Overview and derivation of reflection and transmission between two elastic solids

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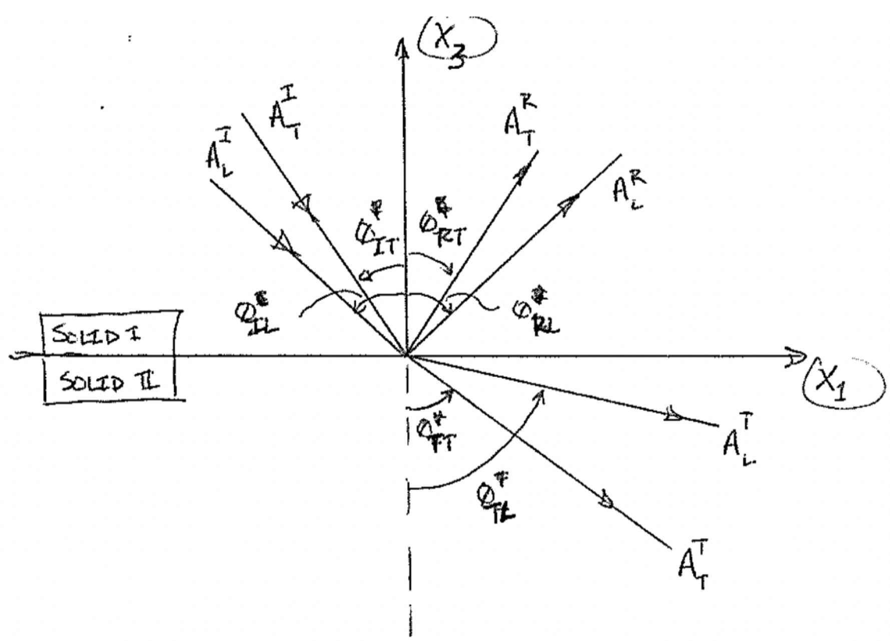
1 Introduction



The sketch above shows either an incident longitudinal (P) or transverse (S) wave (or both). We must establish continuity of velocity (or displacement) and traction at $x_3 = 0$. We define

$$\boldsymbol{v} = v_1\hat{i}_1 + v_3\hat{i}_3, \tag{i}$$

$$\boldsymbol{t} = t_1\hat{i}_1 + t_3\hat{i}_3 = T_{ij}n_j, \tag{iia}$$

v_1 and v_3 are obtained from incident, reflected, and transmitted angles.

We want to know $\boldsymbol{t}$ on the $x_3 = 0$ plane. Therefore, $\boldsymbol{n}$ must be normal to that plane, i.e., $\boldsymbol{n} = \langle 0, 1\rangle$ for 2D. The stress tensor becomes

$$\underline{T} = \begin{bmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{bmatrix}$$

and the tension becomes

$$\boldsymbol{t} = \begin{bmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T_{13} \\ T_{33} \end{bmatrix}$$

So,

$$\boldsymbol{t} = T_{13}\hat{\imath}_1 + T_{33}\hat{\imath}_3 \tag{iib}$$

Each component of $\boldsymbol{v}$ and $\boldsymbol{t}$ must be continuous across the interface:

$$v^{\mathrm{I}}_{1,\mathrm{T}} + v^{\mathrm{I}}_{1,\mathrm{L}} + v^{\mathrm{R}}_{1,\mathrm{T}} + v^{\mathrm{R}}_{1,\mathrm{L}} = v^{\mathrm{T}}_{1,\mathrm{T}} + v^{\mathrm{T}}_{1,\mathrm{L}} \tag{iiia}$$
$$v^{\mathrm{I}}_{3,\mathrm{T}} + v^{\mathrm{I}}_{3,\mathrm{L}} + v^{\mathrm{R}}_{3,\mathrm{T}} + v^{\mathrm{R}}_{3,\mathrm{L}} = v^{\mathrm{T}}_{3,\mathrm{T}} + v^{\mathrm{T}}_{3,\mathrm{L}} \tag{iiib}$$
$$T^{\mathrm{I}}_{13,\mathrm{T}} + T^{\mathrm{I}}_{13,\mathrm{L}} + T^{\mathrm{R}}_{13,\mathrm{T}} + T^{\mathrm{R}}_{13,\mathrm{L}} = T^{\mathrm{T}}_{13,\mathrm{T}} + T^{\mathrm{T}}_{13,\mathrm{L}} \tag{iiic}$$
$$T^{\mathrm{I}}_{33,\mathrm{T}} + T^{\mathrm{I}}_{33,\mathrm{L}} + T^{\mathrm{R}}_{33,\mathrm{T}} + T^{\mathrm{R}}_{33,\mathrm{L}} = T^{\mathrm{T}}_{33,\mathrm{T}} + T^{\mathrm{T}}_{33,\mathrm{L}} \tag{iiid}$$

So, there are four equations for four unknowns: $A^{\mathrm{R}}_{\mathrm{T}}, A^{\mathrm{R}}_{\mathrm{L}}, A^{\mathrm{T}}_{\mathrm{T}}, A^{\mathrm{T}}_{\mathrm{L}}$. The equations are usually solved in terms of reflection and transmission coefficients, which are ratios of the unknown amplitudes to the "known" (or assumed) amplitudes of the incident fields.

Using the notation provided in Rose's Ch. 5, the reflection and transmission coefficients for longitudinal and transverse waves given an incident longitudinal (P-wave) are defined as:

$$R_{\mathrm{LL}} = \frac{A^{\mathrm{R}}_{\mathrm{L}}}{A^{\mathrm{I}}_{\mathrm{L}}},$$
$$R_{\mathrm{LT}} = \frac{A^{\mathrm{R}}_{\mathrm{T}}}{A^{\mathrm{I}}_{\mathrm{L}}},$$
$$T_{\mathrm{LL}} = \frac{A^{\mathrm{T}}_{\mathrm{L}}}{A^{\mathrm{I}}_{\mathrm{L}}},$$
$$T_{\mathrm{LT}} = \frac{A^{\mathrm{T}}_{\mathrm{T}}}{A^{\mathrm{I}}_{\mathrm{L}}}.$$

For shear (S-wave) incidence we define analogous coefficients:

$$R_{\mathrm{TL}} = \frac{A^{\mathrm{R}}_{\mathrm{L}}}{A^{\mathrm{I}}_{\mathrm{T}}},$$
$$R_{\mathrm{TT}} = \frac{A^{\mathrm{R}}_{\mathrm{T}}}{A^{\mathrm{I}}_{\mathrm{T}}},$$
$$T_{\mathrm{TL}} = \frac{A^{\mathrm{T}}_{\mathrm{L}}}{A^{\mathrm{I}}_{\mathrm{T}}},$$
$$T_{\mathrm{TT}} = \frac{A^{\mathrm{T}}_{\mathrm{T}}}{A^{\mathrm{I}}_{\mathrm{T}}}.$$

We now need to relate the velocity to the displacement field, $\boldsymbol{u}$, the stress, $\underline{\boldsymbol{T}}$, and strain, $\underline{\boldsymbol{E}}$, state using following relationships:

$$\boldsymbol{v} = \frac{\partial \boldsymbol{u}}{\partial t} = j\omega \boldsymbol{u} \quad \text{(for time harmonic waves)},$$

and

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The latter relationship can be specialized to yield the following elements of the strain tensor: $E_{11} = u_{1,1}$, $E_{33} = u_{3,3}$, and $E_{13} = \frac{1}{2}(u_{1,3} + u_{3,1})$, which will be used in the stress-strain relationship in order to find expressions for the continuity of traction at the boundary between dissimilar elastic solids.

