appendix A

Complex Analysis

Complex variables are used extensively in the theory of elastic wave propagation. In this appendix we briefly summarize the results from complex analysis used in this book.

A.1 Complex Variables

Here we define a complex variable, give examples of functions of a complex variable, and show how complex variables can be represented by points on a plane. We also define the derivative of a function of a complex variable and analytic functions.

A complex variable is a variable of the form $x + iy$, where x and y are real variables and i is defined to have the property that $i^2 = -1$. The variable x is called the real part of z, and the variable y is called the imaginary part. We denote the real and imaginary parts of a complex variable by the notations

$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z).$$

The magnitude $|z|$ of a complex variable z is defined by

$$|z| = (x^2 + y^2)^{1/2},$$

and the conjugate $\bar{z}$ of z is defined by

$$\bar{z} = x - iy.$$

Functions of a complex variable

A function of a complex variable $f(z)$ is simply a function of z. Functions of a complex variable obey the usual rules of algebra; for example,

$$\begin{aligned} f(z) = z^2 &= (x+iy)^2 \\ &= x^2 + 2xiy + i^2y^2 \\ &= x^2 - y^2 + i2xy. \end{aligned} \tag{A.1}$$

The real and imaginary parts of a function $f(z)$ are denoted by

$$f(z) = u + iv.$$

From Eq. (A.1), we see that the real and imaginary parts of the function $f(z) = z^2$ are $u = x^2 - y^2$ and $v = 2xy$.

Another example of a function of a complex variable is the exponential function e^z, which is defined by

$$e^z = e^x(\cos y + i \sin y). \tag{A.2}$$

The real and imaginary parts of e^z are $u = e^x \cos y$ and $v = e^x \sin y$.

Graphical representation

The values of a complex variable z can be represented by points on a plane by plotting $x = \mathrm{Re}(z)$ along the horizontal axis and $y = \mathrm{Im}(z)$ along the vertical axis (Fig. A.1). This plane is called the *complex plane*. The distance $r = (x^2+y^2)^{1/2}$ from the origin to z is equal to the magnitude $|z|$. The angle θ is called the *argument* of z. Because $x = r\cos\theta$ and $y = r\sin\theta$, we can write z as

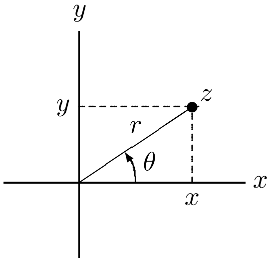


Figure A.1: The complex plane.

$$\begin{aligned} z = x + iy &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta}, \end{aligned} \tag{A.3}$$

where we have used Eq. (A.2). This expression is called the polar form of the complex variable z.

Derivatives and analytic functions

The derivative of a function of a complex variable $f(z)$ is defined in exactly the same way as the derivative of a function of a real variable:

$$\frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}.$$

By writing $f(z) = u(x,y) + iv(x,y)$ and $\Delta z = \Delta x + i\Delta y$, we can express this definition in the form

$$\frac{df(z)}{dz} = \lim_{\Delta x, \Delta y \to 0} \frac{\dfrac{\partial u}{\partial x}\Delta x + \dfrac{\partial u}{\partial y}\Delta y + i\dfrac{\partial v}{\partial x}\Delta x + i\dfrac{\partial v}{\partial y}\Delta y}{\Delta x + i\Delta y}. \tag{A.4}$$

Depending upon how Δx and Δy approach zero, the point $z + \Delta z$ approaches z along different paths in the complex plane. We set $\Delta x = \Delta r \cos\theta$ and $\Delta y = \Delta r \sin\theta$, where Δr is the magnitude of Δz and the angle θ is the argument of Δz (Fig. A.2). Equation (A.4) becomes

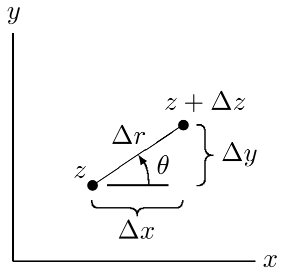


Figure A.2: Points z and $z + \Delta z$ in the complex plane.

$$\frac{df(z)}{dz} = \frac{\dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta + i\dfrac{\partial v}{\partial x}\cos\theta + i\dfrac{\partial v}{\partial y}\sin\theta}{\cos\theta + i\sin\theta}.$$

