mple 2. We can now find the Fourier series for some other functions without more evaluation of coefficients. For example, consider

$$g(x)=\begin{cases}-1, & -\pi<x<0,\\ 1, & 0<x<\pi.\end{cases} \tag{5.13}$$

Sketch this and verify that $g(x)=2f(x)-1$, where $f(x)$ is the function in Example 1. Then from (5.12), the Fourier series for $g(x)$ is

$$g(x)=\frac{4}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right). \tag{5.14}$$

Similarly, verify that $h(x)=f(x+\pi/2)$ is Fig. 5.1 shifted $\pi/2$ to the left (sketch it), and its Fourier series is (replace x in (5.12) by $x+\pi/2$)

$$h(x)=\frac{1}{2}+\frac{2}{\pi}\left(\frac{\cos x}{1}-\frac{\cos 3x}{3}+\frac{\cos 5x}{5}+\cdots\right)$$

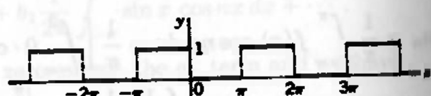
since $\sin(x+\pi/2)=\cos x$, $\sin(x+3\pi/2)=-\cos 3x$, etc.

ROBLEMS, SECTION 5

In each of the following problems you are given a function on the interval $-\pi<x<\pi$. Sketch several periods of the corresponding periodic function of period 2π. Expand the periodic function in a sine-cosine Fourier series.

1. $f(x)=\begin{cases}1, & -\pi<x<0,\\ 0, & 0<x<\pi.\end{cases}$

In this case the sketch is:



Your answer for the series is: $f(x)=\frac{1}{2}-\frac{2}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}\cdots\right)$.
Can you use the ideas of Example 2 to find this result without computation?

2. $f(x)=\begin{cases}0, & -\pi<x<0,\\ 1, & 0<x<\frac{\pi}{2},\\ 0, & \frac{\pi}{2}<x<\pi.\end{cases}$

Answer: $f(x)=\frac{1}{4}+\frac{1}{\pi}\left(\frac{\cos x}{1}-\frac{\cos 3x}{3}+\frac{\cos 5x}{5}\cdots\right)$
$+\frac{1}{\pi}\left(\frac{\sin x}{1}+\frac{2\sin 2x}{2}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}\cdots\right)$.

3. $f(x)=\begin{cases}0, & -\pi<x<\frac{\pi}{2},\\ 1, & \frac{\pi}{2}<x<\pi.\end{cases}$

Answer: $f(x)=\frac{1}{4}-\frac{1}{\pi}\left(\frac{\cos x}{1}-\frac{\cos 3x}{3}+\frac{\cos 5x}{5}\cdots\right)$
$+\frac{1}{\pi}\left(\frac{\sin x}{1}-\frac{2\sin 2x}{2}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}-\frac{2\sin 6x}{6}\cdots\right)$.

4. $f(x)=\begin{cases}-1, & -\pi<x<\frac{\pi}{2},\\ 1, & \frac{\pi}{2}<x<\pi.\end{cases}$

Could you use Problem 3 to solve Problem 4 without computation?

5. $f(x)=\begin{cases}0, & -\pi<x<0,\\ -1, & 0<x<\frac{\pi}{2},\\ 1, & \frac{\pi}{2}<x<\pi.\end{cases}$

6. $f(x)=\begin{cases}1, & -\pi<x<-\frac{\pi}{2}, \text{ and } 0<x<\frac{\pi}{2};\\ 0, & -\frac{\pi}{2}<x<0, \text{ and } \frac{\pi}{2}<x<\pi.\end{cases}$

7. $f(x)=\begin{cases}0, & -\pi<x<0;\\ x, & 0<x<\pi.\end{cases}$

Answer: $f(x)=\frac{\pi}{4}-\frac{2}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right)$
$+\left(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\cdots\right)$.

8. $f(x)=1+x, \quad -\pi<x<\pi.$

Answer: $f(x)=1+2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\cdots\right)$.

9. $f(x)=\begin{cases}-x, & -\pi<x<0,\\ x, & 0<x<\pi.\end{cases}$

Answer: $f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{1}{9}\cos 3x+\frac{1}{25}\cos 5x+\cdots\right)$.

10. $f(x)=\begin{cases}\pi+x, & -\pi<x<0,\\ \pi-x, & 0<x<\pi.\end{cases}$

11. $f(x)=\begin{cases}0, & -\pi<x<0,\\ \sin x, & 0<x<\pi.\end{cases}$

Answer: $f(x)=\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\left(\frac{\cos 2x}{2^2-1}+\frac{\cos 4x}{4^2-1}+\frac{\cos 6x}{6^2-1}+\cdots\right)$.

12. Show that in (5.2) the average values of $\sin mx \sin nx$ and of $\cos mx \cos nx$, $m\neq n$, are zero (over a period), by using the complex exponential forms for the sines and cosines as in (5.3).

13. Write out the details of the derivation of equation (5.10).

IRICHLET CONDITIONS

Now we have a series, but there are still some questions that we ought to get answered. Does it converge, and if so, does it converge to the values of $f(x)$? You will find, if you try, that for most values of x the series in (5.12) does not respond to any of the tests for convergence that we discussed in Chapter 1. What is the sum of the series at $x=0$ where $f(x)$ jumps from 0 to 1? You can see from the series (5.12) that the sum at $x=0$ is $\frac{1}{2}$, but what does this have to do with $f(x)$?