$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = 0$$

$$0 = (1-\lambda)\left[(1-\lambda)^2 - 1\right] - 1\left(1-\lambda-1\right) + 1\left(1-(1-\lambda)\right)$$

$$0 = (1-\lambda)\left[\cancel{1} + \lambda^2 - 2\lambda \cancel{-1}\right] \cancel{-1} + \lambda \cancel{+1} + \cancel{1} \cancel{-1} + \lambda$$

$$= (1-\lambda)(\lambda^2 - 2\lambda) + 2\lambda = 0$$

$$-\lambda^3 + \lambda^2 \cancel{-2\lambda} + 2\lambda^2 \cancel{+2\lambda} = 0$$

$$-\lambda^3 + 3\lambda^2 = 0 \qquad \lambda = 0,\ 0,\ 3$$

$$\lambda^3 - 3\lambda^2 = 0. \qquad \lambda_1 = 0$$

$$\lambda^2(\lambda - 3) = 0. \qquad \lambda_2 = 0$$

$$\lambda_3 = 3.$$

λ_1 and $\lambda_2 = 0$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

not orthogonal !!!

$$a + b + c = 0.$$

$$c = -a - b$$
$$-1 + 1 = 0$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \vec{v}_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad c \overset{?}{=} -a - b = 0 - 1 = -1 \checkmark$$

Could have picked $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ +1 \end{pmatrix}$

$\lambda_3 = 3$.

orthogonal w.r.t. others ($\vec{v}_1$ & $\vec{v}_2$).

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \begin{matrix} 1 \\ -1 \\ = 0 \end{matrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \begin{matrix} 0 \\ 1 \\ -1 \\ \hline 0 \end{matrix}$$

$$-2a + b + c = 0 \qquad a = 1$$
$$a - 2b + c = 0 \qquad b = 1$$
$$a + b - 2c = 0 \qquad c = 1$$

Gram-Schmidt procedure : Orthonormalization.

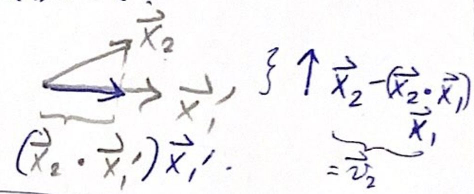
Suppose $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are not ~~orthogonal~~ orthonormal w.r.t. each other. Then use GS procedure as follows:

① $$\vec{x}_1' = \frac{\vec{x}_1}{|\vec{x}|}.$$ (giving $\vec{x}_1'$ length 1).

---

② Project $\vec{x}_2$ on $\vec{x}_1'$ calculated above:

$$(\vec{x}_2 \cdot \vec{x}_1')\, \vec{x}_1'$$



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③ Find $\vec{x}_2 - (\vec{x}_2 \cdot \vec{x}_1')\vec{x}_1'$ ← [orthogonal to $\vec{x}_1'$.
because $(\vec{x}_2 \cdot \vec{x}_1')\vec{x}_1' + \vec{v}_2 = \vec{x}_2$
↑ solve for $\vec{v}_2$.]

---

④ Normalize the above to $|\vec{x}_2 - (\vec{x}_2 \cdot \vec{x}_1')\vec{x}_1|$.

$$\vec{x}_2' = \frac{\vec{x}_2 - (\vec{x}_2 \cdot \vec{x}_1')\vec{x}_1}{|x_2 - (\vec{x}_2 \cdot \vec{x}_1')\vec{x}_1|}.$$

---

⑤ Find $\vec{x}_3 - (\vec{x}_3 \cdot \vec{x}_1')\vec{x}_1' - (\vec{x}_3 \cdot \vec{x}_2')\vec{x}_2'$

---

⑥ Normalize the above to $|\vec{x}_3 - (\vec{x}_3 \cdot \vec{x}_1')\vec{x}_1' - (\vec{x}_3 \cdot \vec{x}_2')\vec{x}_2'|$.

Thus the procedure is complete for the 3-space. i.e., $\vec{x}_1, \vec{x}_2, \vec{x}_3$ which were not orthogonal have been made <u>orthonormal</u>.. $\vec{x}_1', \vec{x}_2', \vec{x}_3'$.

Normalization means division by $|\ldots|$.
Orthogonality means $\vec{u} \cdot \vec{v} = 0$.

Make the eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Orthonormal. Use Gram-Schmit procedure.

① $\vec{v}_1{}' = \dfrac{\vec{v}_1}{|\vec{v}_1|} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

② $(\vec{v}_2 \cdot \vec{v}_1{}') \vec{v}_1{}' = \dfrac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right] \times \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$= \dfrac{1}{2} \left[-1 \right] \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -\dfrac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$

③ $\vec{v}_2 - (\vec{v}_2 \cdot \vec{v}_1{}') \vec{v}_1{}' = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}$

④ Normalize. $\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1} = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{1.5}$

$\vec{v}_2{}' = \dfrac{1}{\sqrt{1.5}} \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix}$.

Check: is $\vec{v}_2{}' \perp \vec{v}_1{}'$? ie, $\vec{v}_2{}' \cdot \vec{v}_1{}' \stackrel{?}{=} 0$.

$\underbrace{\dfrac{1}{\sqrt{1.5}} \dfrac{1}{\sqrt{2}}} \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \stackrel{?}{=} 0$

$\ldots \quad \frac{1}{2} - \frac{1}{2} = 0$ ✓ woohoo!

Is $\vec{v}_2{}'$ an eigenvector of $\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{A}$? for $\lambda = 0$.

$$\underbrace{A - I\lambda = A}_{} $$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Yes! $\frac{1}{2} + \frac{1}{2} - 1 = 0$. ✓

Woohoo!