Multiplying this equation by $\cos\theta - i\sin\theta$, we obtain the result

$$\begin{aligned}\frac{df(z)}{dz} \quad = \quad & \left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)\sin^2\theta + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)\sin\theta\cos\theta \\ & + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\cos^2\theta.\end{aligned} \tag{A.5}$$

This expression is independent of the angle θ (that is, the value of the derivative of $f(z)$ is independent of the path along which $z + \Delta z$ approaches z) only if the coefficients of $\sin^2\theta$ and $\cos^2\theta$ are equal and the coefficient of $\sin\theta\cos\theta$ equals zero. This requires that

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y},\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned} \tag{A.6}$$

These two equations, which are necessary conditions for the derivative of $f(z)$ to have a unique value at z, are called the *Cauchy-Riemann equations.* When they are satisfied, Eq. (A.5) becomes

$$\frac{df(z)}{dz} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}.$$

If the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ are continuous and the Cauchy-Riemann equations are satisfied at a point z, the function $f(z)$ is said to be *analytic* at z. When a function is analytic at every point, we simply say that it is analytic.

Several important classes of functions are analytic or are analytic almost everywhere in the complex plane.

- Polynomials of the form

  $$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_N z^N, \tag{A.7}$$

  where $a_0, a_1, a_2, \ldots, a_N$ are constants, are analytic.

- Rational functions of the form

  $$f(z) = \frac{p(z)}{q(z)} = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_N z^N}{b_0 + b_1 z + b_2 z^2 + \cdots + b_M z^M} \tag{A.8}$$

  are analytic except at points of the complex plane where $q(z) = 0$.

- Analytic functions of analytic functions are analytic.

A.2 Contour Integration

We first introduce the concept of contour integration and state the Cauchy integral theorem. We then define poles and show how to evaluate contour integrals around closed contours that contain poles. We also discuss multiple-valued complex functions and describe the use of branch cuts.

Cauchy integral theorem

A contour integral is an integral of the form

$$I = \int_C f(z)\, dz, \tag{A.9}$$

where C denotes that the integral is evaluated along a prescribed contour in the complex plane (Fig. A.3). By writing $f(z) = u + iv$ and $dz = dx + i\, dy$, we can express this integral in terms of two real line integrals:

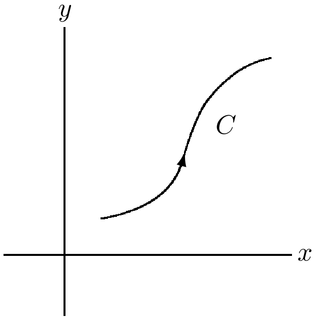


Figure A.3: A contour C in the complex plane.

$$I = \int_C (u\, dx - v\, dy) + i \int_C (v\, dx + u\, dy).$$

For example, suppose that $f(z) = 1 + z$ and the contour C is the curve $y = x^2$ (Fig. A.4). Let us evaluate the integral in Eq. (A.9) from the point $x = 0$, $y = 0$ on the contour to the point $x = 1$, $y = 1$.

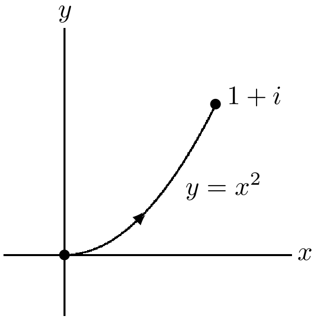


Figure A.4: The contour $y = x^2$.

$$\int_C (1 + z)\, dz \quad = \quad \int_C (1 + x + iy)(dx + i\, dy)$$

$$\begin{aligned} &= \int_C [(1+x)\,dx - y\,dy] + i\int_C [y\,dx + (1+x)\,dy] \\ &= \int_0^1 (1+x)\,dx - \int_0^1 y\,dy + i\int_0^1 x^2\,dx + i\int_0^1 (1+y^{1/2})\,dy \\ &= 1+2i. \end{aligned}$$