We now employ Hooke's law for an isotropic solid to write the following expressions for the components of the stress tensor in terms of the strain

$$\begin{aligned} T_{13} &= 2\mu E_{13} = \mu(u_{1,3} + u_{3,1}), \\ T_{33} &= \lambda(E_{11} + E_{33}) + 2\mu E_{33} = ME_{33} + \lambda E_{11}, \end{aligned}$$

where $M \equiv \lambda + 2\mu$ is the plane wave modulus of the material. By substituting the assumed form of the solution for time-harmonic plane waves form into these relations, we can find a set of four linear equations which are dependent on the material properties and the angle of incidence in matrix-vector form provided below,

$$[M]\boldsymbol{x} = \boldsymbol{b} \tag{1}$$

where

- $[M]$ is a 4×4 matrix of constants $(\mu_1, \lambda_1, \mu_2, \lambda_2, \theta_{\mathrm{RL}}, \theta_{\mathrm{RT}}, \dots)$.
- $\boldsymbol{x}$ is a 4×1 vector of Rs and Ts
- $\boldsymbol{b}$ is a 4×1 vector of "known" inputs $(\mu_1, \lambda_1, \theta_I)$

Those four equations can be solved for the reflection and transmission coefficients for any angle of incidence and for any combination of material half-spaces. Note that $[M]$ is does not change between cases of shear and longitudinal incidence but the vector in inputs, $\mathbf{b}$, will change depending on incident wave type.

Summary of results

The derivations provided in Sec. (2) and (3) arrive at the following results for the system of equations provided in Eq. 1.

The matrix $[M]$ is given by

$$[M] = \begin{bmatrix} \cos\theta_{\mathrm{RL}} & \cos\theta_{\mathrm{TL}} & \sin\theta_{\mathrm{RT}} & -\sin\theta_{\mathrm{TT}} \\ -\sin\theta_{\mathrm{RL}} & \sin\theta_{\mathrm{TL}} & \cos\theta_{\mathrm{RT}} & \cos\theta_{\mathrm{TT}} \\ -Z_{\mathrm{L1}}\cos 2\theta_{\mathrm{RT}} & Z_{\mathrm{L2}}\cos 2\theta_{\mathrm{TT}} & -Z_{\mathrm{T1}}\sin 2\theta_{\mathrm{RT}} & -Z_{\mathrm{T2}}\sin 2\theta_{\mathrm{TT}} \\ Z_{\mathrm{T1}}(c_{\mathrm{T1}}/c_{\mathrm{L1}})\sin 2\theta_{\mathrm{RL}} & Z_{\mathrm{T2}}(c_{\mathrm{T2}}/c_{\mathrm{L2}})\sin 2\theta_{\mathrm{TL}} & -Z_{\mathrm{T1}}\cos 2\theta_{\mathrm{RT}} & Z_{\mathrm{T2}}\cos 2\theta_{\mathrm{TT}} \end{bmatrix} \qquad (2)$$

Incident longitudinal waves

Incident longitudinal waves satisfy

$$[M]\boldsymbol{x} = \boldsymbol{b}_{\mathrm{L}}$$

where

$$\boldsymbol{x} = \left\{ R_{\mathrm{L}} \quad T_{\mathrm{L}} \quad R_{\mathrm{T}} \quad T_{\mathrm{T}} \right\}^T \qquad (3)$$

$$\boldsymbol{b}_{\mathrm{L}} = \left\{ \cos\theta_{\mathrm{IL}} \quad \sin\theta_{\mathrm{IL}} \quad Z_{\mathrm{L1}}\cos(2\theta_{\mathrm{RT}}) \quad Z_{\mathrm{T1}}\left(\frac{c_{\mathrm{T1}}}{c_{\mathrm{L1}}}\right)\sin(2\theta_{\mathrm{IL}}) \right\}^T, \qquad (4)$$

Incident shear waves

Incident shear waves satisfy

$$[M]\boldsymbol{x} = \boldsymbol{b}_{\mathrm{T}}$$

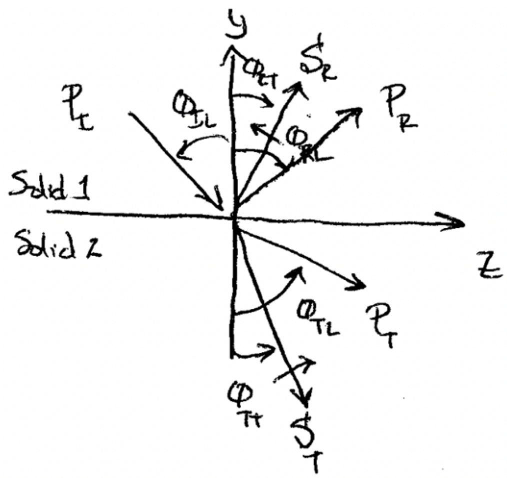
where

$$\boldsymbol{b}_{\mathrm{T}} = \left\{ -\sin\theta_{\mathrm{IT}} \quad \cos\theta_{\mathrm{IT}} \quad -Z_{\mathrm{T1}}\sin(2\theta_{\mathrm{IT}}) \quad Z_{\mathrm{T1}}\cos(2\theta_{\mathrm{IT}}) \right\}^T. \qquad (5)$$

The sections that follow provide the detailed derivation of these matrix equations.

2 Longitudinal (P-wave) incidence

Geometry and material properties



Definitions:

- P_{I} = incident P-wave
- P_{R} = reflected P-wave
- P_{T} = transmitted P-wave
- S_{T} = transmitted S-wave
- θ_{IL} = angle of P-wave incidence (L for *longitudinal*)
- θ_{RT} = angle of S-wave reflection (T for *transverse*)
- θ_{RL} = angle of P-wave reflection
- θ_{TT} = angle of S-wave transmission
- θ_{TL} = angle of P-wave transmission

For $x = 1, 2 \ldots$

- $\rho_x c_{\mathrm{L}x}^2 = (\lambda_x + 2\mu_x) = M_x$
- $\rho_x c_{\mathrm{T}x}^2 = \mu_x$
- $Z_{\mathrm{L}x} = \rho_x c_{\mathrm{L}x} = \sqrt{\rho_x M_x}$
- $Z_{\mathrm{T}x} = \rho_x c_{\mathrm{T}x} = \sqrt{\rho_x \mu_x}$

Continuity Conditions

At $y = 0$,

$$\begin{aligned} u_y^{(1)} &= u_y^{(2)}, \quad u_z^{(1)} = u_z^{(2)} \\ T_{yy}^{(1)} &= T_{yy}^{(2)}, \quad T_{yz}^{(1)} = T_{yz}^{(2)} \end{aligned}$$

with

$$\begin{aligned} &\text{(a)} \quad u_y \implies \quad u_{\text{IL}}^{(y)} + u_{\text{RL}}^{(y)} + u_{\text{RT}}^{(y)} = u_{\text{TL}}^{(y)} + u_{\text{TT}}^{(y)} \\ &\text{(b)} \quad u_z \implies \quad u_{\text{IL}}^{(z)} + u_{\text{RL}}^{(z)} + u_{\text{RT}}^{(z)} = u_{\text{TL}}^{(z)} + u_{\text{TT}}^{(z)} \\ &\text{(c)} \quad T_{yy} \implies T_{yy}^{\text{IL}} + T_{yy}^{\text{RL}} + T_{yy}^{\text{RT}} = T_{yy}^{\text{TL}} + T_{yy}^{\text{TT}} \\ &\text{(d)} \quad T_{yz} \implies T_{yz}^{\text{IL}} + T_{yz}^{\text{RL}} + T_{yz}^{\text{RT}} = T_{yz}^{\text{TL}} + T_{yz}^{\text{TT}} \end{aligned}$$

And we recall:

$$\begin{aligned} T_{yy} &= \lambda(E_{yy} + E_{zz}) + 2\mu E_{yy} = (\lambda + 2\mu)E_{yy} + \lambda E_{zz} \\ T_{yz} &= 2\mu E_{yz} \end{aligned}$$

with $E_{ij} = \frac{1}{2}(u_{y,z} + u_{z,y})$.

