

# Review for the acoustics qualifying exam

Information about the PhD qualifying exam in acoustics at UT Austin can be found [here](#).

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## **Additional resources**

- [Georgia Tech acoustics qualifying questions](#)
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- [Penn State vibrations, acoustics, and math qualifying questions](#)
- [Philip Kaufinger's \*Acoustics Derivations\* YouTube Channel](#)
- [Scott Schoen Jr.'s quals review book](#)

# Physical acoustics

This review follows the chapter-by-chapter organization of *Fundamentals of Physical Acoustics* by D. T. Blackstock. Chapters 13 and 14 are combined into a collective section on diffraction, for which M. F. Hamilton's lecture notes are the primary reference. Some content comes from the [Feynman Lectures](#) (Volume I, chapters 47-52) and A. D. Pierce's [Acoustics: An Introduction to Its Physical Principles and Applications](#). Concepts and problems I find challenging are denoted by the "⊗" symbol. *Note to self: when studying, resist the urge to view the answer before working out the problem yourself!*

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# Chapter 1: What is a wave?

## 1. What is a wave? [ANSWER]

Rather than providing a direct definition, Blackstock provides six qualities that broadly characterize waves.

1. Waves are disturbances from a pre-existing condition.
2. Waves involve time evolution and travel at finite speeds.
3. For mechanical waves, there exists a medium.
4. Waves are distinct from gross movement of matter.
5. Waves can be idealized by a wave equation.
6. Waves need not be oscillatory.

Points (1), (2), and (6) apply to all types of waves. (3) and (4) only apply to mechanical waves in the continuum approximation and could therefore be excluded for a more general perspective. (5) is debatable.

See also: [\*What is a Wave?\* by Scales and Snieder.](#)

## 2. Derive the d'Alembert solution of the 1D wave equation. [ANSWER]

The general solution to the 1D wave equation is  $\xi(x, t) = f(x - ct) + g(x + ct)$ . Write

$$P(x) = \xi(x, 0) = f(x) + g(x) \quad (\text{i})$$

$$Q(x) = \xi'(x, 0) = -cf'(x) + cg'(x). \quad (\text{ii})$$

Integrate equation (ii) and combine with equation (i) to obtain

$$f(x) = \frac{1}{2}P(x) - \frac{1}{2c} \int Q(y)dy \quad (\text{iii})$$

$$g(x) = \frac{1}{2}P(x) + \frac{1}{2c} \int Q(y)dy. \quad (\text{iv})$$

Replace  $x \mapsto x - ct$  in equation (iii) and replace  $x \mapsto x + ct$  in equation (iv) and let the lower limit of the integrals be  $x_0$ :

$$f(x - ct) = \frac{1}{2}P(x - ct) - \frac{1}{2c} \int_{x_0}^{x-ct} Q(y)dy \quad (\text{v})$$

$$g(x + ct) = \frac{1}{2}P(x + ct) + \frac{1}{2c} \int_{x_0}^{x+ct} Q(y)dy. \quad (\text{vi})$$

Finally, add equations (v) and (vi) to obtain the d'Alembert solution,

$$\xi(x, t) = \frac{1}{2} [P(x - ct) + P(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} Q(y)dy.$$

3. What does the d'Alembert solution say about the dependence of the wave speed on the energy of the initial impulse? [ANSWER]

The d'Alembert solution says that the wave speed is *independent of the energy of the initial impulse*, a hallmark of linear wave phenomena.

4. Do Blackstock problems 1B-4 and 1B-5. Are these boundary- or initial-value problems? [ANSWER]

Note that both problems are *forced wave* problems, which are *boundary-value problems* (i.e., "What is the solution to the wave equation given a particular boundary condition?"). Thus the d'Alembert solution derived in question (2) above is not of use here, since that is the solution to the *initial-value problem* (i.e., "What is the solution to the wave equation given an initial displacement and velocity). *The distinction between boundary- and initial-value problems is a good one to keep in mind, since basically all problems in acoustics can be categorized this way. For example, problems involving waveguides, horns, and diffraction are usually formulated as boundary-value problems. Meanwhile, problems involving modal responses in enclosures are usually formulated as initial-value problems (though rarely explicitly). When finding the eigenfunctions and eigenfrequencies of enclosure, these are the functions and frequencies that would be excited due to an impulse.*

1B-4 and 1B-5 are homework problems, so I cannot post the solution, but I will outline what should be done: In 1B-4, the function corresponding to waves traveling to the left should be tossed out. The argument of the initial condition,  $t$ , should simply be replaced with  $t - x/c_0$ . In 1B-5, both functions should be kept; the solution is identical to that of 1B-4 except for the presence of an additional term, which is has the argument  $t + x/c_0$ .

5. In an RLC circuit, which of the following is complex-valued: the impedance, the characteristic impedance, or both? [ANSWER]

For lossy lines,  $e$  and  $i$  are no longer in phase, and thus the *impedance* is complex. However, the characteristic impedance is still  $\sqrt{L/C}$ .

6. In all plane progressive waves, what is the relationship between kinetic and potential energy densities? [ANSWER]

The kinetic and potential energy densities are equal.

For arbitrary waves (standing, progressive, or any combination of the two), a more general relation between the *total* kinetic and potential energies is provided by the virial theorem. See [here](#) for the virial theorem worked out for waves on a string

7. In what sense are tsunamis shallow-water waves? What does this assumption imply about the particle velocity  $\mathbf{u}$ ? [ANSWER]

Tsunamis are shallow-water waves in the sense that the wavelength  $\lambda$  is much greater than the depth of the ocean  $h_0$ , i.e.,  $h_0 \ll \lambda$ . This means that the particle velocity is purely in the  $x$ -direction, i.e.,  $\mathbf{u} = u\mathbf{e}_x$ .

8. Assuming no losses, the power of a shallow-water wave is conserved and is given by  $\mathcal{W} = \rho_0 g^{3/2} h_0^{1/2} \xi_{\text{rms}}^2$ , where  $\rho_0$  is the ambient water density,  $g$  is the acceleration due to gravity,  $h_0$  is the ocean depth, and  $\xi$  is the wave variable. What does this mean about the dependence of  $\xi$  on  $h_0$ ? [ANSWER]

Since  $\mathcal{W}$  is conserved, and since  $\rho_0 g^{3/2}$  is a constant,  $\xi \propto h_0^{-1/4}$ . Thus a large amplification  $(h_{0,\text{deep}}/h_{0,\text{shallow}})^{1/4}$  of the wave height is achieved when a tsunami wave approaches the shore.

9. ✳ Derive the wave equation for finite deformations of a string. [ANSWER]

This is a tough derivation. It is worked out [here](#).

10. ✳ Do Blackstock problem 1-C5, which is to derive the wave equation for a string with variable density per unit length,  $\rho = \rho(x)$ . [ANSWER]

This is another homework problem, so I cannot post the solution here. However, note the error of the sign of the second term in the answer provided in *Fundamentals*. The correct answer is

$$\xi_{xx} + \frac{T_x}{T} \xi_x - \frac{1}{c^2} \xi_{tt} = 0.$$

11. Derive the continuity equation in 1D. [ANSWER]

The mass of fluid inside a tube of cross-sectional area  $S$  and length  $\Delta x$  is  $\rho S \Delta x$ , and the time derivative of this mass must equal the mass inflow + the mass outflow, which is  $\rho u S|_x - \rho u S|_{x+\Delta x}$ . That is,

$$\begin{aligned} \frac{\partial}{\partial t} (S \Delta x \rho) &= \rho u S|_x - \rho u S|_{x+\Delta x} \\ \frac{\partial \rho}{\partial t} &= -\frac{\partial \rho u}{\partial x} \\ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} &= 0. \end{aligned}$$

12. Derive the momentum equation in 1D. [ANSWER]

The momentum of fluid inside a tube of cross-sectional area  $S$  and length  $\Delta x$  is  $\rho u S \Delta x$ , and the time derivative of this momentum must equal the momentum inflow + the momentum outflow + the force on the boundaries, which is  $\rho u^2 S|_x - \rho u^2 S|_{x+\Delta x} + PS|_x - PS|_{x+\Delta x}$ . That is,

$$\begin{aligned} \frac{\partial}{\partial t} (S \Delta x u \rho) &= \rho u^2 S|_x - \rho u^2 S|_{x+\Delta x} + PS|_x - PS|_{x+\Delta x} \\ \frac{\partial (\rho u)}{\partial t} &= -\frac{\partial \rho u^2}{\partial x} - \frac{\partial P}{\partial x} \\ \frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} + \frac{\partial P}{\partial x} &= 0. \end{aligned}$$

13. What is the equation of state as a Taylor expansion of  $P(\rho)$ ? What assumption lets us write  $P(\rho)$  in the first place? What is the definition of the speed of sound  $c$ ? What distinguishes  $c$  from  $c_0$ ?

What are the parameters  $A$  and  $B$ , and what is the significance of the parameter  $B/A$  in nonlinear acoustics? [ANSWER]

Generally, the pressure is given by two other state variables, like density and entropy, i.e.,  $P = P(\rho, s)$ . But in acoustics, we can get by by assuming that even mildly lossy wave propagation is isentropic, according to Dr. Hamilton:

The lossy linear progressive wave equation does indeed describe isentropic wave propagation under the assumed conditions. Until the attenuation is so strong that one is in the gray area of wave propagation versus diffusion (i.e., for an absorption length on the order of the wavelength), the attenuation is a minor perturbation and it introduces a negligible change in the phase speed relative to that in an ideal fluid.

Therefore, for much of acoustics, the isentropic condition lets us assume pressure is only a function of density, i.e.,  $P(\rho)$ . Just as for any function of one variable, the Taylor expansion of  $P(\rho) = P(\rho_0 + \rho')$  about  $\rho_0$  is

$$P(\rho_0 + \rho') = p_0 + \left. \frac{dP}{d\rho} \right|_{\rho=\rho_0} (\rho - \rho_0) + \frac{1}{2!} \left. \frac{d^2P}{d\rho^2} \right|_{\rho=\rho_0} (\rho - \rho_0)^2 + \dots \quad (i)$$

where  $P = p_0 + p$ ,  $P(\rho_0) = p_0$ , and  $\rho = \rho_0 + \rho'$ . Equation (i) can be written as

$$\begin{aligned} P(\rho_0 + \rho') &= p_0 + \left( \rho_0 \left. \frac{dP}{d\rho} \right|_{\rho=\rho_0} \right) \left( \frac{\rho - \rho_0}{\rho_0} \right) + \left( \rho_0^2 \frac{1}{2!} \left. \frac{d^2P}{d\rho^2} \right|_{\rho=\rho_0} \right) \left( \frac{\rho - \rho_0}{\rho_0} \right)^2 + \dots \\ &= p_0 + A \frac{\rho - \rho_0}{\rho_0} + \frac{B}{2!} \left( \frac{\rho - \rho_0}{\rho_0} \right)^2 + \dots \end{aligned} \quad (ii)$$

where

$$\begin{aligned} A &= \rho_0 \left. \frac{dP}{d\rho} \right|_{\rho=\rho_0} = \rho_0 c_0^2 \\ B &= \rho_0^2 \left. \frac{d^2P}{d\rho^2} \right|_{\rho=\rho_0}, \end{aligned}$$

where the definition of the linear speed of sound  $c_0 \equiv (dP/d\rho)|_{\rho=\rho_0}$  has been used. (On the other hand  $c = dP/d\rho$  i.e., not evaluated at at equilibrium density, is the nonlinear speed of sound). Equation (ii) can be rearranged by writing  $P = p_0 + p$  and  $\rho - \rho_0 = \rho'$ , and by factoring out  $A$ , resulting in Blackstock's equation (C-44):

$$p = c_0^2 \rho' \left( 1 + \frac{B}{2!A} \frac{\rho'}{\rho_0} + \dots \right)$$

Thus the ratio  $B/A$  is seen to be the coefficient of the first nonlinear term of the isentropic equation of state.