The *Cauchy integral theorem* states that if a function $f(z)$ is analytic on and within a closed, piecewise smooth contour C,

$$\int_C f(z)\,dz = 0.$$

Poles

If a function $f(z)$ is not analytic at a point z_0 of the complex plane, but is analytic in some neighborhood $0 < |z - z_0| < \epsilon$, where ϵ is a positive real number, the point z_0 is called an *isolated singularity* of the function $f(z)$. If a point z_0 is an isolated singularity of $f(z)$, but the limit of $f(z)$ as $z \to z_0$ exists and is single valued, the point z_0 is called a *removable isolated singularity* of $f(z)$. When this is the case, the function $f(z)$ is analytic at the point z_0 if the value of $f(z)$ at z_0 is defined to be

$$f(z_0) = \lim_{z\to z_0} f(z).$$

Let k be a positive integer. If the function $(z - z_0)^k f(z)$ is analytic or has a removable isolated singularity at z_0, but $(z - z_0)^{k-1} f(z)$ has an isolated singularity at z_0 that is not removable, $f(z)$ is said to have a kth-order pole at z_0.

Suppose that we wish to evaluate the contour integral of a function $f(z)$ about a closed contour C, and let us assume that $f(z)$ is analytic on and within C except at a finite number N of poles (Fig. A.5.a). That is, we want to evaluate the integral

$$I = \int_{C^+} f(z)\,dz, \tag{A.10}$$

where the superscript + indicates that the integral is evaluated in the counterclockwise direction. We cannot use the Cauchy integral theorem to evaluate this integral because the function $f(z)$ is not analytic within the contour. However, we can apply the Cauchy integral theorem to the contour shown in Fig. A.5.b:

$$\int_{C^+} f(z)\,dz + \sum_{k=1}^{N} \int_{C_k^-} f(z)\,dz = 0,$$

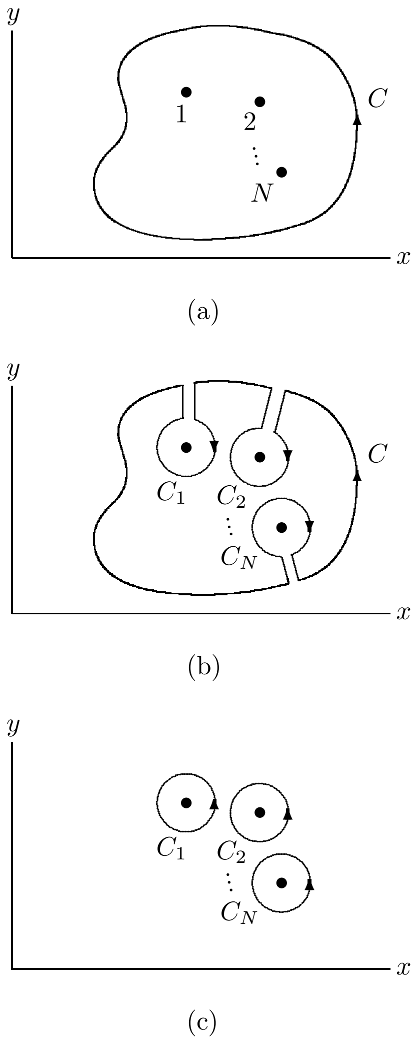


Figure A.5: (a) A contour C containing a finite number N of poles. (b) A contour that doesn't contain the poles. (c) Circular contours around the individual poles.

where C_k^- is a circular contour about the k th pole. The $-$ superscript indicates that the integral is evaluated in the clockwise direction. This result shows that the integral over the contour C can be expressed as the sum of integrals over closed circular contours about the poles of $f(z)$ within C (Fig. A.5.c):

$$\int_{C^+} f(z)\, dz = \sum_{k=1}^{N} \int_{C_k^+} f(z)\, dz.$$

The usefulness of this result derives from the fact that simple expressions can be derived for the contour integral of a function $f(z)$ about a pole.