Assume the displacements have the form...

$$\boldsymbol{u}_{\text{IL}} = \begin{Bmatrix} u_{\text{IL}} \sin\theta_{\text{IL}} \\ -u_{\text{IL}} \cos\theta_{\text{IL}} \end{Bmatrix} e^{j(\omega t + \chi_{\text{IL}})}; \quad \chi_{\text{IL}} = -k_{\text{L1}} z \sin\theta_{\text{IL}} + k_{\text{L1}} y \cos\theta_{\text{IL}}$$

$$\boldsymbol{u}_{\text{RL}} = \begin{Bmatrix} u_{\text{RL}} \sin\theta_{\text{RL}} \\ u_{\text{RL}} \cos\theta_{\text{RL}} \end{Bmatrix} e^{j(\omega t + \chi_{\text{RL}})}; \quad \chi_{\text{RL}} = -k_{\text{L1}} z \sin\theta_{\text{RL}} + k_{\text{L1}} y \cos\theta_{\text{RL}}$$

$$\boldsymbol{u}_{\text{RT}} = \begin{Bmatrix} -u_{\text{RT}} \sin\theta_{\text{RT}} \\ -u_{\text{RT}} \cos\theta_{\text{RT}} \end{Bmatrix} e^{j(\omega t + \chi_{\text{RT}})}; \quad \chi_{\text{RT}} = -k_{\text{T1}} z \sin\theta_{\text{RT}} + k_{\text{T1}} y \cos\theta_{\text{RT}}$$

$$\boldsymbol{u}_{\text{TL}} = \begin{Bmatrix} u_{\text{TL}} \sin\theta_{\text{TL}} \\ -u_{\text{TL}} \cos\theta_{\text{TL}} \end{Bmatrix} e^{j(\omega t + \chi_{\text{TL}})}; \quad \chi_{\text{TL}} = -k_{\text{L2}} z \sin\theta_{\text{TL}} + k_{\text{L2}} y \cos\theta_{\text{TL}}$$

$$\boldsymbol{u}_{\text{TT}} = \begin{Bmatrix} u_{\text{TT}} \sin\theta_{\text{TT}} \\ -u_{\text{TT}} \cos\theta_{\text{TT}} \end{Bmatrix} e^{j(\omega t + \chi_{\text{TT}})}; \quad \chi_{\text{TT}} = -k_{\text{T2}} z \sin\theta_{\text{TT}} + k_{\text{T2}} y \cos\theta_{\text{TT}}$$

with

$$\boldsymbol{u} = \begin{Bmatrix} u_z \\ u_y \end{Bmatrix}$$

The continuity condition (a) becomes...

$$-u_{\rm IL}\cos\theta_{\rm IL}e^{j(\omega t+\chi_{\rm IL})}+u_{\rm RL}\cos\theta_{\rm RL}e^{j(\omega t+\chi_{\rm RL})}+u_{\rm RT}\cos\theta^{j(\omega t+\chi_{\rm RT})}$$
$$=-u_{\rm TL}\cos\theta_{\rm TL}e^{j(\omega t+\chi_{\rm TL})}+u_{\rm TT}\cos\theta_{\rm TT}e^{j(\omega t+\chi_{\rm TT})} \qquad \text{(a)}$$

The common $e^{j\omega t}$ time-dependence cancels out. We also note that for $\chi_{\rm RT}$ to be constant (i.e., phase matching of u_y at all z),

$$\chi_{\rm IL}=\chi_{\rm RL}=\chi_{\rm RT}=\chi_{\rm TL}=\chi_{\rm TT}$$

Expressing the trace wavenumbers in terms of the frequency, speed, and angle,

$$\frac{\omega}{c_{\rm L1}}\sin\theta_{\rm IL}=\frac{\omega}{c_{\rm L1}}\sin\theta_{\rm RL}=\frac{\omega}{c_{\rm T1}}\sin\theta_{\rm RT}=\frac{\omega}{c_{\rm L2}}\sin\theta_{\rm TL}=\frac{\omega}{c_{\rm T2}}\sin\theta_{\rm TT}$$

The first equality above shows that $\theta_{\rm RL}=\theta_{\rm IL}$. Canceling the common ω, the remaining four equalities recover Snell's law:

$$\frac{\sin\theta_{\rm IL}}{c_{\rm L1}}=\frac{\sin\theta_{\rm RT}}{c_{\rm T1}}=\frac{\sin\theta_{\rm TL}}{c_{\rm L2}}=\frac{\sin\theta_{\rm TT}}{c_{\rm T2}}$$

Then, continuity condition (a) becomes

$$-u_{\rm IL}\cos\theta_{\rm IL}+u_{\rm RL}\cos\theta_{\rm RL}+u_{\rm RL}\cos\theta_{\rm RL}=-u_{\rm TL}\cos\theta_{\rm TL}+u_{\rm TT}\cos\theta_{\rm TT}$$

Defining

$$R_{\rm L}=\frac{u_{\rm RL}}{u_{\rm IL}}$$
$$R_{\rm T}=\frac{u_{\rm RT}}{u_{\rm IL}}$$
$$T_{\rm L}=\frac{u_{\rm TL}}{u_{\rm IL}}$$
$$T_{\rm T}=\frac{u_{\rm TT}}{u_{\rm IL}},$$

continuity condition (a) becomes

$$R_{\rm L}\cos\theta_{\rm RL}+R_{\rm T}\sin\theta_{\rm RL}+T_{\rm L}\cos\theta_{\rm TL}-T_{\rm T}\sin\theta_{\rm TT}=\cos\theta_{\rm IL} \qquad \text{(A)}$$

Similarly, continuity condition (b) becomes

$$u_{\rm IL}\sin\theta_{\rm IL}+u_{\rm RL}\sin\theta_{\rm RL}-u_{\rm RT}\cos\theta_{\rm RT}=u_{\rm TL}\cos\theta_{\rm TL}+u_{\rm TT}\cos\theta_{\rm TT}$$

or, in terms of the above-defined reflection and transmission coefficients,

$$R_{\rm L}\sin\theta_{\rm RL}-R_{T}\cos\theta_{\rm RT}-T_{\rm L}\sin\theta_{\rm TL}-T_{\rm T}\cos\theta_{\rm TT}=-\sin\theta_{\rm IL} \qquad \text{(B)}$$

Now for the stress equations. . .

We now need E_{yy}, E_{zz}, E_{yz}. Note that $E_{yy} = u_{y,y} = \frac{\partial u_y}{\partial y}$, which we need for both sides of the interface. Since we have assumed the form $u_y = \left|u_y\right| e^{j\chi(y)}$,

$$E_{yy} = \left|u_y\right| e^{j\chi(y)} \left(j\frac{\partial \chi}{\partial y}\right).$$

So in solid 1, at $y = 0$,

$$\left.\frac{\partial u_y}{\partial y}\right|_{y=0}^{\text{solid 1}} = -u_{\text{IL}} \cos\theta_{\text{IL}}(jk_{\text{L1}}\cos\theta_{\text{IL}}) + u_{\text{RL}}\cos\theta_{\text{RL}}(-jk_{\text{L1}}\cos\theta_{\text{RL}}) + u_{\text{RT}}\cos\theta_{\text{RT}}(-jk_{\text{T1}}\cos\theta_{\text{RT}}),$$

and in solid 2, at $y = 0$

$$\left.\frac{\partial u_y}{\partial y}\right|_{y=0}^{\text{solid 2}} = -u_{\text{TL}}\cos\theta_{\text{TL}}(jk_{\text{L2}}\cos\theta_{\text{TL}}) + u_{\text{TL}}\sin\theta_{\text{TT}}(jk_{\text{TL}}\cos\theta_{\text{TT}}).$$

Likewise, for E_{zz},

$$\left.\frac{\partial u_z}{\partial z}\right|_{y=0}^{\text{solid 1}} = u_{\text{IL}}\sin\theta_{\text{IL}}(-jk_{\text{L1}}\sin\theta_{\text{IL}}) + u_{\text{RL}}\sin\theta_{\text{RL}}(-jk_{\text{L1}}\cos\theta_{\text{RL}}) - u_{\text{RT}}\cos\theta_{\text{RT}}(-jk_{\text{T1}}\sin\theta_{\text{RT}}),$$

and in solid 2, at $y = 0$

$$\left.\frac{\partial u_z}{\partial z}\right|_{y=0}^{\text{solid 2}} = u_{\text{TL}}\sin\theta_{\text{TL}}(-jk_{\text{L2}}\sin\theta_{\text{TL}}) + u_{\text{TL}}\cos\theta_{\text{TT}}(jk_{\text{TL}}\sin\theta_{\text{TT}}).$$