14. Derive the adiabatic and isothermal sound speeds. For the adiabatic sound speed, use the ideal gas law  $P = \rho RT$  to write the sound speed in terms of the temperature. Who first calculated the adiabatic sound speed? Who first calculated the isothermal sound speed? Which came first? *Bonus: Show that  $c_0 = (\gamma/3)^{1/2} v_{rms}$ , where  $v_{rms}$  is the square root of*

the mean square velocity of the molecules in the gas. To do so, use the ideal gas law in terms of the Boltzmann constant  $k$  and number of molecules  $N$ ,  $PV = NkT$  to write the adiabatic sound speed. Note that  $\rho V$  is the mass of the gas, which is equal to  $Nm$ , where  $m$  is the molecular weight. Use the equipartition theorem  $kT = \frac{1}{3}m\langle v^2 \rangle$  to relate the speed of sound to the average speed of the molecules. [ANSWER]

Recall the adiabatic gas law,  $P = p_0(\rho/\rho_0)^\gamma$ . By the definition of linear sound speed,

$$\begin{aligned} c_0^2 &= \left. \frac{dP}{d\rho} \right|_{\rho=\rho_0} \\ &= \gamma \frac{p_0}{\rho_0} \left( \frac{\rho}{\rho_0} \right)^{\gamma-1} \Big|_{\rho=\rho_0} \\ &= \frac{\gamma p_0}{\rho_0} \\ \Rightarrow c_0 &= \sqrt{\frac{\gamma p_0}{\rho_0}} = \sqrt{\gamma RT_0}, \end{aligned}$$

where the second equality in the last line follows from the ideal gas law. This was shown by Laplace.

Next recall the isothermal gas law,  $p/p_0 = \rho/\rho_0$  and repeat the procedure above, giving

$$c_0 = \sqrt{p_0/\rho_0}.$$

One could equivalently set  $\gamma = 1$  in the adiabatic sound speed to recover the isothermal sound speed. Newton had calculated the isothermal sound speed more than a century prior to Laplace's calculation of the adiabatic sound speed.

*Bonus: Use the alternate form of the ideal gas law, in terms of number of molecules and the Boltzmann constant:*

$$\begin{aligned} c_0^2 &= \frac{\gamma P_0}{\rho_0} \\ &= \frac{\gamma P_0 V_0}{\rho_0 V_0} \\ &= \frac{\gamma N k T_0}{Nm} \end{aligned}$$

Noting that  $kT = \frac{1}{3}m\langle v^2 \rangle$ , the above becomes

$$\begin{aligned} c_0^2 &= \frac{\gamma Nm \langle v^2 \rangle}{3Nm} \\ &= \frac{\gamma}{3} \langle v^2 \rangle \end{aligned}$$

Take the square root and note that  $v_{\text{rms}} = \sqrt{\langle v^2 \rangle}$ . Thus

$$c_0 = (\gamma/3)^{1/2} v_{\text{rms}}$$

Where does  $kT = \frac{1}{3}m\langle v^2 \rangle$  come from? It comes from the equipartition theorem, which says that

each quadratic degree of freedom contributes  $kT/2$  to the energy. There are three degrees of freedom for a monatomic gas. Thus  $3kT/2 = \frac{1}{2}m\langle v^2 \rangle$ , which gives the result. From R. P. Feynman's lecture on the topic,

*The speed of sound is of the same order of magnitude as the speed of the molecules, and is actually somewhat less than this average speed. Of course we could expect such a result, because a disturbance like a change in pressure is, after all, propagated by the motion of the molecules.*

15. Derive the linear wave equation in 1D for an isentropic fluid. [ANSWER]

The continuity and momentum equations are linearized, and the linearized state relation  $\rho' = p/c_0^2$  is invoked to eliminate density, resulting in

$$\frac{1}{c_0^2} \frac{\partial p}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0 \quad (\text{i})$$

$$\frac{\partial p}{\partial x} + \rho_0 \frac{\partial u}{\partial t} = 0. \quad (\text{ii})$$

The time derivative of equation (i) and the spatial derivative of equation (ii) are taken, and the resulting equations are subtracted, giving the wave equation,

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} = 0.$$

16. Derive the plane wave impedance relation for linear 1D sound waves. How does the answer change as the direction of the sound reverses? [ANSWER]

Let  $p = f(x - c_0 t)$ . Then by the momentum equation  $\rho_0 \partial u / \partial t = -\partial p / \partial x$ ,

$$\begin{aligned} u &= -\frac{1}{\rho_0} \int f'(x - c_0 t) dt \\ &= -\frac{1}{\rho_0} \left( -\frac{1}{c_0} \right) f \\ &= \frac{1}{\rho_0 c_0} p \\ \Rightarrow \frac{p}{u} &= \rho_0 c_0 \end{aligned}$$

If the direction of the sound is reversed,  $p = f(x + c_0 t)$ , which results in  $p/u = -\rho_0 c_0$ .

17. Derive the spherical wave impedance relation for linear outgoing spherical sound. Write the impedance in polar and rectangular form. How does this impedance change for converging spherical waves? [ANSWER]

Consider the imaginary part of a solution of the spherically symmetric wave equation,

$$p = \frac{\text{Im} e^{j(\omega t - kr)}}{r}. \quad (\text{i})$$

Equation (i) is inserted into the radial component of the momentum equation, where  $\partial u / \partial t = j\omega u$

$$\begin{aligned}
 j\omega \rho_0 u &= -\frac{\partial p}{\partial r} \\
 u &= \frac{1}{j\omega \rho_0} \left( jk + \frac{1}{r} \right) p \\
 \Rightarrow \frac{p}{u} &= \frac{j\omega \rho_0}{jk + 1/r} \\
 \frac{p}{u} &= \frac{j\omega \rho_0 / jk}{1 + 1/jkr} \\
 Z = \frac{p}{u} &= \frac{\rho_0 c_0}{1 + 1/jkr}. \tag{ii}
 \end{aligned}$$

The rectangular and polar forms of equation (ii) are

$$\begin{aligned}
 Z &= \frac{\rho_0 c_0}{1 + 1/k^2 r^2} + j \frac{\rho_0 c_0 / kr}{1 + 1/k^2 r^2} \\
 &= \rho_0 c_0 \frac{kr}{\sqrt{1 + k^2 r^2}} e^{j \arccot kr}.
 \end{aligned}$$

For incoming spherical waves  $p = \frac{\text{Im}e^{j(\omega t + kr)}}{r}$ , equation (ii) obtains two additional negative signs:

$$\begin{aligned}
 j\omega \rho_0 u &= -\frac{\partial p}{\partial r} \\
 u &= \frac{1}{j\omega \rho_0} \left( -jk + \frac{1}{r} \right) p \\
 \Rightarrow \frac{p}{u} &= \frac{j\omega \rho_0}{-jk + 1/r} \\
 \frac{p}{u} &= \frac{-j\omega \rho_0 / jk}{1 - 1/jkr} \\
 \frac{p}{u} &= -\frac{\rho_0 c_0}{1 - 1/jkr}.
 \end{aligned}$$

18. Interpret the  $kr \ll 1$  and  $kr \gg 1$  limits of the spherical wave impedance relation. What limit represents efficient radiation? [ANSWER]

From the above problem, the rectangular form of equation (ii) is

$$Z = \frac{\rho_0 c_0}{1 + 1/k^2 r^2} + j \frac{\rho_0 c_0 / kr}{1 + 1/k^2 r^2},$$

from which it can be seen that for

$$Z = \begin{cases} \rho_0 c_0 & \text{for } kr \gg 1 \\ j\omega \rho_0 r & \text{for } kr \ll 1 \end{cases}$$

The  $kr \gg 1$  limit represents efficient radiation, as the impedance is purely real and recovers the plane wave impedance relation. Meanwhile the  $kr \ll 1$  limit is purely reactive, representing poor radiation.

19. Calculate the pressure field due to a pulsating sphere of radius  $a$ , where the particle velocity is  $u_0 e^{j\omega t}$ . [ANSWER]

We know the solution must have the form

$$p(r, t) = \frac{A}{r} \exp j(\omega t - kr). \quad (\text{i})$$

To determine the constant  $A$ , recall the spherical wave impedance relation,

$$Z(r = a) = \frac{p(r = a)}{u(r = a)} = \frac{\rho_0 c_0}{1 + 1/jka},$$

and match the boundary condition  $u_0 e^{j\omega t}$  at the radius  $a$  to the pressure at  $r = a$ :

$$p(r = a, t) = u_0 e^{j\omega t} \frac{\rho_0 c_0}{1 + 1/jka} \quad (\text{ii})$$

Setting equations (i) and (ii) equal gives

$$\begin{aligned} \frac{A}{a} \exp j(\omega t - ka) &= u_0 e^{j\omega t} \frac{\rho_0 c_0}{1 + 1/jka} \\ \Rightarrow A &= \frac{\rho_0 c_0 u_0 a}{1 + 1/jka} \exp jka \end{aligned} \quad (\text{iii})$$

The pressure field is therefore given by combining equations (i) and (iii):

$$p(r, t) = \frac{\rho_0 c_0 u_0}{1 + 1/jka} \frac{a}{r} \exp j[\omega t - k(r - a)].$$

20. Derive the expression for the instantaneous intensity  $\mathbf{I}$  of a linear sound wave. [ANSWER]

Two approaches are provided, the second being more formal than the first.

**Informal approach.** Denoting  $W = \text{power} = dw/dt$ , where  $w$  is work, the magnitude of the instantaneous intensity  $I$  is equal to the power  $W$  divided by the surface area  $S$

$$\begin{aligned} I &= W/S \\ &= \frac{dw}{dt} \frac{1}{S} \\ &= \frac{F dx}{S} \frac{1}{dt} \\ &= \frac{F}{S} \frac{dx}{dt} \\ &= pu \end{aligned}$$

This quick approach does not retain the vectorial nature of the intensity vector. The formal approach below is therefore preferred.