Residues

Suppose that a function $f(z)$ has a first-order pole at a point z_0. From the definition of a first-order pole, this means that the function $\phi(z) = (z - z_0)f(z)$ has a removable isolated singularity at z_0. We make the function $\phi(z)$ analytic at z_0 by defining

$$\phi(z_0) = \lim_{z\to z_0} (z - z_0)f(z).$$

Our objective is to evaluate the integral

$$\int_C f(z)\, dz = \int_C \frac{\phi(z)}{z - z_0}\, dz \tag{A.11}$$

for a closed circular contour C of radius R around z_0 (Fig. A.6.a). (We choose R so that the function $\phi(z)$ is analytic on and within C.) Let z be a point on the contour C. We can express $z - z_0$ in polar form as (Fig. A.6.b)

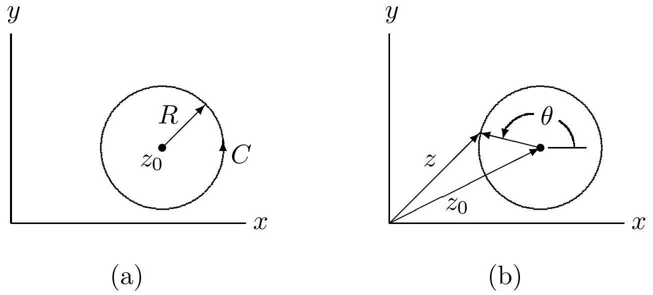


Figure A.6: (a) Circular contour around a pole z_0. (b) A point z on the contour.

$$z - z_0 = Re^{i\theta}.$$

Using this expression, we can write Eq. (A.11) as

$$\int_C f(z)\,dz = \int_0^{2\pi} i\phi(z_0 + Re^{i\theta})\,d\theta. \tag{A.12}$$

To evaluate the integral on the right side, we express the function $\phi(z_0 + Re^{i\theta})$ in a Taylor series:

$$\begin{aligned} \phi(z_0 + Re^{i\theta}) &= \phi(z_0) + \frac{d\phi}{dz}(z_0)Re^{i\theta} + \tfrac{1}{2}\frac{d^2\phi}{dz^2}(z_0)R^2e^{i2\theta} + \cdots \\ &= \phi(z_0) + \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n\phi}{dz^n}(z_0)R^n e^{in\theta}. \end{aligned}$$

We substitute this result into Eq. (A.12), obtaining the equation

$$\int_C f(z)\,dz = i\phi(z_0)\int_0^{2\pi} d\theta + \sum_{n=1}^{\infty} \frac{i}{n!}\frac{d^n\phi}{dz^n}(z_0)R^n \int_0^{2\pi} e^{in\theta}\,d\theta.$$

The integrals

$$\int_0^{2\pi} e^{in\theta}\,d\theta = 0$$

for $n = 1, 2, \ldots$ Therefore we obtain the result

$$\int_C f(z)\,dz = 2\pi i\phi(z_0) = 2\pi i \lim_{z\to z_0}(z - z_0)f(z).$$

This equation gives the value of the contour integral of a function $f(z)$ when the function is analytic on and within the contour except at a first-order pole z_0 within the contour. This result can be extended to the case when $f(z)$ is analytic on and within the contour except at a kth-order pole z_0:

$$\int_C f(z)\,dz = 2\pi i \frac{1}{(k-1)!}\lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}[(z - z_0)^k f(z)].$$

The term

$$\frac{1}{(k-1)!}\lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}[(z - z_0)^k f(z)] \tag{A.13}$$

is called the *residue* of a kth-order pole. Using this terminology, we can write the value of the contour integral of a function $f(z)$ that is analytic on and within the contour except at a finite number of poles in the simple form

$$\int_C f(z)\,dz = 2\pi i \sum \text{Residues}.$$

As an example, we evaluate the integral

$$\int_C f(z)\,dz = \int_C \frac{2z^2}{z^2 - 2z - 3}\,dz$$

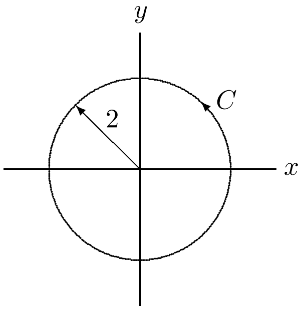


Figure A.7: A closed contour C.

for the closed contour C shown in Fig. A.7. The integrand is a rational function (see Eq. (A.8)) and is analytic except at points where the denominator equals zero. It can be written

$$f(z) = \frac{2z^2}{(z+1)(z-3)}.$$

This function has first-order poles at $z = -1$ and at $z = 3$. The pole at $z = 3$ is not within the contour. The residue of the pole at $z = -1$ is

$$\text{Residue } = \lim_{z \to -1} (z+1) \frac{2z^2}{(z+1)(z-3)} = -\tfrac{1}{2}.$$

The value of the integral is $2\pi i(-\frac{1}{2}) = -\pi i$.