So condition (c) is written

$$\left.T_{yy}\right|_{y=0}^{\text{solid 1}} = \left.T_{yy}\right|_{y=0}^{\text{solid 2}}$$

By the stress-strain relation,

$$(\lambda_1 + 2\mu_1)E_{yy}^{\text{solid 1}} + \lambda_1 E_{zz}^{\text{solid 1}} = (\lambda_2 + 2\mu_u)E_{yy}^{\text{solid 2}} + \lambda_2 E_{zz}^{\text{solid 2}}$$

Substituting in the derivatives taken above,

$$
\begin{aligned}
(\lambda_1 + 2\mu_1) &\left(-j\frac{\omega}{c_{\mathrm{L1}}}\cos^2\theta_{\mathrm{IL}}u_{\mathrm{IL}} - j\frac{\omega}{c_{\mathrm{L1}}}\cos^2\theta_{\mathrm{RL}}u_{\mathrm{RL}} - j\frac{\omega}{c_{\mathrm{T1}}}\cos^2\theta_{\mathrm{RT}}u_{\mathrm{RT}}\right)\\
&+ \lambda_1\left(-j\frac{\omega}{c_{\mathrm{L1}}}\sin^2\theta_{\mathrm{IL}} - j\frac{\omega}{c_{\mathrm{L1}}}\sin^2\theta_{\mathrm{RL}}u_{\mathrm{RL}} + j\frac{\omega}{c_{\mathrm{T1}}}\sin\theta_{\mathrm{RT}}\cos\theta_{\mathrm{RT}}u_{\mathrm{RT}}\right)\\
= (\lambda_2+2\mu_2) &\left(-j\frac{\omega}{c_{\mathrm{L2}}}\cos^2\theta_{\mathrm{TL}}u_{\mathrm{TL}} + j\frac{\omega}{c_{\mathrm{T2}}}\sin\theta_{\mathrm{TT}}\cos\theta_{\mathrm{TT}}u_{\mathrm{TT}}\right)\\
&+ \lambda_2\left(-j\frac{\omega}{c_{\mathrm{L2}}}\sin^2\theta_{\mathrm{TL}}u_{\mathrm{TL}} - j\frac{\omega}{c_{\mathrm{T2}}}\sin\theta_{\mathrm{TT}}\cos\theta_{\mathrm{TT}}u_{\mathrm{TT}}\right)
\end{aligned}
$$

Rearranging yields

$$
\begin{aligned}
&\lambda_1\left(-(\cos^2\theta_{\mathrm{IL}} + \sin^2\theta_{\mathrm{IL}})\frac{u_{\mathrm{IL}}}{c_{\mathrm{L1}}} - (\cos^2\theta_{\mathrm{RL}} + \sin^2\theta_{\mathrm{RL}})\frac{u_{\mathrm{RL}}}{c_{\mathrm{L1}}} - (\sin\theta_{\mathrm{RT}}\cos\theta_{\mathrm{RT}} - \sin\theta_{\mathrm{RT}}\cos\theta_{\mathrm{RT}})\frac{u_{\mathrm{RT}}}{c_{\mathrm{T1}}}\right)\\
&\quad + 2\mu_1\left(-\cos^2\theta_{\mathrm{IL}}\frac{u_{\mathrm{IL}}}{c_{\mathrm{L1}}} - \cos^2\theta_{\mathrm{RL}}\frac{u_{\mathrm{RL}}}{c_{\mathrm{L1}}} - \sin\theta_{\mathrm{RT}}\cos_{\mathrm{RT}}\frac{u_{\mathrm{RT}}}{c_{\mathrm{T1}}}\right)\\
&= \lambda_2\left(-(\cos^2\theta_{\mathrm{TL}} + \sin^2\theta_{\mathrm{TL}})\frac{u_{\mathrm{TL}}}{c_{\mathrm{L2}}} + (\sin\theta_{\mathrm{TT}}\cos\theta_{\mathrm{TT}} - \sin\theta_{\mathrm{TT}}\cos\theta_{\mathrm{TT}})\frac{u_{\mathrm{TT}}}{c_{\mathrm{T2}}}\right)\\
&\quad + 2\mu_1\left(-\cos^2\theta_{\mathrm{TL}}\frac{u_{\mathrm{TL}}}{c_{\mathrm{L2}}} + \sin\theta_{\mathrm{TT}}\cos_{\mathrm{TT}}\frac{u_{\mathrm{TT}}}{c_{\mathrm{T2}}}\right)
\end{aligned}
$$

So, we have

$$
\begin{aligned}
-(\lambda_1+2\mu_1\cos^2\theta_{\mathrm{IL}})\frac{u_{\mathrm{IL}}}{c_{\mathrm{L1}}} - (\lambda_1 + 2\mu_1\cos^2\theta_{\mathrm{RL}})\frac{u_{\mathrm{RL}}}{c_{\mathrm{L1}}} - \mu_1\sin(2\theta_{\mathrm{RT}})\frac{u_{\mathrm{RT}}}{c_{\mathrm{T1}}}\\
= -(\lambda_2 + 2\mu_2\cos^2\theta_{\mathrm{TL}})\frac{u_{\mathrm{TL}}}{c_{\mathrm{L2}}} + \mu_2\sin(2\theta_{\mathrm{TT}})\frac{u_{\mathrm{TT}}}{c_{\mathrm{T2}}}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\lambda_1 + 2\mu_1\cos^2\theta_{\mathrm{RL}}}{c_{\mathrm{L1}}}R_{\mathrm{L}} - \frac{\mu_1}{c_{\mathrm{T1}}}\sin(2\theta_{\mathrm{RT}})R_{\mathrm{T}} + \frac{\lambda_2 + 2\mu_2\cos^2\theta_{\mathrm{TL}}}{c_{\mathrm{L2}}}T_{\mathrm{L}} - \frac{\mu_2}{c_{\mathrm{T2}}}\sin(2\theta_{\mathrm{TT}})T_{\mathrm{T}}\\
= \frac{\lambda_1 + 2\mu_1\cos^2\theta_{\mathrm{IL}}}{c_{\mathrm{L1}}} \qquad (\dagger)
\end{aligned}
$$

Then, Achenbach chapter 5 (following equation 5.86) gives

$$
\frac{\lambda_1 + 2\mu_1\cos^2\theta_{\mathrm{RL}}}{\mu_1} = \left(\frac{c_{\mathrm{L1}}}{c_{\mathrm{T1}}}\right)^2\cos 2\theta_{\mathrm{RT}},
$$

which yields

$$
\frac{\lambda_1 + 2\mu_1\cos^2\theta_{\mathrm{RL}}}{c_{\mathrm{L1}}} = Z_{\mathrm{L1}}\cos 2\theta_{\mathrm{RT}}
$$

Likewise, we find

$$\frac{\lambda_2 + 2\mu_2 \cos^2\theta_{\mathrm{RL}}}{c_{\mathrm{L2}}} = Z_{\mathrm{L2}} \cos 2\theta_{\mathrm{TT}}$$

and

$$\frac{\lambda_1 + 2\mu_1 \cos^2\theta_{\mathrm{IL}}}{c_{\mathrm{L1}}} = Z_{\mathrm{L1}} \cos 2\theta_{\mathrm{RT}}$$

Then, equation (†) becomes

$$\begin{aligned} -Z_{\mathrm{L1}} \cos(2\theta_{\mathrm{RT}}) R_{\mathrm{L}} - Z_{\mathrm{T1}} \sin(2\theta_{\mathrm{RT}}) R_{\mathrm{T}} + Z_{\mathrm{L2}} \cos(2\theta_{\mathrm{TT}}) T_{\mathrm{L}} - Z_{\mathrm{T2}} \sin(2\theta_{\mathrm{TT}}) T_{\mathrm{T}} \\ = Z_{\mathrm{L1}} \cos(2\theta_{\mathrm{RT}}) \end{aligned} \tag{C}$$

Note that in Rose and Nagy, Chimenti, Rokhlin [*Physical Ultrasonics of Composites*, Oxford Univ. Press, (2011)], the argument of cosine on the right-hand-side is incorrectly written as $2\theta_{\mathrm{IL}}$.