**Formal approach.** Define the power  $W$  to be the time derivative of the work  $w$  done by the sound wave:

$$W = \frac{dw}{dt}.$$

Note that the differential work done is  $dw = \mathbf{F} \cdot d\mathbf{s}$ . Thus

$$W = \frac{d}{dt} (\mathbf{F} \cdot d\mathbf{s}) = \frac{d\mathbf{F}}{dt} \cdot d\mathbf{s} + \mathbf{F} \cdot \frac{d\mathbf{s}}{dt}$$

The first term,  $\frac{d\mathbf{F}}{dt} \cdot d\mathbf{s}$ , is infinitesimally small. And  $\frac{d\mathbf{s}}{dt} = \mathbf{u}$ . Thus

$$W = \mathbf{F} \cdot \mathbf{u}$$

Next consider an area  $S$  with inward unit normal  $\mathbf{n}$ , and suppose that  $\mathbf{F} = F\mathbf{n}$ . Then, the power can be written as

$$W = F\mathbf{n} \cdot \mathbf{u} \frac{1}{S} \oint dS$$

Assuming  $F$  and  $u$  are constant along the surface, they can be moved inside the integral:

$$W = \frac{1}{S} \oint F\mathbf{n} \cdot \mathbf{u} dS = \oint (F\mathbf{n}/S) \cdot \mathbf{u} dS$$

Then identifying  $F\mathbf{n}/S$  to be the acoustic pressure times the normal vector  $p\mathbf{n}$  yields

$$W = \oint p\mathbf{u} \cdot \mathbf{n} dS$$

The integrand  $p\mathbf{u}$  is therefore identified to be the instantaneous intensity vector:

$$\mathbf{I} = p\mathbf{u}$$

21. Show that the time-averaged magnitude of the intensity of a plane wave is  $\langle I \rangle = p_{\text{rms}}^2 / \rho_0 c_0$ .  
[ANSWER]

From the previous problem, the time-averaged intensity is given by

$$\begin{aligned} \langle I \rangle &= \langle pu \rangle \\ &= \frac{1}{\rho_0 c_0} \langle p^2 \rangle \\ &= \frac{1}{\rho_0 c_0} \int_0^{t_{\text{av}}} p^2 dt \\ &= \frac{p_{\text{rms}}^2}{\rho_0 c_0} \end{aligned}$$

where the root-mean-square of the pressure is defined as

$$p_{\text{rms}} = \sqrt{\int_0^{t_{\text{av}}} p^2 dt}$$

22. Calculate the time-averaged intensity due to a spherically symmetric outgoing wave,  $p = \frac{A}{r} \exp j(\omega t - kr)$ . What about this result is often misquoted? [ANSWER]

The time-averaged intensity is given by

$$\begin{aligned}
 \langle I \rangle &= \langle pu \rangle \\
 &= \frac{1}{2} \operatorname{Re}(pu^*) && \text{(See problem 28)} \\
 &= \frac{1}{2\rho_0 c_0} \operatorname{Re}(pp^*(1 + 1/jkr)) \\
 &= \frac{1}{\rho_0 c_0} \frac{|p|^2}{2} \\
 &= \frac{p_{\text{rms}}^2}{\rho_0 c_0}
 \end{aligned}$$

23. Find  $p_{\text{rms}}$  for a sinusoidal wave and for an N-wave, both of amplitude  $A$ . [ANSWER]

By taking the integral  $p_{\text{rms}}^2 = \frac{1}{t_{\text{av}}} \int_0^{t_{\text{av}}} p^2 dt$ , one finds that  $p_{\text{rms}} = A/\sqrt{2}$  for a sinusoidal wave and  $p_{\text{rms}} = A/\sqrt{3}$  for N-waves.

24. State the definitions of the sound pressure level (SPL), intensity level (IL), and sound power level (PWL). What is the IL in a purely standing wave field? [ANSWER]

The definitions are

$$\begin{aligned}
 \text{SPL} &= 20 \log_{10}(p_{\text{rms}}/p_{\text{ref}}) \\
 \text{IL} &= 10 \log_{10}(\langle I \rangle/I_{\text{ref}}) \\
 \text{PWL} &= 10 \log_{10}(W/W_{\text{ref}}),
 \end{aligned}$$

where Blackstock writes " $I$ " for the time-averaged intensity vector instead of  $\langle I \rangle$ . SPL and IL are functions of distance from a sound source (because, e.g., sound pressure is proportional to  $1/r$  and intensity is proportional to  $1/r^2$  in a spherical wave) whereas PWL is the total power radiated by a sound source, wherever it goes and however it diverges. The IL in a purely standing wave field is  $-\infty$ , because  $I = 0$ .

It is a good thing to be able to show that  $\langle I \rangle = 0$  in a standing wave field. Consider the standing wave  $p(x) = A \cos(kx)$ . Then by the momentum equation,

$$u = \frac{jka}{\rho_0} \int \cos kx dt = \frac{jkA}{\rho_0} (t - \phi) \cos kx,$$

where  $\phi$  is a phase. Then the time-averaged intensity is given by (see problem 28 for the derivation of the first line below)

$$\begin{aligned}
 \langle I \rangle &= \frac{1}{2} \operatorname{Re}(pu^*) \\
 &= \frac{1}{2} \operatorname{Re}\left(j \frac{kA^2}{\rho} (t - \phi) \cos^2 kx\right) \\
 &= 0.
 \end{aligned}$$

25. ✳ Calculate the energy density of linear sound. What is the relationship between kinetic and potential energy densities in a progressive plane wave? State and derive the acoustic energy corollary. [ANSWER]

This is another hard one. See [here](#). I doubt we are responsible for these kinds of clever manipulations, but it's good to know the main result, that the kinetic and potential energy densities are equal in a progressive plane wave. The potential and kinetic energies being equal for progressive waves recovers a special case of the virial theorem.

26. What is the difference between acoustic, specific acoustic, and mechanical impedance? [ANSWER]

The definitions are

$$\begin{aligned} Z_{\text{sp ac}} &= \frac{p}{u} \\ Z_{\text{ac}} &= \frac{p}{q} = \frac{p}{Su} = Z_{\text{sp ac}}/S \\ Z_{\text{mech}} &= \frac{F}{u} = \frac{Sp}{u} = Z_{\text{sp ac}}S. \end{aligned}$$

27. For plane progressive waves in air, why is the intensity level (IL) roughly numerically equal to the sound pressure level (SPL)? [ANSWER]

The intensity level can be expressed as

$$\begin{aligned} \text{IL} &= 10 \log_{10} \frac{\langle I \rangle}{I_{\text{ref}}} \\ &= 10 \log_{10} \frac{p_{\text{rms}}^2 / \rho_0 c_0}{I_{\text{ref}}} \\ &= 10 \log_{10} \frac{p_{\text{rms}}^2}{p_{\text{ref}}^2} \frac{p_{\text{ref}}^2}{\rho_0 c_0 I_{\text{ref}}} \\ &= 10 \log_{10} \frac{p_{\text{rms}}^2}{p_{\text{ref}}^2} + 10 \log_{10} \frac{p_{\text{ref}}^2}{\rho_0 c_0 I_{\text{ref}}} \\ &= 20 \log_{10} \frac{p_{\text{rms}}}{p_{\text{ref}}} + 10 \log_{10} \frac{p_{\text{ref}}^2}{\rho_0 c_0 I_{\text{ref}}} \\ &\approx \text{SPL} \end{aligned}$$

It turns out that in air at STP the term  $10 \log_{10} \frac{p_{\text{ref}}^2}{\rho_0 c_0 I_{\text{ref}}}$  is small, like  $-0.16$

28. ✳ For two time-harmonic functions  $f$  and  $g$  represented by the real parts of the complex-valued functions  $\tilde{f}(t) = f_0 e^{j(\omega t + \phi_f)} = \tilde{f}_\omega e^{j\omega t}$  and  $\tilde{g}(t) = g_0 e^{j(\omega t + \phi_g)} = \tilde{g}_\omega e^{j\omega t}$  (where  $\tilde{f}_\omega = f_0 e^{j\phi_f}$  and  $\tilde{g}_\omega = g_0 e^{j\phi_g}$ ), show that the time average of their product,  $\langle fg \rangle$ , is given by  $\langle \text{Re}(\tilde{f}) \text{Re}(\tilde{g}) \rangle = \frac{1}{2} \text{Re}(\tilde{f}_\omega \tilde{g}_\omega^*) = \frac{1}{2} \text{Re}(\tilde{f}_\omega^* \tilde{g}_\omega)$ , where "Re" denotes "real part." [ANSWER]

Note that  $\text{Re}(\tilde{f}) = f_0 \cos(\omega t + \phi_f)$  and  $\text{Re}(\tilde{g}) = g_0 \cos(\omega t + \phi_g)$ . Thus

$$\begin{aligned} \langle \text{Re}(\tilde{f}) \text{Re}(\tilde{g}) \rangle &= \langle f_0 \cos(\omega t + \phi_f) g_0 \cos(\omega t + \phi_g) \rangle \\ &= f_0 g_0 \langle \cos(\omega t + \phi_f) \cos(\omega t + \phi_g) \rangle. \end{aligned} \quad (\text{i})$$

Since  $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$  and  $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ , Eq. (i) becomes (by letting  $A = \omega t + \phi_f$  and  $B = \omega t + \phi_g$ )

$$\langle \text{Re}(\tilde{f}) \text{Re}(\tilde{g}) \rangle = f_0 g_0 \langle \cos(2\omega t + \phi_f + \phi_g) + \sin(\omega t + \phi_f) \sin(\omega t + \phi_g) \rangle. \quad (\text{ii})$$

Noting that  $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ , Eq. (ii) becomes

$$\begin{aligned} \langle \text{Re}(\tilde{f}) \text{Re}(\tilde{g}) \rangle &= f_0 g_0 \langle \cos(2\omega t + \phi_f + \phi_g) - \frac{1}{2} \cos(2\omega t + \phi_f + \phi_g) + \frac{1}{2} \cos(\phi_f - \phi_g) \rangle \\ &= f_0 g_0 \langle \frac{1}{2} \cos(2\omega t + \phi_f + \phi_g) + \frac{1}{2} \cos(\phi_f - \phi_g) \rangle. \end{aligned} \quad (\text{iii})$$

The time-averaging operation amounts to an integral, which is a linear operation. Thus Eq. (iii) becomes

$$\langle \text{Re}(\tilde{f}) \text{Re}(\tilde{g}) \rangle = \frac{1}{2} f_0 g_0 \langle \cos(2\omega t + \phi_f + \phi_g) \rangle + \frac{1}{2} f_0 g_0 \langle \cos(\phi_f - \phi_g) \rangle.$$

The first term on the left-hand side is 0. Meanwhile, the second term does not depend on time, and therefore its time average is itself:

$$\langle \text{Re}(\tilde{f}) \text{Re}(\tilde{g}) \rangle = \frac{1}{2} f_0 g_0 \cos(\phi_f - \phi_g). \quad (\text{iv})$$

Noting that  $f_0 g_0 \cos(\phi_f - \phi_g)$  is  $\text{Re}[f_0 g_0 e^{i(\phi_f - \phi_g)}]$ , which by the relations  $\tilde{f}_\omega = f_0 e^{i\phi_f}$  and  $\tilde{g}_\omega = g_0 e^{i\phi_g}$  is  $\text{Re}(\tilde{f}_\omega \tilde{g}_\omega^*)$ , Eq. (iv) becomes

$$\langle f g \rangle = \langle \text{Re}(\tilde{f}) \text{Re}(\tilde{g}) \rangle = \frac{1}{2} \text{Re}(\tilde{f}_\omega \tilde{g}_\omega^*) = \frac{1}{2} \text{Re}(\tilde{f}_\omega^* \tilde{g}_\omega), \quad (\text{v})$$

where the final equality holds by noting that  $\cos(\phi_f - \phi_g) = \cos(\phi_g - \phi_f)$ .

Letting  $\text{Re}(f) = p$  and  $\text{Re}(g) = \mathbf{u}$  immediately leads to  $\langle I \rangle = \frac{1}{2} \text{Re}(p \mathbf{u}^*) = \frac{1}{2} \text{Re}(p^* \mathbf{u})$ .

29. Evaluate the integral for time-averaged intensity in *Fundamentals of Physical Acoustics*, Sec. 1E-3 (page 50), and thereby recover the result above. [ANSWER]

See [here](#) for the evaluation, noting that Blackstock suppresses the angle brackets to denote time average, i.e., his  $\mathbf{I}$  is our  $\langle \mathbf{I} \rangle$ .

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## Chapter 2: Detailed development of the acoustic wave equation

See *An Introduction to Thermal Physics* by Daniel Schroeder for more on thermodynamics.

1. To what particle does the "particle velocity"  $\mathbf{u}$  refer? [ANSWER]

"Particle velocity" does not refer to a single molecule. Rather, it refers to a "fluid particle" within the continuum approximation, "a large enough collection of molecules that the average of their random motions is zero," as Blackstock writes (see page 27-28).