Exercises

Exercise A.1 Show that the magnitude of a complex variable z is given by the relation

$$|z| = (z\bar{z})^{1/2}.$$

Exercise A.2 Show that for any two complex numbers z_1 and z_2,

$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

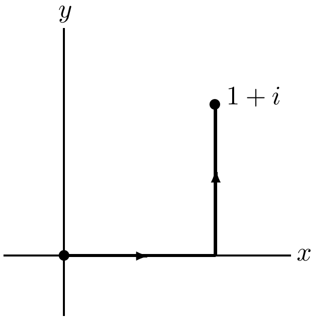
Discussion—Express z_1 and z_2 in polar form.

Exercise A.3

(a) Show that the exponential function $f(z) = e^z$ is analytic.

(b) Show that the function $f(z) = \bar{z}$ is not analytic anywhere.

Exercise A.4



Evaluate the contour integral

$$\int_C (1+z)\,dz$$

along the contour shown from the point $z = 0$ to the point $z = 1 + i$. Compare your result with the result of the example on page 357.

Exercise A.5

(a) Show that the function

$$f(z) = \frac{1}{z-1}$$

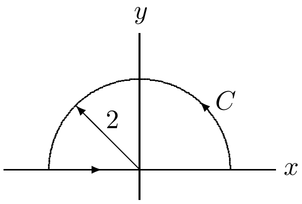
has a first-order pole at $z = 1$.

(b) Show that the function

$$f(z) = \frac{1}{(z-1)^2}$$

has a second-order pole at $z = 1$.

Exercise A.6



Evaluate the contour integral

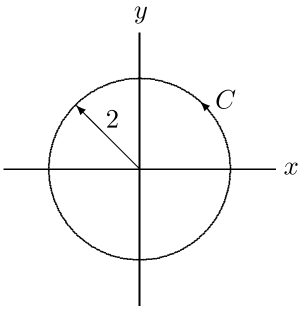
$$\int_C \frac{dz}{z^2+1}$$

for the closed contour shown.

Discussion—Find the poles of the integrand by locating the points where $z^2 + 1 = 0$.

Answer: The value of the integral is π.

EXERCISE A.7



Evaluate the contour integral

$$\int_C \frac{dz}{z(z-1)(z-3)^2}$$

for the closed circular contour shown.

Answer: The value of the integral is $\frac{5}{18}\pi i$.

A.3 Multiple-Valued Functions

Some functions $f(z)$ can have more than one value for a given value of z. In this section we describe the most common multiple-valued functions and show how *branch cuts* are used to insure that they remain single valued.

Examples

Only three types of multiple-valued functions occur commonly in complex analysis: roots, logarithms, and complex variables with complex exponents.

Roots

The square root of the complex variable z is defined by the expression

$$z^{1/2} = (re^{i\theta})^{1/2} = r^{1/2}e^{i\theta/2}. \tag{A.14}$$

That is, the square root of z is defined in terms of its magnitude r and its argument θ, and we see that

$$\left(r^{1/2}e^{i\theta/2}\right)\left(r^{1/2}e^{i\theta/2}\right) = re^{i\theta} = z.$$

If we increase the argument by 2π, the magnitude r and the argument $\theta + 2\pi$ define the same point z in the complex plane. However, this magnitude and argument give us a different value of the square root of z:

$$z^{1/2} = [re^{i(\theta + 2\pi)}]^{1/2} = r^{1/2}e^{i(\theta + 2\pi)/2}.$$

It is easy to confirm that this square root is equal to the negative of the square root in Eq. (A.14):

$$r^{1/2}e^{i(\theta + 2\pi)/2} = -r^{1/2}e^{i\theta/2},$$

which is analogous to the square root of a real number.