For E_{yz}, we have

$$\left.\frac{\partial u_y}{\partial z}\right|_{y=0}^{\text{solid 1}} = -u_{\mathrm{IL}} \cos\theta_{\mathrm{IL}}(-jk_{\mathrm{L1}} \sin\theta_{\mathrm{IL}}) + u_{\mathrm{RL}} \cos\theta_{\mathrm{RL}}(-jk_{\mathrm{L1}} \sin\theta_{\mathrm{RL}}) + u_{\mathrm{RT}} \sin\theta_{\mathrm{RT}}(-jk_{\mathrm{T1}} \sin\theta_{\mathrm{RT}}),$$

$$\left.\frac{\partial u_z}{\partial y}\right|_{y=0}^{\text{solid 1}} = u_{\mathrm{IL}} \sin\theta_{\mathrm{IL}}(-jk_{\mathrm{L1}} \cos\theta_{\mathrm{IL}}) + u_{\mathrm{RL}} \sin\theta_{\mathrm{RL}}(-jk_{\mathrm{L1}} \cos\theta_{\mathrm{RL}}) - u_{\mathrm{RT}} \cos\theta_{\mathrm{RT}}(-jk_{\mathrm{T1}} \cos\theta_{\mathrm{RT}}),$$

$$\left.\frac{\partial u_z}{\partial y}\right|_{y=0}^{\text{solid 2}} = u_{\mathrm{TL}} \sin\theta_{\mathrm{TL}}(jk_{\mathrm{L2}} \cos\theta_{\mathrm{TL}}) + u_{\mathrm{TT}} \cos\theta_{\mathrm{TT}}(jk_{\mathrm{T2}} \cos\theta_{\mathrm{TT}}),$$

$$\left.\frac{\partial u_y}{\partial z}\right|_{y=0}^{\text{solid 2}} = -u_{\mathrm{TL}} \cos\theta_{\mathrm{TL}}(jk_{\mathrm{L2}} \cos\theta_{\mathrm{TL}}) + u_{\mathrm{TL}} \sin\theta_{\mathrm{TT}}(jk_{\mathrm{TL}} \cos\theta_{\mathrm{TT}}).$$

So, for continuity of shear waves at $y = 0$, we have

$$2\mu_1 \frac{1}{2}(u_{y,z} + u_{z,y})\Big|_{y=0}^{\text{solid 1}} = 2\mu_2 \frac{1}{2}(u_{y,z} + u_{z,y})\Big|_{y=0}^{\text{solid 2}}$$

Inserting the above-calculated derivatives,

$$\begin{aligned} &\mu_1 \left(j\frac{2\omega}{c_{\mathrm{L1}}} u_{\mathrm{IL}} \sin\theta_{\mathrm{IL}} \cos\theta_{\mathrm{IL}} - j\frac{2\omega}{c_{\mathrm{L1}}} u_{\mathrm{RL}} \sin\theta_{\mathrm{RL}} \cos\theta_{\mathrm{RL}} + j\frac{\omega}{c_{\mathrm{T1}}} u_{\mathrm{RT}} (\cos^2\theta_{\mathrm{RT}} - \sin^2\theta_{\mathrm{RT}}) \right) \\ &\quad = \mu_2 \left(j\frac{2\omega}{c_{\mathrm{L2}}} u_{\mathrm{TL}} \sin\theta_{\mathrm{TL}} \cos\theta_{\mathrm{TL}} + j\frac{\omega}{c_{\mathrm{T2}}} u_{\mathrm{TT}} (\cos^2\theta_{\mathrm{TT}} - \sin^2\theta_{\mathrm{TT}}) \right) \end{aligned}$$

Canceling the common j and ω,

$$\frac{\mu_1}{c_{\rm L1}}\sin(2\theta_{\rm IL}) - \frac{\mu_1}{c_{\rm L1}}R_{\rm L}\sin(2\theta_{\rm IL}) + \frac{\mu_1}{c_{\rm T1}}R_{\rm L}\cos(2\theta_{\rm RL}) = \frac{\mu_2}{c_{\rm L2}}T_{\rm L}\sin(2\theta_{\rm TL}) + \frac{\mu_2}{c_{\rm T2}}T_{\rm T}\cos(2\theta_{\rm TT})$$

Noting that

$$\frac{\mu_1}{c_{\rm T1}} = \sqrt{\frac{\rho_1}{\lambda_1 + 2\mu_1}}\mu_1\sqrt{\frac{\rho_1}{\mu_1}}\sqrt{\frac{\mu_1}{\rho_1}} = \frac{c_{\rm T1}}{c_{\rm L1}}\sqrt{\rho_1\mu_1} = Z_{\rm T1}\left(\frac{c_{\rm T1}}{c_{\rm L1}}\right),$$

we have

$$-Z_{\rm T1}\left(\frac{c_{\rm T1}}{c_{\rm L1}}\right)\sin(2\theta_{\rm TL})R_{\rm L} + Z_{\rm T1}\cos(2\theta_{\rm RT})R_{\rm T} - Z_{\rm T2}\left(\frac{c_{\rm T2}}{c_{\rm L2}}\right)\sin(2\theta_{\rm TL})T_{\rm L} - Z_{\rm T2}\cos(2\theta_{\rm TT})T_{\rm T} \\ = -Z_{\rm T1}\left(\frac{c_{\rm T1}}{c_{\rm L1}}\right)\sin(2\theta_{\rm IL})$$

Rearranging,

$$R_{\rm L}Z_{\rm T1}\left(\frac{c_{\rm T1}}{c_{\rm L1}}\right)\sin(2\theta_{\rm TL}) - R_{\rm T}Z_{\rm T1}\cos(2\theta_{\rm RT}) + T_{\rm L}Z_{\rm T2}\left(\frac{c_{\rm T2}}{c_{\rm L2}}\right)\sin(2\theta_{\rm TL}) - T_{\rm T}Z_{\rm T2}\cos(2\theta_{\rm TT}) \\ = Z_{\rm T1}\left(\frac{c_{\rm T1}}{c_{\rm L1}}\right)\sin(2\theta_{\rm IL}) \qquad \text{(D)}$$

Putting everything together...

If we write

$$\boldsymbol{x} = \left\{R_{\rm L} \quad T_{\rm L} \quad R_{\rm T} \quad T_{\rm T}\right\}^T$$

and

$$\boldsymbol{b}_{\rm L} = \left\{\cos\theta_{\rm IL} \quad \sin\theta_{\rm IL} \quad Z_{\rm L1}\cos(2\theta_{\rm RT}) \quad Z_{\rm T1}\left(\frac{c_{\rm T1}}{c_{\rm L1}}\right)\sin(2\theta_{\rm IL})\right\}^T,$$

the full reflection-transmission problem can be written as

$$[M]\boldsymbol{x} = \boldsymbol{b}_{\rm L}$$

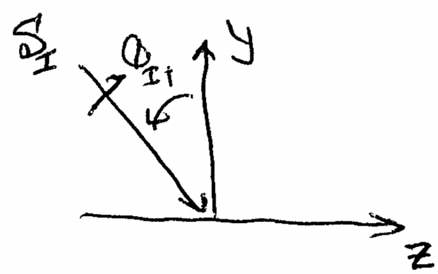
where

$$[M] = \begin{bmatrix} \cos\theta_{\rm RL} & \cos\theta_{\rm TL} & \sin\theta_{\rm RT} & -\sin\theta_{\rm TT} \\ -\sin\theta_{\rm RL} & \sin\theta_{\rm TL} & \cos\theta_{\rm RT} & \cos\theta_{\rm TT} \\ -Z_{\rm L1}\cos 2\theta_{\rm RT} & Z_{\rm L2}\cos 2\theta_{\rm TT} & -Z_{\rm T1}\sin 2\theta_{\rm RT} & -Z_{\rm T2}\sin 2\theta_{\rm TT} \\ Z_{\rm T1}(c_{\rm T1}/c_{\rm L1})\sin 2\theta_{\rm RL} & Z_{\rm T2}(c_{\rm T2}/c_{\rm L2})\sin 2\theta_{\rm TL} & -Z_{\rm T1}\cos 2\theta_{\rm RT} & Z_{\rm T2}\cos 2\theta_{\rm TT} \end{bmatrix}$$

3 Transverse (S-wave) incidence

Geometry and material properties

The material properties are the same as in the previous section. The definitions are also the same, with the addition of θ_{IT} denoting the angle of the incident shear (T for *transverse*) wave.