2. Write the exact mass and momentum equations in integral form. [ANSWER]

The conservation of mass in integral form is

$$\frac{\partial}{\partial t} \int_V \rho dV + \oint_S \rho \mathbf{u} \cdot d\mathbf{S} = 0,$$

while the conservation of momentum in integral form is

$$\frac{\partial}{\partial t} \int_V \rho \mathbf{u} dV = \int_V \mathbf{B} \rho dV - \oint_S P d\mathbf{S} - \oint_S (\rho \mathbf{u}) \mathbf{u} \cdot d\mathbf{S}.$$

3. State the 0th, 1st, 2nd, and 3rd laws of thermodynamics. [ANSWER]

**Zeroth law:** the transitive property. If A is in thermal equilibrium with B, and B is in thermal equilibrium with C, then A is in thermal equilibrium with C.

**First law:** the conservation of energy. The change in internal energy is the heat plus the work done on the system:  $\Delta U = Q + W$ .

**Second law:** entropy. The entropy of the universe tends towards a maximum, i.e.,  $dS \geq \delta Q/T$ , where the equality is fulfilled by quasistatic processes.

**Third law:** absolute zero. The entropy approaches a constant in the limit that the temperature goes to absolute zero.

4. How many quadratic degrees of freedom  $f$  does helium have? Oxygen? Nitrogen? *Bonus: What are the seven diatomic gases?* [ANSWER]

Note that quadratic degrees of freedom include translational, rotational, and vibrational types of energies, which are quadratic in velocity, angular velocity, and displacement respectively. Helium is monatomic and therefore has only three translational degrees of freedom:  $f = 3$ . Oxygen and nitrogen are both diatomic and thus have three translational degrees of freedom and two rotational degrees of freedom:  $f = 5$ .

The seven diatomic gases can be remembered with the expression, "BrINCiHOF!", which indicates that bromine, iodine, nitrogen, chlorine, hydrogen, oxygen, and fluorine are diatomic

gases.

5. State the equipartition theorem. [ANSWER]

The equipartition theorem states that the average energy of any quadratic degree of freedom is  $\frac{1}{2}kT$ , where  $k$  is the Boltzmann constant. For example, a system of  $N$  molecules, each with  $f$  degrees of freedom, has a total thermal energy of  $U = NfkT/2$ .

Note that for three dimensional motion for a monatomic gas,  $U = 3kT/2$  by the equipartition theorem. As all the energy is kinetic,  $U = m\langle v^2 \rangle / 2$ , where  $\langle v^2 \rangle$  is the average of the square of the velocities. Therefore,  $\sqrt{\langle v^2 \rangle} = 3kT/m = v_{\text{rms}}$ .

6. Derive the adiabatic gas law. [ANSWER]

Start with the first law of thermodynamics, and note that the no heat is flows into or out of the gas in an adiabatic process:

$$\Delta U = Q + W = W.$$

By the equipartition theorem,

$$U = \frac{f}{2}NkT$$
$$\Rightarrow dU = \frac{f}{2}Nk dT.$$

Setting the above relation equal to the infinitesimal work done by the gas during a compression (because  $dU = dW$ ) results in

$$\frac{f}{2}Nk dT = -PdV.$$

Substitution of the ideal gas law  $P = NkT/V$  on the right-hand side and rearrangement gives

$$\frac{f}{2} \frac{dT}{T} = -\frac{dV}{V}.$$

Integration gives

$$\frac{f}{2} \ln T/T_0 = -\ln V/V_0$$
$$\ln[(T/T_0)^{f/2}] = -\ln V/V_0$$

Exponentiation and rearrangement gives

$$VT^{f/2} = V_0T_0^{f/2}.$$

Invoking the ideal gas law to eliminate temperature gives

$$P^{f/2}V^{\frac{f+2}{2}} = P_0^{f/2}V_0^{\frac{f+2}{2}}.$$

Raising both sides to the power of  $2/f$  gives

$$PV^{\frac{f+2}{f}} = P_0V_0^{\frac{f+2}{f}} .$$

Identifying  $\gamma \equiv (f + 2)/f$ , and noting that  $V \propto 1/\rho$ , gives the desired result:

$$\begin{aligned} P\rho_0^\gamma &= P_0^{\frac{f+2}{f}} \rho_0^\gamma \\ P/P_0 &= (\rho/\rho_0)^\gamma \end{aligned}$$

7. Calculate the work done in an adiabatic compression. [ANSWER]

For an adiabatic compression,  $P = P_0(V_0/V)^\gamma$ , as derived above. The work done on the gas is

$$\begin{aligned} W &= - \int_{V_0}^{V_1} P dV \\ &= - \int_{V_0}^{V_1} P_0 (V_0/V)^\gamma dV \\ &= -P_0 V_0^\gamma \int_{V_0}^{V_1} V^{-\gamma} dV \\ &= - \frac{P_0 V_0^\gamma}{1-\gamma} V^{1-\gamma} \Big|_{V_0}^{V_1} \\ &= - \frac{P_0 V_0^\gamma}{1-\gamma} (V_0^{1-\gamma} - V_1^{1-\gamma}) \\ &= \frac{P_0 V_0}{\gamma-1} [(V_1/V_0)^{1-\gamma} - 1] \\ &= \frac{P_0 V_0}{\gamma-1} [(V_0/V_1)^{\gamma-1} - 1] \end{aligned}$$

8. Is every adiabatic process isentropic? Is every isentropic process adiabatic? [ANSWER]

Every adiabatic process is *not* isentropic, but every isentropic process *is* adiabatic. This is because an isentropic process is defined to be a reversible adiabatic process.

Also note that every reversible process is quasistatic, but not every quasistatic process is reversible "if there is also heat flowing in or out or if entropy is being created in some other way" (Schroeder).

9. Use the entropy equation [see Blackstock equation (2-A-43), or the beginning of *Nonlinear Acoustics* by Hamilton and Blackstock, or *Fluid Mechanics* by Lifshitz and Landau],

$$\rho T \frac{\partial s}{\partial t} = \kappa \nabla^2 T + \text{miscellaneous terms} , \quad (1)$$

to show that low-frequency sound is isentropic, while high-frequency sound is isothermal.

[ANSWER]

The thermodynamic quantities in equation (1) are first expressed as ambient + perturbation quantities, i.e.,

$$\begin{aligned} T &= T_0 + T' \\ s &= s_0 + s, . \end{aligned}$$

Thus equation (1) becomes

$$\rho_0 T_0 \frac{\partial s'}{\partial t} \sim \kappa \nabla^2 T',$$

or, assuming time-harmonic solutions (i.e.,  $\partial/\partial t \mapsto j\omega$  and  $\nabla^2 \mapsto k^2 = \omega^2/c^2$ ),

$$j\omega \rho_0 T_0 s' \sim \kappa \frac{\omega^2}{c_0^2} T'.$$

Rearranging this result gives

$$\rho_0 c_0^2 \frac{T_0}{\kappa} \frac{s'}{T'} \sim \omega,$$

from which it can be seen that

$$\begin{aligned} s' &\rightarrow 0 & \text{as } \omega &\rightarrow 0, & \text{(Isentropic)} \\ T' &\rightarrow 0 & \text{as } \omega &\rightarrow \infty, & \text{(Isothermal).} \end{aligned}$$

Feynman provides some insightful commentary:

Newton was the first to calculate the rate of change of pressure with density, and he supposed that the temperature remained unchanged. He argued that the heat was conducted from one region to the other so rapidly that the temperature could not rise or fall. This argument gives the isothermal speed of sound, and it is wrong. The correct deduction was given later by Laplace, who put forward the opposite idea—that the pressure and temperature change adiabatically in a sound wave. The heat flow from the compressed region to the rarefied region is negligible so long as the wavelength is long compared with the mean free path. Under this condition the slight amount of heat flow in a sound wave does not affect the speed, although it gives a small absorption of the sound energy. We can expect correctly that this absorption increases as the wavelength approaches the mean free path, but these wavelengths are smaller by factors of about a million than the wavelengths of audible sound.

10. What is the meaning of  $D/Dt$ ? [ANSWER]

This is the material derivative, defined by

$$\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla.$$

11. ✪ Explain the meaning of each term in

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \quad (\text{i})$$

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla P = (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u} \quad (\text{ii})$$

$$\rho C_v \frac{DT}{Dt} + P \nabla \cdot \mathbf{u} = \Phi^{(\text{visc})} + \kappa \nabla^2 T \quad (\text{iii})$$

$$P = R\rho T, \quad (\text{iv})$$

which correspond to Blackstock's eqs. A-47–A-49. [\[ANSWER\]](#)

All four of the equations above are exact, i.e.,  $\rho$ ,  $P$ , and  $T$  are the exact density, pressure, and temperature. Equation (i) is the continuity equation. Abandoning the material derivative, it is written

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

Equation (ii) is the momentum equation.  $\lambda$  is the dilatational viscosity coefficient, and  $\mu$  is the shear viscosity coefficient. (Sometimes the two are combined as  $\nabla + 2\mu = \tilde{V}$ , where  $\tilde{V}$  is the viscosity number). Abandoning the material derivative, this equation becomes

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla P = (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u}$$

Equation (iii) is the energy equation.  $C_v$  is the heat capacity at constant volume.  $\Phi^{(\text{visc})}$  is the nonlinear viscous dissipation function, and  $\kappa$  is the thermal conduction coefficient. Equation (iv) is the ideal gas law.  $R$  is the gas constant, given by  $C_p - C_v$ . Since  $\gamma = C_p/C_v$ , the gas constant can also be written as

$$R = (\gamma - 1)C_v.$$

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## Chapter 3: Reflection and transmission for normal incidence

1. Derive the reflection and transmission pressure coefficients for normal incidence of plane waves on a boundary. Check the results by assessing the limits. *[ANSWER]*

On one side of the boundary (impedance  $Z_1$ ) there is a pressure of  $p_i + p_r$ . On the other side of the boundary (impedance  $Z_2$ ) there is a pressure  $p_t$ . Begin by noting that there cannot be any force at the boundary. That is,  $p_i + p_r = p_t$ , or dividing by  $p_i$ ,

$$1 + R = T. \quad (\text{i})$$

Also note that the particle velocity must be continuous at the boundary:  $u_i + u_r = u_t$ . Using the plane wave impedance relation  $u = p/Z$ , the above relation becomes

$$\begin{aligned} \frac{p_i}{Z_1} - \frac{p_r}{Z_1} &= \frac{p_t}{Z_2} \\ p_i - p_r &= \frac{Z_1}{Z_2} p_t \end{aligned}$$

Dividing the above equation by  $p_i$  gives

$$1 - R = \frac{Z_1}{Z_2} T \quad (\text{ii})$$

Combining equations (i) and (ii) yields

$$T = \frac{2}{1 + Z_1/Z_2}. \quad (\text{iii})$$

Combining equation (iii) with equation (i) gives

$$R = \frac{1 - Z_1/Z_2}{1 + Z_1/Z_2}. \quad (\text{iv})$$

Equations (iii) and (iv) make sense because they match the following limits:

When  $Z_1/Z_2 \rightarrow 0$ , the incident wave experiences a rigid boundary, and an in-phase reflection and pressure doubling at boundary is expected. Indeed, equation (iv) goes to 1 while equation (iii) goes to 2.

When  $Z_1/Z_2 \rightarrow \infty$  the incident wave experiences a pressure-release boundary, and an out-of-phase reflection and zero transmission at boundary is expected. Indeed, equation (iv) goes to  $-1$  while equation (iii) goes to 0.

Finally, when  $Z_1/Z_2 = 1$ , the incident wave does not experience a boundary, and no reflection and perfect transmission is expected. Indeed, equation (iv) goes to 0 and equation (iii) goes to 1.