A natural question is whether there are more values of the square root of z. Although the magnitude r and the argument $\theta + 2\pi n$, where n is any integer, also define the same point z in the complex plane, it is easy to confirm that there are only two values of the square root of z. In fact, the complex variable z raised to the $1/n$ power has exactly n roots:

$$z^{1/n} = r^{1/n}e^{i(\theta + 2\pi m)/n}, \quad m = 0, 1, \cdots, n-1. \tag{A.15}$$

Logarithms

The natural logarithm of the complex variable z is defined in terms of its magnitude r and argument θ by

$$\ln z = \ln r + i\theta. \tag{A.16}$$

Because the argument of z is uniquely defined only within a multiple of 2π, this definition has infinitely many values for a given value of z:

$$\ln z = \ln r + i(\theta + 2\pi m),$$

where m is any integer.

Complex variables with complex exponents

The value of the complex variable z raised to a complex exponent α is defined by the expression

$$z^{\alpha} = e^{\alpha \ln z}. \tag{A.17}$$

Because $\ln z$ is a multiple-valued function, the function z^{α} is multiple valued.

Branch cuts

The definitions given in Eqs. (A.14), (A.16), and (A.17) yield single-valued functions only if the range of the argument θ is less than 2π. This has an important implication when we evaluate contour integrals using the Cauchy integral theorem; if the integrand contains multiple-valued functions, the ranges of their arguments must be prevented from exceeding 2π. If this is not done, the values of these functions are discontinuous and the theorem is invalid.

For example, the function

$$z^{1/2} = r^{1/2} e^{i\theta/2}$$

defines a single-valued function if the range of the argument is $0 < \theta < 2\pi$. If the argument does not remain in this range, the function is multiple valued. This can be illustrated as shown in Fig. A.8.a, which shows what are called the *Riemann sheets* of the function $z^{1/2}$. On the "front" sheet, the value of the function is

$$z^{1/2} = r^{1/2} e^{i\theta/2},$$

and on the "rear" sheet, its value is

$$z^{1/2} = r^{1/2} e^{i(\theta + 2\pi)/2}.$$

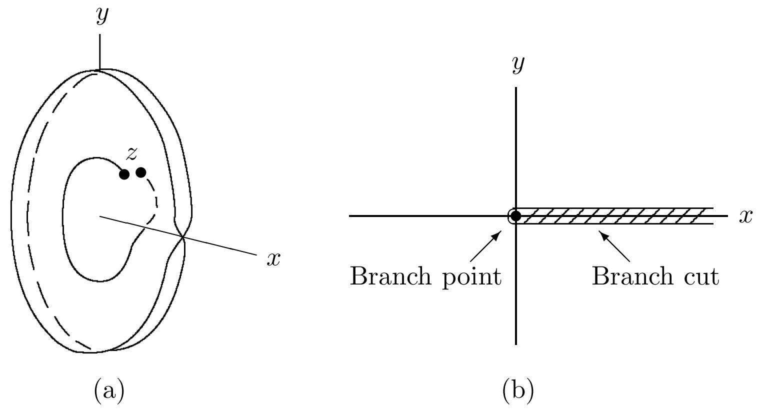


Figure A.8: (a) The Riemann sheets of the function $f(z) = z^{1/2}$. (b) A branch cut in the complex plane.

In the figure we show an integration path in the complex plane that goes from a point z with argument $\pi/4$ to the same point z with argument $\pi/4 + 2\pi$. Observe that the function is multiple valued at z. As a consequence, its value on the integration path is discontinuous.

Multiple-valued functions are avoided by using what are called *branch cuts* in the complex plane. A branch cut is simply a line in the complex plane that integration paths are not permitted to cross. This line is said to be "cut" from the complex plane. Figure A.8.b shows a branch cut along the positive real axis. On any integration path that does not cross this branch cut, the argument of z remains in the range $0 < \theta < 2\pi$ and the function $z^{1/2}$ is single valued. The endpoint of a branch cut is called a *branch point*. Notice that any continuous line that begins at $z = 0$ and extends to infinity is an acceptable branch cut to make the function $z^{1/2}$ single valued.