The incident S-wave displayed above is given by

$$\boldsymbol{u}_{\text{IT}} = \begin{Bmatrix} u_z \\ u_y \end{Bmatrix} = \begin{Bmatrix} u_{\text{IT}} \cos\theta_{\text{IT}} \\ u_{\text{IT}} \sin\theta_{\text{IT}} \end{Bmatrix} e^{j(\omega t + \chi_{\text{IT}})}$$

where

$$\chi_{\text{IT}} = -k_{\text{T1}} z \sin\theta_{\text{IT}} + k_{\text{T1}} y \cos\theta_{\text{IT}}$$

Here we note that the angle of propagation, $\boldsymbol{n}$, is captured in the phase χ_{IT}, while the polarization direction, $\boldsymbol{n}^{\text{p}}$, is captured in the components of the displacement vector.

Applying the continuity conditions...

Continuity of (a) now requires incident amplitudes to be

$$u_{\text{IT}} \sin\theta_{\text{IT}}$$

Matching the phases for all z at $y = 0$ means

$$\chi_{\text{IT}} = \chi_{\text{RL}} = \chi_{\text{RT}} = \chi_{\text{TL}} = \chi_{\text{TT}}$$

The first and third equalities recover the law of reflection:

$$\theta_{\text{RT}} = \theta_{\text{IT}}$$

and the remaining four equalities recover Snell's law:

$$\frac{\sin\theta_{\text{IT}}}{c_{\text{T1}}} = \frac{\sin\theta_{\text{RL}}}{c_{\text{L1}}} = \frac{\sin\theta_{\text{TL}}}{c_{L2}} = \frac{\sin\theta_{\text{TT}}}{c_{\text{T2}}}$$

So, continuity condition (a) becomes

$$R_{\mathrm{L}}^{\mathrm{T}} \cos\theta_{\mathrm{RL}} + R_{\mathrm{T}}^{\mathrm{T}} \sin\theta_{\mathrm{RT}} + T_{\mathrm{L}}^{\mathrm{T}} \cos\theta_{\mathrm{TL}} - T_{\mathrm{T}}^{\mathrm{T}} \sin\theta_{\mathrm{TT}} = -\sin\theta_{\mathrm{IT}} \tag{A†}$$

where

$$\begin{aligned} R_{\mathrm{L}}^{\mathrm{T}} &= \frac{u_{\mathrm{RL}}}{u_{\mathrm{IT}}} \\ R_{\mathrm{T}}^{\mathrm{T}} &= \frac{u_{\mathrm{RT}}}{u_{\mathrm{IT}}} \\ T_{\mathrm{L}}^{\mathrm{T}} &= \frac{u_{\mathrm{TL}}}{u_{\mathrm{IT}}} \\ T_{\mathrm{T}}^{\mathrm{T}} &= \frac{u_{\mathrm{TT}}}{u_{\mathrm{IT}}} \end{aligned}$$

Continuity equation (b) requires the incident z-amplitude to be

$$u_{\mathrm{IT}} \cos\theta_{\mathrm{IT}}$$

which yields

$$-R_{\mathrm{L}}^{\mathrm{T}} \sin\theta_{\mathrm{RL}} + R_{\mathrm{T}}^{\mathrm{T}} \cos\theta_{\mathrm{RT}} + T_{\mathrm{L}}^{\mathrm{T}} \sin\theta_{\mathrm{TL}} - T_{\mathrm{T}}^{\mathrm{T}} \cos\theta_{\mathrm{TT}} = \cos\theta_{\mathrm{IT}} \tag{B†}$$

For continuity of normal stress,

$$T_{yy}\Big|_{y=0}^{\text{solid 1}} = T_{yy}\Big|_{y=0}^{\text{solid 2}}$$

So in solid 1, at $y = 0$,

$$\frac{\partial u_y}{\partial y}\Bigg|_{y=0}^{\text{solid 1}} = u_{\mathrm{IT}} \sin\theta_{\mathrm{IT}}(jk_{\mathrm{T1}} \cos\theta_{\mathrm{IT}}) + u_{\mathrm{RL}} \cos\theta_{\mathrm{RL}}(-jk_{\mathrm{L1}} \cos\theta_{\mathrm{RL}}) + u_{\mathrm{RT}} \cos\theta_{\mathrm{RT}}(-jk_{\mathrm{T1}} \cos\theta_{\mathrm{RT}}),$$

and in solid 2, at $y = 0$

$$\frac{\partial u_y}{\partial y}\Bigg|_{y=0}^{\text{solid 2}} = -u_{\mathrm{TL}} \cos\theta_{\mathrm{TL}}(jk_{\mathrm{L2}} \cos\theta_{\mathrm{TL}}) + u_{\mathrm{TL}} \sin\theta_{\mathrm{TT}}(jk_{\mathrm{TL}} \cos\theta_{\mathrm{TT}}).$$

Likewise, for E_{zz},

$$\frac{\partial u_z}{\partial z}\Bigg|_{y=0}^{\text{solid 1}} = u_{\mathrm{IT}} \cos\theta_{\mathrm{IT}}(-jk_{\mathrm{T1}} \sin\theta_{\mathrm{IT}}) + u_{\mathrm{RL}} \sin\theta_{\mathrm{RL}}(-jk_{\mathrm{L1}} \cos\theta_{\mathrm{RL}}) - u_{\mathrm{RT}} \cos\theta_{\mathrm{RT}}(-jk_{\mathrm{T1}} \sin\theta_{\mathrm{RT}}),$$

and in solid 2, at $y = 0$

$$\left.\frac{\partial u_z}{\partial z}\right|_{y=0}^{\text{solid 2}} = u_{\text{TL}} \sin\theta_{\text{TL}}(-jk_{\text{L2}} \sin\theta_{\text{TL}}) + u_{\text{TL}} \cos\theta_{\text{TT}}(jk_{\text{TL}} \sin\theta_{\text{TT}}).$$

Therefore, the u_{IT} term becomes

$$\frac{\lambda_1 + 2\mu_1}{c_{\text{T1}}} \sin\theta_{\text{IT}} \cos\theta_{\text{IT}} u_{\text{IT}} - \frac{\lambda_1}{c_{\text{T1}}} \sin\theta_{\text{IT}} \cos\theta_{\text{IT}} u_{\text{IT}}$$

Regrouping,

$$\frac{\lambda_1}{c_{\text{T1}}}(\sin\theta_{\text{IT}} \cos\theta_{\text{IT}} - \sin\theta_{\text{IT}} \cos\theta_{\text{IT}}) - Z_{\text{T1}} \sin 2\theta_{\text{IT}}$$

So, continuity equation (c) becomes

$$\begin{aligned} -Z_{\text{L1}} R_{\text{L}}^{\text{T}} \cos 2\theta_{\text{RT}} - Z_{\text{T1}} R_{\text{T}}^{\text{T}} \sin 2\theta_{\text{RT}} + Z_{\text{L2}} T_{\text{L}}^{\text{T}} \cos 2\theta_{\text{TT}} - Z_{\text{T2}} T_{\text{T}}^{\text{T}} \sin 2\theta_{\text{TT}} \\ = -Z_{\text{T1}} \sin 2\theta_{\text{TT}} \end{aligned} \tag{C†}$$