2. Derive the power reflection and transmission coefficients. How does the result relate to the first law of thermodynamics. [ANSWER]

Recall that the power is given by  $W = \oint \mathbf{I} \cdot d\mathbf{S}$ , which for a plane wave simply reads  $p_{\text{rms}}^2 S/Z$ . What is curious about this result is that the power in a plane wave is infinite, because the pressure extends over the infinite surface, and the integral of a constant over all space diverges. However, never mind this curiosity; it is resolved because the *ratio* of powers is taken, and thus the surface area cancels. The power reflection coefficient is

$$r = \frac{W_r}{W_i} = \frac{p_{r,\text{rms}}^2 S/Z_1}{p_{i,\text{rms}}^2 S/Z_1}$$

Critically, note that the ratio of the rms pressures equals the ratio of the pressures, because the reflected wave has the same waveform as the incident wave *in linear acoustics*. The reflection coefficient is therefore simply

$$r = \frac{p_r^2}{p_i^2} = R^2 .$$

Meanwhile, the transmission coefficient is

$$\begin{aligned} \tau &= \frac{W_t}{W_i} \\ &= \frac{p_{t,\text{rms}}^2 S/Z_2}{p_{i,\text{rms}}^2 S/Z_1} \\ &= \frac{p_t^2/Z_2}{p_i^2/Z_1} \\ &= T^2 \frac{Z_1}{Z_2} . \end{aligned}$$

Evidently, since

$$\begin{aligned} R^2 &= \frac{1 + (Z_1/Z_2)^2 - 2Z_1/Z_2}{1 + (Z_1/Z_2)^2 + 2Z_1/Z_2} \\ T^2 &= \frac{4}{1 + (Z_1/Z_2)^2 + 2Z_1/Z_2} \end{aligned}$$

the conservation of power is recovered, i.e.,

$$\begin{aligned} r + \tau &= R^2 + T^2 \frac{Z_1}{Z_2} \\ &= \frac{1 + (Z_1/Z_2)^2 - 2Z_1/Z_2}{1 + (Z_1/Z_2)^2 + 2Z_1/Z_2} + \frac{4Z_1/Z_2}{1 + (Z_1/Z_2)^2 + 2Z_1/Z_2} \\ &= \frac{1 + (Z_1/Z_2)^2 + 2Z_1/Z_2}{1 + (Z_1/Z_2)^2 + 2Z_1/Z_2} \\ &= 1 , \end{aligned}$$

and the result obeys the first law of thermodynamics.

3. Derive  $R$  and  $T$  due to change in cross-sectional surface area. [ANSWER]

If there is a change in surface area in a medium, then  $1 + R = T$  as before (force-free surface), but now the *volume velocity*  $q$  must be conserved at the junction,

$$\begin{aligned}q_i + q_r &= q_t \\S_1 u_i + S_1 u_r &= S_2 u_t,\end{aligned}$$

where the definition of volume velocity has been used to obtain the second line above. Invoking the plane wave impedance relation gives

$$\begin{aligned}\frac{S_1}{Z} p_i - \frac{S_1}{Z} p_r &= \frac{S_2}{Z} p_t \\p_i - p_r &= \frac{S_2}{S_1} p_t \\1 - R &= \frac{S_2}{S_1} T\end{aligned}$$

Combining the above with  $1 + R = T$  gives

$$\begin{aligned}T &= \frac{2S_1}{S_1 + S_2} \\&= \frac{2}{1 + S_2/S_1}\end{aligned}$$

and

$$R = \frac{1 - S_2/S_1}{1 + S_2/S_1}.$$

Again a physical interpretation is provided:

For the case that the boundary is replaced by a rigid wall,  $S_2/S_1 \rightarrow 0$ . Appropriately,  $R \rightarrow 1$  and  $T \rightarrow 2$ .

For the case that the boundary is replaced by a pressure release surface,  $S_2/S_1 \rightarrow \infty$ . Appropriately,  $R \rightarrow -1$  and  $T \rightarrow 0$ .

Finally, for the degenerate case of no boundary,  $S_2/S_1 \rightarrow 1$ . Appropriately,  $R \rightarrow 0$  and  $T \rightarrow 1$ .

4. What is an implicit assumption about the expressions for  $R$  and  $T$  due to change in cross-sectional surface area? [ANSWER]

An implicit assumption is that the wavelength is much longer than the characteristic length scale of the change in area. This is because the acoustic wave must maintain its plane-wave nature at the junction for the simple derivation above to hold. For example, if the change in surface area is due to sharp corners at the junction of two pipes of two different cross sectional areas, then the scattering off these corners must be neglected; the scattering is minimal for the case when the corners are of a dimension much smaller than a wavelength.

5. Derive  $R$  and  $T$  due to change in both surface area and impedance. [ANSWER]

If there is a change in both surface area *and* impedance, then  $1 + R = T$  and

$$\begin{aligned} q_i + q_r &= q_t \\ S_1 u_i + S_1 u_r &= S_2 u_t, \end{aligned}$$

where the definition of volume velocity has been used to obtain the second line above. Invoking the plane wave impedance relation gives

$$\begin{aligned} \frac{S_1}{Z_1} p_i - \frac{S_1}{Z_1} p_r &= \frac{S_2}{Z_2} p_t \\ p_i - p_r &= \frac{S_2 Z_1}{S_1 Z_2} p_t \\ 1 - R &= \frac{S_2 Z_1}{S_1 Z_2} T \end{aligned}$$

Combining the above with  $1 + R = T$  gives

$$\begin{aligned} T &= \frac{2S_1 Z_2}{S_1 Z_2 + S_2 Z_1} \\ &= \frac{2}{1 + Z_1 S_2 / Z_2 S_1} \end{aligned}$$

and

$$R = \frac{1 - Z_1 S_2 / Z_2 S_1}{1 + Z_1 S_2 / Z_2 S_1}.$$

The above results can easily be recast using the notion of *acoustic impedance*, which is defined as  $Z_{ac} = Z_{sp, ac} / S$ :

$$\begin{aligned} T &= \frac{2}{1 + Z_{ac,1} / Z_{ac,2}} \\ R &= \frac{1 - Z_{ac,1} / Z_{ac,2}}{1 + Z_{ac,1} / Z_{ac,2}}. \end{aligned}$$

6. Derive the power reflection and transmission coefficients for the general case in problem (5).

[ANSWER]

The setup is much the same as in problem (3), only this time accounting for the change in surface area as well as the change in impedance at the boundary:

$$r = \frac{p_r^2}{p_i^2} = R^2.$$

Meanwhile, the transmission coefficient is

$$\begin{aligned}
\tau &= \frac{W_t}{W_i} \\
&= \frac{p_{t,\text{rms}}^2 S_2 / Z_2}{p_{i,\text{rms}}^2 S_1 / Z_1} \\
&= \frac{p_t^2 / Z_{\text{ac},2}}{p_i^2 / Z_{\text{ac},1}} \\
&= T^2 \frac{Z_{\text{ac},1}}{Z_{\text{ac},2}}.
\end{aligned}$$

You can check for yourself that power is conserved, but the algebra is identical to that of problem (3).

7. Calculate pressure in a semi-infinite shock tube with a rigid termination at  $x = 0$  due to a finite step shock. The initial conditions are

$$\begin{aligned}
p(x, 0) &= A[H(x) - H(x - L)] \\
u(x, 0) &= 0
\end{aligned}$$

and the boundary condition is  $u(0, t) = 0$ . [ANSWER]

The general pressure solution is

$$p(x, t) = f(x - c_0 t) + g(x + c_0 t).$$

Using the plane wave impedance relation, the corresponding particle velocity is

$$u(x, t) = \frac{1}{\rho_0 c_0} [f(x - c_0 t) - g(x + c_0 t)].$$

Matching the initial velocity condition gives

$$u(x, 0) = \frac{1}{\rho_0 c_0} [f(x) - g(x)] = 0$$

which means that  $f(x) = g(x)$ . Thus the pressure and particle velocity solution reads

$$\begin{aligned}
p(x, t) &= f(x - c_0 t) + f(x + c_0 t) \\
u(x, t) &= \frac{1}{\rho_0 c_0} [f(x - c_0 t) - f(x + c_0 t)].
\end{aligned}$$

Next, matching the initial pressure condition  $p(x, 0) = A[H(x) - H(x - L)]$  gives

$$p(x, 0) = f(x) = \frac{A}{2} [H(x) - H(x - L)].$$

The pressure and particle velocity solution then reads

$$\begin{aligned}
p(x, t) &= \frac{A}{2} [H(x - c_0 t) - H(x - L - c_0 t) + H(x + c_0 t) - H(x - L + c_0 t)] \\
u(x, t) &= \frac{A}{2\rho_0 c_0} [H(x - c_0 t) - H(x - L - c_0 t) - H(x + c_0 t) + H(x - L + c_0 t)],
\end{aligned}$$

or, in terms of the "rect" function (see footnote on page 10 of Blackstock for the definition),

$$p(x, t) = \frac{A}{2} \left[ \text{rect} \left( \frac{x - c_0 t}{2L} \right) + \text{rect} \left( \frac{x + c_0 t}{2L} \right) \right]$$

$$u(x, t) = \frac{A}{2\rho_0 c_0} \left[ \text{rect} \left( \frac{x - c_0 t}{2L} \right) - \text{rect} \left( \frac{x + c_0 t}{2L} \right) \right].$$

The solution for negative  $x$  is disregarded. Note that the above particle velocity solution satisfies the boundary condition  $u(0, t) = 0$ . I don't know Dr. Blackstock makes a fuss about satisfying the boundary condition on page 120 ("But this is not a sufficient definition of  $f$  to satisfy the last condition...").

8. What is the volume velocity at the center of a spherically converging wave? [ANSWER]

The volume velocity vanishes at the center of a spherical wave. A spherically converging wave is spherically symmetric, and therefore the particle velocities (which only have a radial component) coming from all directions towards the focus cancel each other out at the focus.

9. ✳ Calculate pressure in bursting balloon due to a finite step shock set off by the initial conditions

$$p(r, 0) = A[H(r) - H(r - r_0)] \quad (\text{i})$$

$$u(r, 0) = 0. \quad (\text{ii})$$

Note that the volume velocity  $q = Su$  must vanish at  $r = 0$  (see previous problem for explanation):

$$\lim_{r \rightarrow 0} q = \lim_{r \rightarrow 0} Su = \lim_{r \rightarrow 0} 4\pi r^2 u = 0. \quad (\text{iii})$$

*Outline: Use the velocity potential  $\phi$ , recalling that  $p = -\rho_0 \phi_t$  and  $u = \phi_r$ . Match the initial velocity condition and conclude that  $g(r) = -f(r)$ . Apply this result  $p(r, t)$  and  $u(r, t)$ , calculate  $q(r, t) = \pi r^2 u(r, t)$ , and apply the limit above to show that  $f$  is even and  $f'$  is odd. Then apply the initial pressure condition to  $p(r, t)$ , solve for  $f'(t)$ , replace  $r$  with  $r \pm c_0 t$ . Substitute the resulting  $f'(r \pm c_0 t)$  into  $p(r, t)$ . [ANSWER]*

Since the sound obeys the spherically symmetric wave equation, the velocity potential is of the form

$$\phi = \frac{f(r - c_0 t)}{r} + \frac{g(r + c_0 t)}{r}.$$

The pressure is therefore

$$p(r, t) = -\rho_0 \phi_t = \rho_0 c_0 \frac{f'(r - c_0 t) - g'(r + c_0 t)}{r}, \quad (\text{iv})$$

and the particle velocity is

$$u(r, t) = \phi_r = -\frac{f(r - c_0 t) + g(r + c_0 t)}{r^2} + \frac{f'(r - c_0 t) + g'(r + c_0 t)}{r}. \quad (\text{v})$$

Applying the initial condition given by equation (ii) on equation (v) gives

$$\frac{f(r) + g(r)}{r^2} = \frac{f'(r) + g'(r)}{r}$$

This equality is guaranteed if  $g(r) = -f(r)$ , because this implies that  $g'(r) = -f'(r)$  (though the converse is not necessarily true). Therefore, equation (iv) becomes

$$p(r, t) = \rho_0 c_0 \frac{f'(r - c_0 t) + f'(r + c_0 t)}{r}, \quad (\text{vi})$$

and equation (v) becomes

$$u(r, t) = -\frac{f(r - c_0 t) - f(r + c_0 t)}{r^2} + \frac{f'(r - c_0 t) - f'(r + c_0 t)}{r}.$$

The volume velocity is therefore

$$\begin{aligned} q &= Su = 4\pi r^2 u \\ &= -4\pi [f(r - c_0 t) - f(r + c_0 t)] + 4\pi r [f'(r - c_0 t) - f'(r + c_0 t)]. \end{aligned} \quad (\text{vii})$$

The condition given by equation (iii) is applied to equation (vii):

$$\begin{aligned} \lim_{r \rightarrow 0} q &= -4\pi [f(-c_0 t) - f(c_0 t)] = 0 \\ \Rightarrow f(-c_0 t) &= f(c_0 t). \end{aligned} \quad (\text{viii})$$

Taking the derivative of equation (viii) gives

$$-f'(-c_0 t) = f'(c_0 t), \quad (\text{ix})$$

i.e., that  $f'$  is **odd**.