As a second example, consider the function

$$f(z) = (z - z_0)^{1/2}.$$

By writing z as $z = z_0 + re^{i\theta}$ (Fig. A.9.a), we can see that $f(z)$ has the two values

$$f(z) = \begin{cases} r^{1/2}e^{i\theta/2}, \\ r^{1/2}e^{i(\theta + 2\pi)/2}. \end{cases}$$

We can make this function single valued by introducing the branch cut shown in Fig. A.9.b. It insures that θ remains in the range $-3\pi/2 < \theta < \pi/2$. Observe

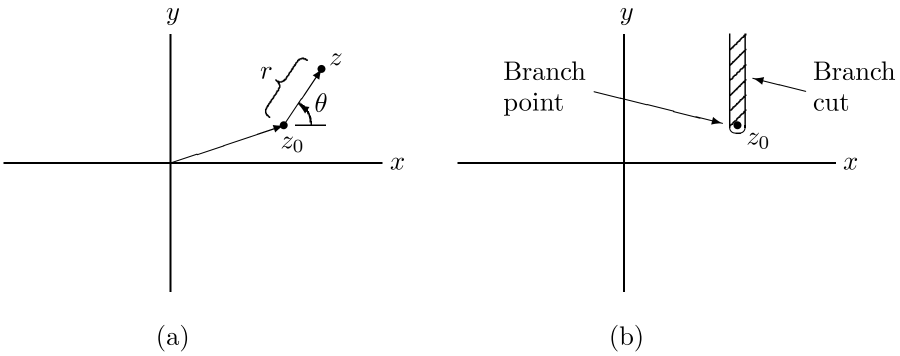


Figure A.9: (a) The points z_0 and z. (b) A branch cut in the complex plane.

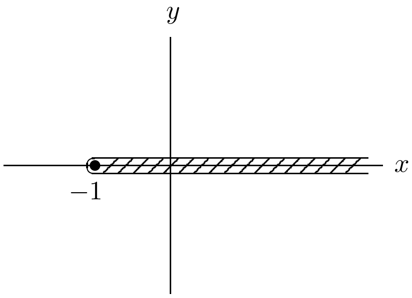
that the branch point is the point at which the expression within the square root equals zero.

Exercises

Exercise A.8 By using the definition of the square root of the complex variable z given in Eq. (A.15), show that:
(a) the two values of the square root of 1 are $+1$ and -1;
(b) the two values of the square root of i are $\frac{1}{2}\sqrt{2}(1+i)$ and $-\frac{1}{2}\sqrt{2}(1+i)$.

Exercise A.9 By using the definition of the square root of the complex variable z given in Eq. (A.15), show that there are only two values of the square root of z.

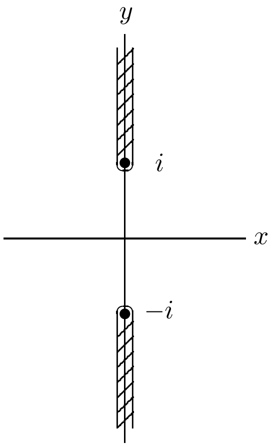
Exercise A.10 Suppose that you make the function $f(z) = (z+1)^{1/2}$ single valued by using the branch cut



If $f(i) = 1.099 + 0.455i$, what is the value of $f(-i)$?

Answer: $f(-i) = -1.099 + 0.455i$.

Exercise A.11 Suppose that you make the function $f(z) = (z^2 + 1)^{1/2}$ single valued by using the branch cuts



If $f(1 + 2i) = 1.112 + 1.799i$, what is the value of $f(-1 + 2i)$?

Answer: $f(-i) = 1.112 - 1.799i$.

Appendix B

Tables of Material Properties

Table B.1: Relationships between the elastic constants, where $R = \sqrt{E^2 + 9\lambda^2 + 2E\lambda}$

	E	ν	μ	λ
E, ν			$\dfrac{E}{2(1+\nu)}$	$\dfrac{E\nu}{(1+\nu)(1-2\nu)}$
E, μ		$\dfrac{E-2\mu}{2\mu}$		$\dfrac{\mu(E-2\mu)}{3\mu-E}$
E, λ		$\dfrac{2\lambda}{E+\lambda+R}$	$\dfrac{E-3\lambda+R}{4}$	
μ, ν	$2\mu(1+\nu)$			$\dfrac{2\mu\nu}{1-2\nu}$
ν, λ	$\dfrac{\lambda(1+\nu)(1-2\nu)}{\nu}$		$\dfrac{\lambda(1-2\nu)}{2\nu}$	
μ, λ	$\dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\dfrac{\lambda}{2(\lambda+\mu)}$		