For continuity of shear stress,

$$\left.T_{yz}\right|_{y=0}^{\text{solid 1}} = \left.T_{yz}\right|_{y=0}^{\text{solid 2}}.$$

The strains are slightly altered because of the different polarization of the incident waves:

$$\left.\frac{\partial u_y}{\partial z}\right|_{y=0}^{\text{solid 1}} = -u_{\text{IT}} \sin\theta_{\text{IT}}(-jk_{\text{L1}} \sin\theta_{\text{IT}}) + u_{\text{RL}} \cos\theta_{\text{RL}}(-jk_{\text{L1}} \sin\theta_{\text{RL}}) + u_{\text{RT}} \sin\theta_{\text{RT}}(-jk_{\text{T1}} \sin\theta_{\text{RT}}),$$

$$\left.\frac{\partial u_z}{\partial y}\right|_{y=0}^{\text{solid 1}} = u_{\text{IT}} \cos\theta_{\text{IT}}(jk_{\text{T1}} \cos\theta_{\text{IT}}) + u_{\text{RL}} \sin\theta_{\text{RL}}(-jk_{\text{L1}} \cos\theta_{\text{RL}}) - u_{\text{RT}} \cos\theta_{\text{RT}}(-jk_{\text{T1}} \cos\theta_{\text{RT}})$$

Continuity of shear stress still requires

$$\mu_1(u_{y,z} + u_{z,y})_{y=0}^{\text{solid 1}} = \mu_2(u_{y,z} + u_{z,y})_{y=0}^{\text{solid 2}}$$

The only changed components are those detailed below:

$$\begin{aligned} (u_{y,z} + u_{z,y})_{\text{IT}} &= \left(-j\frac{\omega}{c_{\text{T1}}} \sin^2\theta_{\text{IT}} + j\frac{\omega}{c_{\text{T1}}} \cos^2\theta_{\text{IT}}\right) u_{\text{IT}} \\ &= \frac{j\omega}{c_{\text{T1}}}(\cos^2\theta_{\text{IT}} - \sin^2\theta_{\text{IT}}) u_{\text{IT}} \\ &= \frac{j\omega}{c_{\text{T1}}} \cos(2\theta_{\text{IT}}) u_{\text{IT}} \end{aligned}$$

Substituting this into the full relationship for the continuity of shear stress, we find

$$\frac{\mu_1}{c_{\mathrm{T1}}}\cos(2\theta_{\mathrm{IT}}) - \frac{\mu_1}{c_{\mathrm{L1}}}R^{\mathrm{T}}_{\mathrm{L}}\sin(2\theta_{\mathrm{IL}}) + \frac{\mu_1}{c_{\mathrm{T1}}}R^{\mathrm{T}}_{\mathrm{T}}\cos(2\theta_{\mathrm{RT}}) = \frac{\mu_2}{c_{\mathrm{L2}}}T^{\mathrm{T}}_{\mathrm{L}}\sin(2\theta_{\mathrm{IT}}) + \frac{\mu_2}{c_{\mathrm{T2}}}T^{\mathrm{T}}_{\mathrm{T}}\cos(2\theta_{\mathrm{TT}})$$

Identifying $Z_{\mathrm{T1}} = \mu_1/c_{\mathrm{T1}}$ and $Z_{\mathrm{T2}} = \mu_2/c_{\mathrm{T2}}$, the above becomes

$$R^{\mathrm{T}}_{\mathrm{L}}Z_{\mathrm{T1}}\left(\frac{c_{\mathrm{T1}}}{c_{\mathrm{L1}}}\right)\sin(2\theta_{\mathrm{RL}}) - R^{\mathrm{T}}_{\mathrm{T}}Z_{\mathrm{T1}}\cos(2\theta_{\mathrm{RT}}) + T^{\mathrm{T}}_{\mathrm{L}}Z_{\mathrm{T2}}\left(\frac{c_{\mathrm{T2}}}{c_{\mathrm{L2}}}\right)\sin(2\theta_{\mathrm{TL}}) + T^{\mathrm{T}}_{\mathrm{T}}Z_{\mathrm{T2}}\left(\frac{c_{\mathrm{T1}}}{c_{\mathrm{L1}}}\right)\cos(2\theta_{\mathrm{TT}}) = Z_{\mathrm{T1}}\cos(2\theta_{\mathrm{IT}}) \qquad (\mathrm{D}\dagger)$$

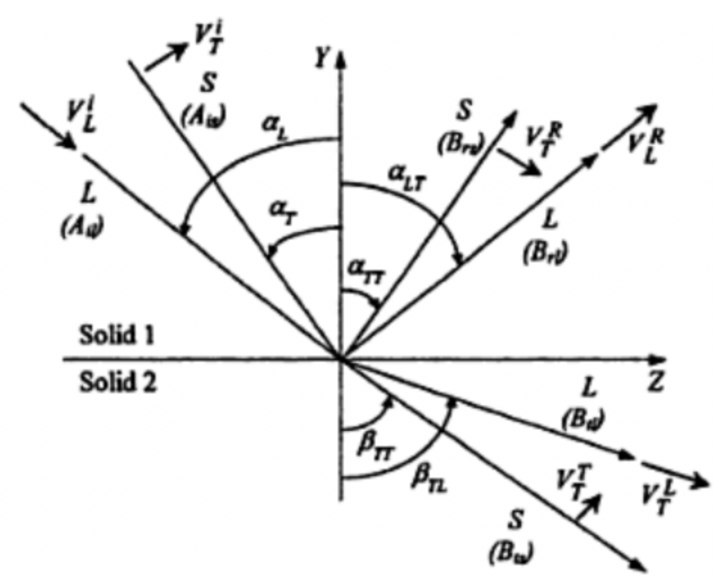
Putting everything together...

Combining (A†), (B†), (C†), (D†) yields a different $\boldsymbol{b}$ vector, $\boldsymbol{b}_{\mathrm{T}}$, which satisfies $[M]\boldsymbol{x} = \boldsymbol{b}_{\mathrm{T}}$, where $[M]$ and $\boldsymbol{x}$ are as defined in the previous section.

$$\boldsymbol{b}_{\mathrm{T}} = \Big\{ -\sin\theta_{\mathrm{IT}} \quad \cos\theta_{\mathrm{IT}} \quad -Z_{\mathrm{T1}}\sin(2\theta_{\mathrm{IT}}) \quad Z_{\mathrm{T1}}\cos(2\theta_{\mathrm{IT}}) \Big\}^T,$$

4 Comments on Rose's presentation of this topic (Ch. 5)

Figure 5.1 of Rose gives the following conventions for the incident, reflected, and transmitted waves for a two medium problem where both media are isotropic elastic solids.