Meanwhile, the initial condition given by equation (i) is applied to equation (vi):

$$A[H(r) - H(r - r_0)] = 2\rho_0 c_0 \frac{f'(r)}{r}$$

Solving the above for  $f'(r)$  gives

$$f'(r) = \frac{rA[H(r) - H(r - r_0)]}{2\rho_0 c_0} \quad (\text{x})$$

Enforcing equation (ix) (the oddness of  $f'$ ) on equation (x) requires that  $f'$  is defined for  $-r$  as well as  $+r$ . This can be achieved using the rectangle function, where  $\text{rect}\left(\frac{x-x_0}{w}\right) = H(x - x_0 + w/2) - H(x - x_0 - w/2)$ :

$$f'(r) = \frac{rA}{2\rho_0 c_0} \text{rect}\left(\frac{r}{2r_0}\right)$$

Therefore,

$$f'(r \pm c_0 t) = \frac{A}{2\rho_0 c_0} (r \pm c_0 t) \operatorname{rect}\left(\frac{r \pm c_0 t}{2r_0}\right) \quad (\text{xi})$$

Substituting equation (xi) into equation (vi) gives the solution:

$$p(r, t) = \frac{A}{2r} \left[ (r - c_0 t) \operatorname{rect}\left(\frac{r - c_0 t}{2r_0}\right) + (r + c_0 t) \operatorname{rect}\left(\frac{r + c_0 t}{2r_0}\right) \right]$$

10. ✳ Consider a sphere of radius  $r_0$ . At  $r = r_0$ , the incident pressure wave is given by  $p_{\text{in}}(t)$ . The pressure solution is therefore of the form

$$p = \frac{r_0}{r} p_{\text{in}}(t + r/c_0) + \frac{F(t - r/c_0)}{r}, \quad (\text{i})$$

where  $F(t - r/c_0)/r$  corresponds to the wave emerging through the focus. Determine  $F(t - r/c_0)$  in terms of  $p_{\text{in}}$ . What is the pressure at the center of a sphere? *Outline: Given equation (i), find  $u$ , where  $\tilde{F}$  is the antiderivative of  $F$  and  $\tilde{p}$  is the antiderivative of  $p$ . Take the limit as  $r \rightarrow 0$ , for which the volume velocity  $4\pi r^2 u = 0$ . Solve for  $\tilde{F}$  and obtain  $F(t) = -r_0 p_{\text{in}}(t)$ . [ANSWER]*

First, apply the momentum equation for a spherical wave,  $\rho_0 \dot{u} = -p_r$ , to equation (i):

$$\rho_0 \frac{\partial u}{\partial t} = \frac{F'(t - r/c_0) - r_0 p'_{\text{in}}(t + r/c_0)}{c_0 r} + \frac{F(t - r/c_0) + r_0 p_{\text{in}}(t + r/c_0)}{r^2}$$

Solving the above for  $u$  by integration over time gives

$$\begin{aligned} u &= -\frac{1}{\rho_0} \int \frac{\partial p}{\partial t} dt \\ &= \frac{F(t - r/c_0) - r_0 p_{\text{in}}(t + r/c_0)}{\rho_0 c_0 r} + \frac{\tilde{F}(t - r/c_0) + r_0 \tilde{p}_{\text{in}}(t + r/c_0)}{\rho_0 r^2}, \end{aligned} \quad (\text{ii})$$

where  $\tilde{p}$  is the antiderivative of  $p$ , and  $\tilde{F}$  is the antiderivative of  $F$ . When the boundary condition  $\lim_{r \rightarrow 0} q = \lim_{r \rightarrow 0} 4\pi r^2 u = 0$  is applied to equation (ii), the first term of equation (ii) vanishes, and the second term gives

$$\frac{4\pi}{\rho_0} [\tilde{F}(t) + r_0 \tilde{p}_{\text{in}}(t)] = 0.$$

Solving the above for  $\tilde{F}(t)$  gives

$$\tilde{F}(t) = -r_0 \tilde{p}_{\text{in}}(t) \quad \Rightarrow \quad F(t) = -r_0 p_{\text{in}}(t).$$

Substituting  $F(t) = -r_0 p_{\text{in}}(t)$  into equation (i) gives the solution

$$p = \frac{r_0}{r} p_{\text{in}}(t + r/c_0) - \frac{r_0}{r} p_{\text{in}}(t - r/c_0), \quad (\text{iii})$$

The first term corresponds to the incoming wave, and the second term corresponds to the outgoing wave. *Note that the outgoing wave is out of phase with respect to the incoming wave.*

What happens at  $r = 0$  (the focus)? The limit of equation (iii) is taken:

$$\begin{aligned}
\lim_{r \rightarrow 0} p &= \lim_{r \rightarrow 0} \frac{r_0}{r} \left[ p_{\text{in}}(t + r/c_0) - p_{\text{in}}(t - r/c_0) \right] \\
&= \lim_{r \rightarrow 0} \frac{r_0}{r} \left[ p_{\text{in}}(t) + \frac{r}{c_0} p'_{\text{in}}(t) - p_{\text{in}}(t) + \frac{r}{c_0} p'_{\text{in}}(t) \right] \\
&= \frac{2r_0}{c_0} p'_{\text{in}}(t)
\end{aligned}$$

In the second equality above, the function is Taylor expanded to first order, and the higher-order terms are dropped. The conclusion is that the pressure at the center of the sphere is proportional to the time derivative of the incident pressure.

$$p(r = 0, t) = \frac{2r_0}{c_0} p'_{\text{in}}(t)$$

11. Given the previous result, what happens to the phase at the focal point of a focused sound beam?

[ANSWER]

The phase undergoes a 180° inversion.

12. ✳ Derive the reflection and transmission coefficients  $R$  and  $T$  for the three-medium problem, where the three media are labelled I, II, and III, as described on p. 168 of *Fundamentals of Physical Acoustics*. Assume that  $P_{\text{I}} = A_1 e^{-jk_1 x} + B_1 e^{jk_1 x}$ ,  $P_{\text{II}} = A_2 e^{-jk_2 x} + B_2 e^{jk_2 x}$ , and  $P_{\text{III}} = A_3 e^{-jk_3(x-l)}$  and apply the boundary conditions at the two interfaces. [ANSWER]

The problem is worked out [here](#).

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## Chapter 4: Impedance tubes and $k\ell \ll 1$ impedances

1. ✳ How is the impedance  $Z_n$  of an unknown material measured in an impedance tube, where the impedance of the medium (e.g., air) is  $Z_0$ ? [ANSWER]

Start with the general solution  $P(x)$  and  $U(x)$  of the 1D Helmholtz equation for plane traveling waves. The general solution for pressure is

$$P = Ae^{-jkx} + Be^{jkx},$$

and division by  $Z_0$ , the impedance of the medium, gives the particle velocity:

$$U = \frac{A}{Z_0}e^{-jkx} - \frac{B}{Z_0}e^{jkx}.$$

Define  $x = l - d$ , where  $x = 0$  corresponds to the location of the source, and  $d = 0$  corresponds to the location of the load, i.e.,  $d$  is a coordinate pointing in the opposite direction as  $x$  and offset by a distance  $l$ . The pressure and particle velocity become

$$\begin{aligned} P &= Ae^{-jkl}e^{jkd} + Be^{jkl}e^{-jkd} \\ U &= \frac{A}{Z_0}e^{-jkl}e^{jkd} - \frac{B}{Z_0}e^{jkl}e^{-jkd} \end{aligned}$$

Call  $P_i \equiv Ae^{-jkl}$  and  $P_r \equiv Be^{jkl}$ . Then the above becomes

$$\begin{aligned} P &= P_i e^{jkd} + P_r e^{-jkd} \\ &= P_i e^{jkd} (1 + R e^{-2jkd}). \\ U &= \frac{P_i}{Z_0} e^{jkd} - \frac{P_r}{Z_0} e^{-jkd} \\ &= \frac{P_i}{Z_0} e^{jkd} (1 - R e^{-2jkd}) \end{aligned}$$

Note from the above that  $P(d = 0) = P_i(1 + R)$ , and that  $U(d = 0) = \frac{P_i}{Z_0}(1 - R)$ . Calculate  $P(d = 0)/U(d = 0)$  to find  $Z_n$ , the impedance of the load, in terms of the reflection coefficient  $R$ :

$$Z_n = Z_0 \frac{1 + R}{1 - R}$$

If  $R$  (both magnitude and phase) can be found, then the complex impedance  $Z_n$  of the unknown load can be found.

2. How is the *magnitude*  $|R|$  of the reflection coefficient  $R$  found? [ANSWER]

$|R|$  is found by measuring the "standing wave ratio," or SWR. The standing wave ratio is the ratio  $|P_{\max}|/|P_{\min}|$ , where  $P_{\max} = P_i(1 + R)$  is the maximum pressure magnitude and where  $P_{\min} = P_i(1 - R)$  the minimum pressure magnitude. Therefore,

$$\text{SWR} = \frac{1 + |R|}{1 - |R|}.$$

This relation is inverted for  $|R|$ :

$$|R| = \frac{\text{SWR} - 1}{\text{SWR} + 1}.$$

3. How is the *phase*  $\psi$  of  $R = |R|e^{j\psi}$  determined? [ANSWER]

Take the magnitude of  $P = P_i e^{jkd} (1 - R e^{-2jkd})$  by writing  $R = \rho e^{j\psi}$ :

$$\begin{aligned} |P| &= P_i \sqrt{(1 + \rho e^{j(\psi-2kd)})(1 + \rho e^{-j(\psi-2kd)})} \\ &= P_i \sqrt{1 + \rho^2 + \rho[e^{-j(\psi-2kd)} + e^{j(\psi-2kd)}]} \\ &= P_i \sqrt{1 + \rho^2 + 2\rho \cos(\psi - 2kd)}. \end{aligned}$$

Apparently,  $|P|$  attains a maximum for  $\cos(\psi - 2kd) = 1$ , or

$$(kd)_{\max} = \frac{\psi}{2},$$

and a minimum for  $\cos(\psi - 2kd) = 0$ , or

$$(kd)_{\min} = \frac{\psi}{2} \pm \frac{\pi}{2}.$$

Thus the phase of reflection coefficient is found by finding the distance  $d$  to either the maximum or the minimum in the pressure field.

