$2x^2 y'' - xy' + (1+x)y = 0.$ ← want to solve (1)

→ of the form $a(x)y'' + b(x)y' + c(x)y = 0.$

x_0 is a singular pt. if $a(x_0) = 0.$

x_0 is regular singular if

$$\lim_{x \to x_0} \frac{b(x)}{a(x)}(x - x_0) = \text{finite}$$

and $$\lim_{x \to x_0} \frac{c(x)}{a(x)}(x - x_0)^2 = \text{finite}.$$

} memorize these criteria for qualifier

In this case, $b(x) = -x$, $a(x) = 2x^2$, and $(1+x) = c(x)$,

and $x = 0$ is the singular pt.

$$\lim_{x \to 0} \frac{-x}{2x^2}(x) = \lim_{x \to 0} -\frac{1}{2}\frac{x}{x} = -\frac{1}{2}$$

$$\lim_{x \to 0} \frac{1+x}{2x^2}(x)^2 = \lim_{x \to 0} \frac{1}{2}(1+x) = \frac{1}{2}$$

} Regular.

Therefore use Frobenius's method to solve (1).

Let $y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}.$

then $y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$

and $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}.$

Substitute the derivatives into (1).

$$2x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} - x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + (1+x) \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$2\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$= 0.$

$$\sum_{n=0}^{\infty}\left[2a_n(n+r)(n+r-1)x^{n+r} - a_n(n+r)x^{n+r} + a_n x^{n+r}\right] + \sum_{n=1}^{\infty} a_{n-1}x^{n+r} = 0.$$

In order to combine the summations, the $n=0$ term needs to be removed from the first summation, i.e., (first rewrite it)

$$\sum_{n=0}^{\infty}\left[2(n+r)(n+r-1) - (n+r) + 1\right]a_n x^{n+r} + \sum_{n=1}^{\infty} a_{n-1}x^{n+r} = 0.$$

$$\left[2r(r-1) - r + 1\right]a_0 x^r + \sum_{n=1}^{\infty}\Big\{\underbrace{\left[2(n+r)(n+r-1) - (n+r) + 1\right]a_n + a_{n-1}}_{\star}\Big\}x^{n+r} = 0$$

Every term above (i.e., every power of x) must be 0 because the RHS is 0. Therefore, since $a_0 \neq 0$.

"indicial eq." $\longrightarrow$ $2r(r-1) - r + 1 = 0.$

$$2r^2 - 2r - r + 1 = 0$$
$$2r^2 - 3r + 1 = 0.$$
$$(2r-1)(r-1) = 0.$$

$$\boxed{\begin{aligned} r &= \tfrac{1}{2} \\ r &= 1 \end{aligned}}$$

"exponents @ the singularity"

The characteristic equation from setting $\star = 0$ must be considered for the exponents at the singularity:

(2) ---- $$a_n = \frac{-a_{n-1}}{2(n+r)(n+r-1) - (n+r) + 1}, \quad \text{for } n \geq 1.$$

We will consider first eq. (2) for $r = 1$:

for $r = 1$, equation (2) becomes

$$a_n = \frac{-a_{n-1}}{2(n+1)n - (n+1) + 1} = \frac{-a_{n-1}}{2n^2 + 2n - n}$$

$$\longrightarrow \quad a_n = \frac{-a_{n-1}}{n(2n+1)}, \text{ for } n \geq 1$$

Choose $a_0 = 1$. then ...

$n=1 \qquad a_1 = \frac{-a_0}{1(2\cdot 1+1)} = -\frac{1}{1\cdot 3}$

$n=2 \qquad a_2 = \frac{-a_1}{2(2\cdot 2+1)} = \frac{-a_1}{2\cdot 5} = +\frac{1}{1\cdot 3\cdot 2\cdot 5}$

$n=3 \qquad a_3 = \frac{-a_2}{3(6+1)} = -\frac{a_2}{3\cdot 7} = -\frac{1}{1\cdot 3\cdot 2\cdot 5\cdot 3\cdot 7}.$

$n=4 \qquad a_4 = \frac{-a_3}{4(8+1)} = +\frac{1}{1\cdot 3\cdot 2\cdot 5\cdot 3\cdot 7\cdot 4\cdot 9}$

$n!$

$3\cdot 5\cdot 7 \cdot 2n+1$

So $a_n = \frac{(-1)^n}{n!\,(3\cdot 5\cdot 7 \cdots 2n+1)}$.

The first sol. is thus $\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+1}$

$$y_1 = \cancel{a_0}\,1\, x + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\, 3\cdot 5\cdot 7\cdots(2n+1)} x^{n+1}.$$

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for $r = \frac{1}{2}$, eq. (2) becomes

$$a_n = \frac{-a_{n-1}}{2(n+\frac{1}{2})(n+\frac{1}{2}-1) - (n+\frac{1}{2}) + 1}$$

$$= \frac{-a_{n-1}}{2n^2 - \frac{1}{2} - n - \frac{1}{2} + 1} = \frac{-a_{n-1}}{2n^2 - n}$$

$$= \frac{-a_{n-1}}{n(2n-1)}.$$

Again pick $a_0 = 1$

$n=1$ $\quad a_1 = \dfrac{-a_0}{2-1} = -1$

$n=2$ $\quad a_2 = \dfrac{-a_1}{2(4-1)} = \dfrac{1}{2\cdot 3}$

$n=3$ $\quad a_3 = \dfrac{-a_2}{3(6-1)} = -\dfrac{1}{2\cdot 3\cdot 3\cdot 5}$

$n=4$ $\quad a_4 = \dfrac{-a_3}{4(8-1)} = \dfrac{1}{2\cdot 3\cdot 3\cdot 5\cdot 4\cdot 7}$

$n!$ $\qquad 3\cdot 5\cdots(2n-1)$.

So $a_n = \dfrac{(-1)^n}{n!\; 3\cdot 5\cdot 7\cdots(2n-1)}$

So $y_2 = \displaystyle\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} \frac{(-1)^n \, x^{n+\frac{1}{2}}}{n!\, 3\cdot 5\cdot 7 \cdots (2n-1)} + \underbrace{x^{1/2}}$

$n=0$ term.

The sum $y_1 + y_2$ is the general solution.