The text gives a matrix relationship $[M]\boldsymbol{x} = \boldsymbol{a}$ for the reflection-transmission problem where $\boldsymbol{x}$ and $[M]$ can be written in the same form regardless of the incident wave type (longitudinal or vertically-polarized shear (SV) waves) as long as both materials are elastic solids. Slight modification to Rose's notation allow us to write those terms as and

$$[M] = \begin{bmatrix} -\cos\alpha_{LT} & \sin\alpha_{\mathrm{TT}} & -\cos\beta_{\mathrm{TL}} & \sin\beta_{\mathrm{TT}} \\ -\sin\alpha_{LT} & \cos\alpha_{\mathrm{TT}} & \sin\beta_{\mathrm{TL}} & \cos\beta_{\mathrm{TT}} \\ -k_{\mathrm{L1}}(\lambda_1+2\mu_1)\cos 2\alpha_{\mathrm{TT}} & k_{\mathrm{T1}}\mu_1\sin 2\alpha_{\mathrm{TT}} & k_{\mathrm{L2}}(\lambda_2+\mu_2)\cos 2\beta_{\mathrm{TT}} & k_{\mathrm{T2}}\mu_2\sin 2\beta_{\mathrm{TT}} \\ -k_{\mathrm{L1}}\mu_1\sin 2\alpha_{LT} & -k_{\mathrm{T1}}\mu_1\cos 2\alpha_{\mathrm{TT}} & -k_{\mathrm{L2}}\mu_1\sin 2\beta_{\mathrm{TL}} & -k_{\mathrm{T2}}\mu_2\cos 2\beta_{\mathrm{TT}} \end{bmatrix}$$

The vector $\boldsymbol{x}$ here replaced the one provided by Rose because we will always have reflected and transmitted longitudinal and transverse waves whether we have a longitudinal or transverse wave incident, making the first subscript in his notation superfluous. The adopted symbol for the transmission coefficient, T, is employed rather than Rose's D. Now, since we are only concerned with incident longitudinal waves, the relevant $\boldsymbol{a}$ vector provided by Rose is

$$\boldsymbol{a} = \begin{bmatrix} -\cos\alpha_{\mathrm{L}} \\ \sin\alpha_{\mathrm{L}} \\ k_{\mathrm{L1}}(\lambda_1+2\mu_1)\cos 2\alpha_{\mathrm{L}} \\ -k_{\mathrm{L1}}\mu_1\sin 2\alpha_{\mathrm{L}} \end{bmatrix}.$$

Unfortunately, there are several errors in this matrix expression. One annoying detail is that the final two rows of $[M]$ and the vector $\boldsymbol{a}$ contain wavenumbers, $k = \omega/c$, when the common ω cancels.

The equations below provide the corrected form of $[M]$ and $\boldsymbol{a}$ for longitudinal wave incidence which can be simplified for the case of a fluid-solid configuration to yield the correct approximation of the reflected and transmitted displacement amplitudes. First, let's re-define the angles using a more intuitive naming convention:

$$\begin{array}{rr} \alpha_{\mathrm{L}} \to \theta_{\mathrm{IL}} & \alpha_{\mathrm{T}} \to \theta_{\mathrm{IT}} \\ \alpha_{\mathrm{LT}} \to \theta_{\mathrm{RL}} & \beta_{\mathrm{TL}} \to \theta_{\mathrm{TL}} \\ \alpha_{\mathrm{TT}} \to \theta_{\mathrm{RT}} & \beta_{\mathrm{TT}} \to \theta_{\mathrm{TT}} \end{array}$$

Here the first subscript represents incident, reflected, or transmitted as I, R, and T, respectively, and the second subscript denotes the wave type with L = longitudinal and T = transverse/shear. Using this convention and correcting mistakes in Rose (detailed in the scanned notes and above) $[M]$ and $\boldsymbol{a}$ in Rose's form for longitudinal wave incidence become

$$[M] = \begin{bmatrix} -\cos\theta_{\mathrm{RL}} & -\sin\theta_{\mathrm{RT}} & -\cos\theta_{\mathrm{TL}} & \sin\theta_{\mathrm{TT}} \\ -\sin\theta_{\mathrm{RL}} & \cos\theta_{\mathrm{RT}} & \sin\theta_{\mathrm{TL}} & \cos\theta_{\mathrm{TT}} \\ -Z_{\mathrm{L1}}\cos 2\theta_{\mathrm{RT}} & -Z_{\mathrm{T1}}\sin 2\theta_{\mathrm{RT}} & Z_{\mathrm{L2}}\cos 2\theta_{\mathrm{TT}} & -Z_{\mathrm{T2}}\sin 2\theta_{\mathrm{TT}} \\ -Z_{\mathrm{T1}}(c_{\mathrm{T1}}/c_{\mathrm{L1}})\sin 2\theta_{\mathrm{RL}} & Z_{\mathrm{T1}}\cos 2\theta_{\mathrm{RT}} & -Z_{\mathrm{T2}}(c_{\mathrm{T2}}/c_{\mathrm{L2}})\sin 2\theta_{\mathrm{TL}} & -Z_{\mathrm{T2}}\cos 2\theta_{\mathrm{TT}} \end{bmatrix}$$

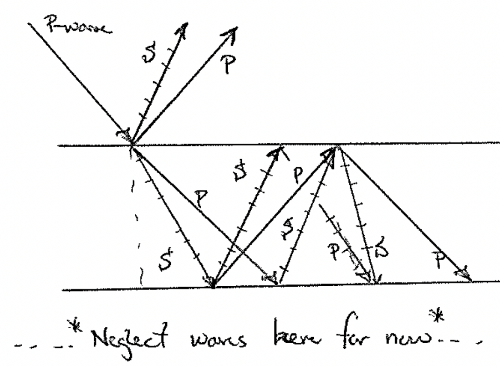
and

$$
\boldsymbol{a} = \begin{bmatrix} -\cos\theta_{\mathrm{IL}} \\ \sin\theta_{\mathrm{IL}} \\ Z_{\mathrm{L1}}\cos 2\theta_{\mathrm{RT}} \\ -Z_{\mathrm{T1}}(c_{\mathrm{T1}}/c_{\mathrm{L1}})\sin 2\theta_{\mathrm{IL}} \end{bmatrix}.
$$

Note that this matrix looks different from the one provided in Eq. 1. This is primarily due to the fact that Eq. 1 assumes that the vector of unknowns is $\boldsymbol{x} = [R_{\mathrm{L}}\ T_{\mathrm{L}}\ R_{\mathrm{T}}\ T_{\mathrm{T}}]^T$ while Rose's text uses the order $\boldsymbol{x} = [R_{\mathrm{L}}\ R_{\mathrm{T}}\ T_{\mathrm{L}}\ T_{\mathrm{T}}]^T$. We further observe that the inclusion of θ_{RT} in $\boldsymbol{a}$ is *not* a typo, but due to a relationship following Eq. 5.86 provided in chapter 5 of *Wave Propagation in Elastic Solids* by J.D. Achenchach. The various angles of incidence can be found using expressions for Snell's law that were provided in class. Direct derivation of the reflection and transmission coefficients verifies that this expression can be correctly simplified for the case where an incident pressure (longitudinal) wave in a fluid is incident on an elastic solid.

5 Additional observations

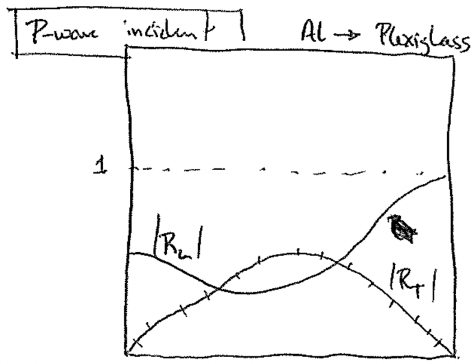
For multiple layers, this can be handled sameas with fluids, by introducing a phase Fortunately, this is somewhat simpler than expected since we can essentially solve for $R - T$ conditions when pairs of incident waves are incident on an interface. [Thompson-Haskell Method] (Rose chapter 5.3).



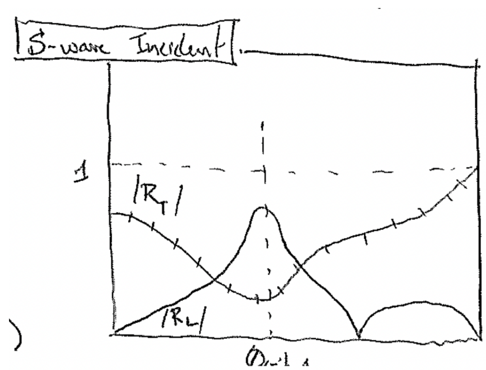
This creates an elastic wave guide $\implies$ Lamb modes, i.e. plate waves. We will discuss this in more detail later in the class.

We can solve the problem of incident P- and S-wave pairs as well, but we will look at the problem in detail as Lamb waves come into the picture.

What do R and T look like vs. angle of incidence? Usually, smooth curves are expected, unless the critical angle is subtended (see Rose pg. 58).



Sum fields due to incident longitudinal and shear waves at each interface.



Note that this is useful for 1-sided material testing.