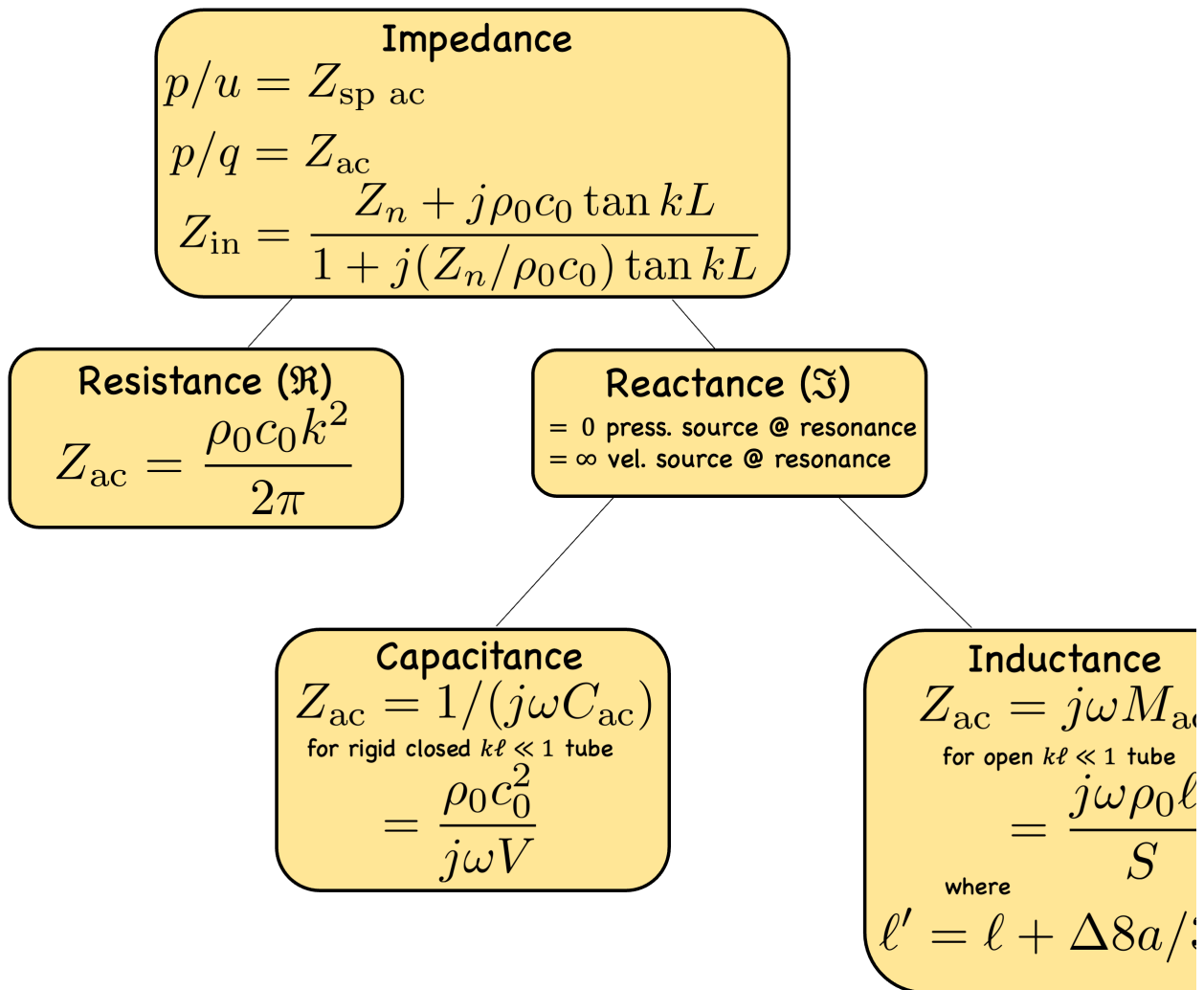
4. How is the attenuation coefficient  $\alpha$  measured? [ANSWER]

By definition,  $\alpha$  is the ratio of absorbed to incident power. Note that by energy conservation, the incident power is the sum of reflected power and the absorbed power. Thus,

$$\alpha = \frac{W_{\text{absorbed}}}{W_{\text{incident}}} = \frac{W_{\text{incident}} - W_{\text{reflected}}}{W_{\text{incident}}} = 1 - \frac{W_{\text{reflected}}}{W_{\text{incident}}} = 1 - |R|^2.$$

5. What is the relationship between impedance, reactance, resistance, inductance, and capacitance?

[ANSWER]



6. For a pressure source, the impedance at the resonance frequency goes to what quantity? *[ANSWER]*

For a pressure source, the impedance goes to 0 for resonance. A small amount of effort causes a huge flow, and impedance is effort/flow.

7. For a velocity source, the impedance at the resonance frequency goes to what quantity? *[ANSWER]*

For a velocity source, the impedance goes to  $\infty$  for resonance. A small amount of flow at the source results in a huge pressure field.

8. Derive  $Z_{ac}$  for mass-like and spring-like behaviours, appealing to Newton's law of motion and Hooke's constitutive relation. *[ANSWER]*

Recall that the acoustic impedance is  $Z_{ac} = p/q$ , where  $q = Su$  is the volume velocity. Therefore, for mass-like behaviour,

$$Z_{ac} = \frac{p}{Su} = \frac{f/S}{Su} = \frac{M\dot{u}}{S^2 u} = \frac{j\omega M}{S^2} = j\omega M_{ac}.$$

Meanwhile, for spring-like behaviour,

$$Z_{ac} = \frac{p}{Su} = \frac{f/S}{S\dot{x}} = \frac{Kx}{S^2 j\omega x} = \frac{K}{S^2 j\omega} = \frac{K_{ac}}{j\omega} = 1/j\omega C_{ac},$$

where  $M_{ac}$  is the acoustic mass,  $K_{ac}$  is the acoustic stiffness, and  $C_{ac}$  is the acoustic compliance.

9. In question (1) of this section,  $Z_n$  was calculated by evaluating  $P/U$  at  $d = 0$  (or  $x = L$  which corresponds to the location of the load). Now calculate the input impedance  $Z_{in}$ , which is  $P/U$  at  $d = L$  (or  $x = 0$ ) in terms of  $Z_n$ , the impedance of the load. Let  $P(x) = A \cos kx + B \sin kx$ .

[ANSWER]

Using the momentum equation, the particle velocity is found:

$$U(x) = -\frac{1}{j\omega\rho_0} \frac{dP}{dx} = \frac{A}{j\rho_0 c_0} \sin kx - \frac{B}{j\rho_0 c_0} \cos kx.$$

*Why must the momentum equation must be used here, while the impedance relation could be used in problem (1)? It's because these are standing waves (left-going and right-going components are already baked in), while those were traveling waves (which superpose to give standing waves). The impedance relation  $p/u = Z$  applies to what type of wave: standing or traveling?*

The coefficients  $A$  and  $B$  are found by applying the boundary conditions. At  $x = L$ ,

$$\begin{aligned} Z_n &= \frac{P(L)}{U(L)} = j\rho_0 c_0 \frac{A \cos kL + B \sin kL}{A \sin kL - B \cos kL} \\ &= j\rho_0 c_0 \frac{A/B + \tan kL}{A/B \tan kL - 1} \\ \Rightarrow \frac{A}{B} &= j \frac{Z_n / \rho_0 c_0 + j \tan kL}{1 + j(Z_n / \rho_0 c_0) \tan kL}. \end{aligned}$$

Meanwhile, at  $x = 0$ ,

$$\begin{aligned} Z_{in} &= \frac{P(0)}{U(0)} = -j\rho_0 c_0 \frac{A}{B} \\ \Rightarrow \frac{A}{B} &= j \frac{Z_{in}}{\rho_0 c_0} \end{aligned}$$

Setting the above two equations for  $\frac{A}{B}$  equal and solving for  $\frac{Z_{in}}{\rho_0 c_0}$  gives the input impedance in dimensionless form:

$$\frac{Z_{in}}{\rho_0 c_0} = \frac{Z_n / \rho_0 c_0 + j \tan kL}{1 + j(Z_n / \rho_0 c_0) \tan kL}.$$

10. In the previous problem it was found that the input impedance in terms of the impedance of the load is

$$\frac{Z_{in}}{\rho_0 c_0} = \frac{Z_n / \rho_0 c_0 + j \tan kL}{1 + j(Z_n / \rho_0 c_0) \tan kL}. \quad (2)$$

What are the three special cases that follow from equation (2)? [ANSWER]

1. For  $Z_n = 0$  (i.e., pressure release),

$$Z_{\text{in}} = j\rho_0 c_0 \tan kL.$$

2. For  $|Z_n| \rightarrow \infty$  (i.e., rigid)

$$Z_{\text{in}} = -j\rho_0 c_0 \cot kL.$$

3. For  $Z_n = \rho_0 c_0$  (i.e., impedance matching),

$$Z_{\text{in}} = 1$$

11. Use equation (2) to derive  $Z_{\text{in}}$  for a short closed cavity. "Short" means  $kL \ll 1$  and "closed" means  $Z_n \rightarrow \infty$ . Find the *acoustic impedance* also. [ANSWER]

Note that  $\tan kL \simeq kL$  for small  $kL$  and equation (2) becomes  $Z_{\text{in}} = -j\rho_0 c_0 \cot kL$  for  $Z_n \rightarrow \infty$ . Then,  $Z_{\text{in}} = \frac{\rho_0 c_0}{jkL}$ , or

$$Z_{\text{in}} = \frac{\rho_0 c_0^2}{j\omega L}.$$

However, the quantity that should be memorized for acoustic circuit analysis is the *acoustic impedance*,

$$Z_{\text{ac}} = Z_{\text{sp ac}}/S = \rho_0 c_0^2 / j\omega L S^2 = \rho_0 c_0^2 / j\omega V.$$

See question (5) for the impedances that should be memorized.

12. What is the end correction for a flanged tube? Qualitatively describe the origin of the end correction. What about the end correction for an unflanged tube? [ANSWER]

For a flanged tube, the end correction is  $\Delta L = 8a/3\pi$ . That is to say, the effective length of a flanged tube is  $L + \Delta L$ , and the effective length of an orifice is  $2\Delta L$ . This end correction originate from the fact that the impedance due to a circular piston contains resistive (real) and reactive (imaginary) parts, of which the reactance dominates. Therefore, when sound arrives at the open end of a tube, "it sees, not zero load, but rather a load consisting mainly of a short continuation of the tube,  $\Delta L = 8a/3\pi$ " (Blackstock, pg. 151-152).

The unflanged tube has an end correction of  $\Delta L = 0.6133a$ . This is a 20th-century result (Schwinger).

13. Use equation (2) to find the input impedance  $Z_{\text{in}}$  for short open cavity, where "short" means  $kL \ll 1$  and "open" means  $Z_n \rightarrow 0$ . Also find the *acoustic impedance*. [ANSWER]

Equation (2) in these limits reads

$$Z_{\text{in}} = j\rho_0 c_0 kL',$$

where  $L' = L + \Delta L$  (the end correction). The version that should be memorized, though, is the *acoustic impedance*, which is

$$Z_{ac} = j\rho_0 c_0 kL/S.$$

14. Calculate the acoustic impedance of a Helmholtz resonator. What is the resonance frequency for a pressure source? [ANSWER]

The Helmholtz resonator consists of capacitive (from the chamber), inductive (from the mass in the neck), and resistive (from the opening) elements. The three corresponding impedances should be added in series since the components appear in series in the resonator (bad reasoning, I know):

$$\begin{aligned} Z_{ac} &= Z_{ac}^{\text{cap}} + Z_{ac}^{\text{ind}} + Z_{ac}^{\text{res}} \\ &= \frac{\rho_0 c_0^2}{j\omega V} + \frac{j\omega \rho_0 L'}{S} + \frac{\rho_0 c_0 k^2}{2\pi}. \end{aligned}$$

For a pressure source, the resonance frequency is found by setting the impedance (neglecting resistance) to 0:

$$\begin{aligned} \frac{\rho_0 c_0^2}{j\omega V} + \frac{j\omega \rho_0 L'}{S} &= 0 \\ \omega_0 &= c_0 \sqrt{S/L'V}. \end{aligned}$$

*How would you find the resonance frequency for a velocity source, for which  $Z_{in} = \infty$ ?*

15. Provide a few definitions of the quality factor  $Q$ . [ANSWER]

One common definition is

$$Q = \frac{\omega}{\Delta\omega}$$

where  $\Delta\omega$  is the bandwidth.

