$x^2 y'' + x y' + x^2 y = 0$ [$\nu = 0$ Bessel's equation]

$y = \sum_{n=0}^{\infty} a_n x^{n+r}$ because $x=0$ is reg. singular:

$\lim_{x \to 0} \frac{x}{x^2} x = 1$, $\lim_{x \to 0} \frac{x^2}{x^2} x^2 = 0$. both finite.

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0.$$

Want to combine all the summations.

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + (n+r) a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0.$$

write first two terms ($n=0$ and $n=1$) outside the summation

$$r(r-1) a_0 x^r + r a_0 x^r + (1+r) r a_1 x^{1+r} + (1+r) a_1 x^{1+r} + \sum_{n=2}^{\infty} \{ [(n+r)(n+r-1) + (n+r)] a_n + a_{n-2} \} x^{n+r} = 0. \quad (*)$$

Each powers of $x \to 0$ because the RHS is 0:
(term with different)

$r(r-1) + r = 0$

$r^2 - r + r = 0 \longrightarrow r = 0$
$r = 0$.

focus on this one for now.

$(1+r) r + 1 + r = 0$.
$r^2 + 2r + 1 = 0$
$r = -1$
$r = -1$. ??

What about this one

see next.

Setting $\sum_{n=2}\{\ \} = 0$, gives
(each term of)

$$a_n = \frac{-a_{n-2}}{(n+r)(n+r-1)+n+r}$$

for $r=0$:

$$a_n = \frac{-a_{n-2}}{(n)(n-1)+n} = \frac{-a_{n-2}}{n^2}. \rightarrow \text{Recursion relation.}$$

~~Fich~~ Then $a_1 = 0$ necessarily because

$[1+r+(1+r)r]a_1 = 0$ (from the other term of ⊛).

$(1+0+0)a_1 = 0 \longrightarrow a_1 = 0$, Let $a_0 = 1$.

So
$$\left.\begin{aligned} a_2 &= -\frac{a_0}{2^2} = -\frac{1}{4} \\ a_3 &= -\frac{a_1}{3^2} = 0. \\ a_4 &= -\frac{a_2}{4^2} = +\frac{1}{2^2 4^2}\ldots \end{aligned}\right\} \text{Thus } a_k = \frac{(-1)^k}{2^2 4^2 \cdots (2k)^2}.$$

Thus $y_1 = \sum_{k=1}^{\infty} \frac{(-1)^k}{2^2 4^2 \cdot (2k)^2} x^{2k} + 1.$

Can include in the summation $\longleftarrow$ $k=0$ term to account for $a_0 = 1$.

$$y_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^2 4^2 \cdot (2k)^2} x^{2k}.$$

$$= J_0(x).$$ Bessel function

I think using $r=-1$ would give Neumann function $N_0(x)$ but not sure.