Table B.2: Properties of various solids

Material	ρ (Mg/m^3)	α (km/s)	β (km/s)	E (GPa)	ν	λ (GPa)	μ (GPa)
Aluminum (rolled)	2.70	6.42	3.04	67.6	0.35	61.3	24.9
Beryllium	1.87	12.8	8.88	309	0.04	15.7	147
Bismuth	9.80	2.20	1.10	31.6	0.33	23.7	11.8
Brass (yellow)	8.64	4.70	2.10	104	0.37	114	38.1
Cadmium	8.60	2.80	1.50	50.2	0.29	28.7	19.3
Copper (rolled)	8.93	5.01	2.27	126	0.37	132	46.0
Epon 828	1.21	2.83	1.23	5.06	0.38	6.02	1.83
Fused silica	2.20	5.70	3.75	69.2	0.11	9.60	30.9
Glass (pyrex)	2.24	5.64	3.28	59.9	0.24	23.0	24.0
Gold (hard drawn)	19.7	3.24	1.20	80.5	0.42	150	28.3
Ice	0.91	3.99	1.98	9.61	0.33	7.40	3.59
Iron	7.69	5.90	3.20	203	0.29	110	78.7
Iron (cast)	7.22	4.60	2.60	123	0.26	55.1	48.8
Lead	11.2	2.20	0.70	15.8	0.44	43.2	5.48
Magnesium	1.73	5.80	3.00	41.2	0.31	27.1	15.6
Molybdenum	10.0	6.30	3.40	299	0.29	165	115
Nickel	8.84	5.60	3.00	206	0.29	118	79.5
Platinum	21.4	3.26	1.73	167	0.30	99.3	64.0
Silver	10.6	3.60	1.60	74.7	0.37	83.1	27.1
Steel (mild)	7.80	5.90	3.20	206	0.29	111	79.8
Steel (stainless)	7.89	5.79	3.10	196	0.29	112	75.8
Titanium carbide	5.15	8.27	5.16	323	0.18	77.9	137
Tungsten	19.4	5.20	2.90	415	0.27	198	163
Zircaloy	9.36	4.72	2.36	139	0.33	104	52.1
Nylon (6/6)	1.12	2.60	1.10	3.77	0.39	4.86	1.35
Polyethylene	0.90	1.95	0.54	0.76	0.45	2.89	0.26
Polystyrene	1.05	2.40	1.15	3.75	0.35	3.27	1.38

Table B.3: Properties of various liquids

Material	ρ (Mg/m^3)	α (km/s)	λ (GPa)
Acetonyl acetone	0.72	1.40	1.42
Alcohol (ethanol 25°C)	0.79	1.20	1.15
Alcohol (methanol)	0.79	1.10	0.96
Argon (87K)	1.43	0.84	1.00
Carbon tetrachloride (25°C)	1.59	0.92	1.36
Chloroform (25°C)	1.49	0.98	1.45
Fluorinert (FC-40)	1.86	0.64	0.76
Gasoline	0.80	1.25	1.25
Glycol (polyethylene 200)	1.08	1.62	2.85
Helium-4 (2K)	0.14	0.22	0.00
Honey (Sue Bee orange)	1.42	2.03	5.85
Kerosene	0.81	1.32	1.41
Nitromethane	1.13	1.33	1.99
Oil (baby)	0.82	1.43	1.67
Oil (jojoba)	1.17	1.45	2.47
Oil (olive)	0.91	1.44	1.91
Oil (peanut)	0.91	1.43	1.88
Oil (SAE 30)	0.87	1.70	2.51
Oil (sperm)	0.88	1.44	1.82
Turpentine (25°C)	0.88	1.25	1.38
Water (20°C)	1.10	1.40	2.16

Table B.4: Properties of various gases

Material	ρ (kg/m^3)	α (km/s)	λ (MPa)
Air (dry at 0°C)	1.29	0.33	0.14
Carbon dioxide (0°C)	1.97	0.25	0.13
Helium (0°C)	0.17	0.96	0.16
Nitrogen (0°C)	1.25	0.33	0.13
Oxygen (0°C)	1.42	0.31	0.14
Oxygen (20°C)	1.32	0.32	0.14