Another definition is

$$Q = \frac{P_{\text{max}}}{P_{\text{min}}},$$

which is subtly different from the definition of SWR. (But how? I actually don't know. Is it that  $Q$  is a complex quantity here, while the SWR is real?)

Yet another definition (Dr. Wilson's favourite), is

$$Q = \text{number of oscillations required to return to steady state}$$

16. Write the impedance of a bubble using the low- $ka$  approximation of the spherical wave impedance and find the resonance frequency in terms of the ratio of specific heats,  $\gamma$ . Note from the definition of adiabatic sound speed that  $\rho_0 c_0^2 = \gamma p_0$ . [ANSWER]

The spherical wave impedance is

$$Z = \frac{\rho_0 c_0}{1 + 1/jka}$$

$$\rightarrow jka \rho_0 c_0, \quad ka \ll 1.$$

This is a specific acoustic impedance. To obtain an acoustic impedance, divide by  $S = 4\pi a^2$ :

$$Z_{ac}^{ind} = \frac{j\omega a \rho_0}{4\pi a^2} = \frac{j\omega \rho_0}{4\pi a} \quad (i)$$

Noting that this in the form of a mass-like impedance, a compliance-like impedance is sought to set up a resonance phenomenon:

$$Z_{ac}^{cap} = \frac{\rho_0 c_0^2}{j\omega V} = \frac{3\rho_0 c_0^2}{4j\omega \pi a^3}. \quad (ii)$$

The resonance frequency is found by combining equations (i) and (ii) and setting the total impedance equal to 0. At this juncture, also note that  $\rho_0 c_0^2 = \gamma p_0$ , since an expression involving this quantity is requested in the problem:

$$Z_{ac}^{tot} = \frac{3\gamma p_0}{4j\omega \pi a^3} + \frac{j\omega \rho_0}{4\pi a} = 0$$

Solving for  $\omega$  gives

$$\omega = \frac{1}{a} \sqrt{\frac{3\gamma p_0}{\rho_0}}$$

$$f = \frac{1}{2\pi a} \sqrt{\frac{3\gamma p_0}{\rho_0}}.$$

This is a decent approximation because  $ka = 0.014$  for an air-water bubble.

17. What is the acoustic impedance of an orifice (a hole in a plate)? *[ANSWER]*

The orifice has a resistive part due to the opening on both ends, as well as an inductive part:

$$Z_{ac} = \frac{2\rho_0 c_0 k^2}{2\pi} + \frac{j\omega \rho_0 L'}{S},$$

where the effective length of the orifice is  $L' = 2\Delta L$  (because the orifice has no length of its own), where  $\Delta L = 8a/3\pi$ .

18. Derive the  $R$ ,  $T_2$ , and  $T_3$  for a side branch, where each pipe has a different cross-sectional area and contains a different medium. Let the incident wave originate from a region with cross-sectional area  $S_1$ , and let where the two branches have cross-sectional area of  $S_2$  and  $S_3$  respectively. *[ANSWER]*

Chris has an nice way of doing this using circuit analysis. Here is how to find the reflection and transmission coefficients using the force-free boundary condition and continuity of volume velocity:

At the junction, there are two conditions, as in the case with a single change of cross-sectional surface area. The first condition is that there is no force at the junction, and the second condition is that volume velocity at the junction is continuous:

$$\begin{aligned} p_i + p_r &= p_{t2} = p_{t3} \\ q_i + q_r &= q_{t2} + q_{t3} \end{aligned}$$

Divide the first equation by  $p_i$  and call  $p_{t2}/p_i = p_{t3}/p_i \equiv T$ :

$$1 + R = T .$$

Meanwhile, the second equation is written as

$$S_1 u_i + S_1 u_r = S_2 u_{t2} + S_3 u_{t3}$$

Noting that  $Z = p/u$ , gives

$$S_1 \frac{p_i}{Z_1} - S_1 \frac{p_r}{Z_1} = S_2 \frac{p_{t2}}{Z_2} + S_3 \frac{p_{t3}}{Z_3} .$$

Dividing by  $p_i$  results in

$$\frac{S_1}{Z_1} (1 - R) = T \frac{S_2}{Z_2} + T \frac{S_3}{Z_3} .$$

The above quantities are written in terms of their *acoustic impedances*.

$$1 - R = Z_{ac,1} T (Z_{ac,2}^{-1} + Z_{ac,3}^{-1})$$

The above equation is added to  $1 + R = T$ :

$$\begin{aligned} 2 &= Z_{ac,1} T (Z_{ac,2}^{-1} + Z_{ac,3}^{-1}) + T \\ 2 &= T [1 + Z_{ac,1} (Z_{ac,2}^{-1} + Z_{ac,3}^{-1})] \end{aligned}$$

Solving for  $T$  gives

$$T = \frac{2Z_{ac,1}^{-1}}{Z_{ac,1}^{-1} + Z_{ac,2}^{-1} + Z_{ac,3}^{-1}}$$

Thus the reflection coefficient is

$$\begin{aligned} R &= T - 1 = \frac{2Z_{ac,1}^{-1}}{Z_{ac,1}^{-1} + Z_{ac,2}^{-1} + Z_{ac,3}^{-1}} - 1 \\ &= \frac{Z_{ac,1}^{-1} - Z_{ac,2}^{-1} - Z_{ac,3}^{-1}}{Z_{ac,1}^{-1} + Z_{ac,2}^{-1} + Z_{ac,3}^{-1}} . \end{aligned}$$

19. How can a Helmholtz resonator be used as a filter? [ANSWER]

Let a Helmholtz resonator be attached a side branch to a pipe, and let the medium be uniform. Thus  $Z_{ac,1} = Z_{ac,3}$  the reflection coefficient is

$$\begin{aligned}
R &= -\frac{Z_{ac,2}^{-1}}{2Z_{ac,1}^{-1} + Z_{ac,2}^{-1}} \\
&= -\frac{Z_{ac,2}^{-1}Z_{ac,2}}{2Z_{ac,1}^{-1}Z_{ac,2} + Z_{ac,2}^{-1}Z_{ac,2}} \\
&= -\frac{1}{2Z_{ac,1}^{-1}Z_{ac,2} + 1}
\end{aligned}$$

The impedance of the Helmholtz resonator goes to 0 at resonance. Thus the reflection coefficient becomes  $R = -1$ . Therefore, at the resonance frequency of the Helmholtz resonator, all of the sound is sent back down the pipe from which it originated.

20. The reflection and transmission coefficients for the three-medium problem are

$$R = \frac{(1 - Z_1/Z_3) \cos k_2\ell + j(Z_2/Z_3 - Z_1/Z_2) \sin k_2\ell}{(1 + Z_1/Z_3) \cos k_2\ell + j(Z_2/Z_3 + Z_1/Z_2) \sin k_2\ell} \quad (3)$$

$$T = \frac{2}{(1 + Z_1/Z_3) \cos k_2\ell + j(Z_2/Z_3 + Z_1/Z_2) \sin k_2\ell} \quad (4)$$

Find  $R$  and  $T$  for the special cases of  $k_2\ell = n\pi$  and  $k_2\ell = (2n - 1)\pi/2$ . [ANSWER]

For  $k_2\ell = n\pi$ , equations (3) and (4) become

$$\begin{aligned}
R &= \frac{(1 - Z_1/Z_3)}{(1 + Z_1/Z_3)} \\
T &= (-1)^n \frac{2}{(1 + Z_1/Z_3)}.
\end{aligned}$$

For  $k_2\ell = (2n - 1)\pi/2$ :

$$\begin{aligned}
R &= \frac{Z_2/Z_3 - Z_1/Z_2}{Z_2/Z_3 + Z_1/Z_2} \\
T &= j(-1)^n \frac{2}{Z_2/Z_3 + Z_1/Z_2}.
\end{aligned}$$

21. How do equations (3) and (4) change for a homogeneous medium of three different cross-sectional areas  $S_1$ ,  $S_2$ , and  $S_3$ ? [ANSWER]

The reflection and transmission coefficients become

$$\begin{aligned}
R &= \frac{(1 - S_3/S_1) \cos k\ell + j(S_3/S_2 - S_2/S_1) \sin k\ell}{(1 + S_3/S_1) \cos k\ell + j(S_3/S_2 + S_2/S_1) \sin k\ell} \\
T &= \frac{2}{(1 + S_3/S_1) \cos k\ell + j(S_3/S_2 + S_2/S_1) \sin k\ell}
\end{aligned}$$

22. Use equation (4) to find  $T$  and TL for the case that  $k_2\ell \ll 1$  and  $Z_2 \gg Z_1 = Z_3$ . What might these conditions represent physically? [ANSWER]

For  $k_2\ell \ll 1$ , equation (4) becomes

$$\begin{aligned}
 T &= \frac{2}{(1 + 1) + j(Z_2/Z_1)k_2\ell} \\
 &= \frac{2}{2 + jk_2\ell\rho_2c_2/\rho_0c_0} \\
 &= \frac{2}{2 + j\omega\ell\rho_2/\rho_0c_0} \\
 &= \frac{1}{1 + j\omega m/2\rho_0c_0}
 \end{aligned}$$

where  $\ell\rho_2$  has been identified as the mass of medium 2. The transmission loss is therefore

$$\text{TL} = -10 \log_{10} \tau = 10 \log_{10} \left[ 1 + \left( \frac{\omega m}{2\rho_0c_0} \right)^2 \right]$$

These conditions represent a thin wall or barrier of  $m = \text{mass/unit area}$  in air.

23. Take the high-frequency limit of the TL obtained in the previous question. How many SPL dB/octave does this correspond to? What is this "law" referred to as? Which frequencies therefore tend to pass through a wall? How much more massive must the wall be to make the sound passing through a wall half as loud? [ANSWER]

For  $\omega m/2\rho_0c_0 \gg 1$ , equation the transmission loss becomes

$$\begin{aligned}
 \text{TL} &\approx 10 \log_{10} \left[ \left( \frac{\omega m}{2\rho_0c_0} \right)^2 \right] \\
 &= 10 \log_{10} \left( \frac{\pi f m}{\rho_0c_0} \right)^2 \\
 &= 20 \log_{10} \frac{\pi f m}{\rho_0c_0}
 \end{aligned}$$

Thus the transmission loss is  $20 \log_{10} 2 = 6$  dB/octave. This is the so-called *mass law* for normal incidence. So the bass frequencies are heard through the wall. The criterion "half as loud" roughly corresponds to  $12 \text{ dB} \approx 20 \log_{10} 4$ , which would require the wall to be four times as massive.

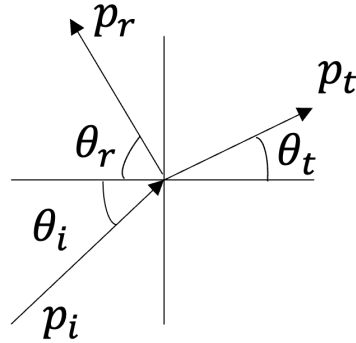
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## Chapter 5: $R$ - $T$ for oblique incidence

1. Consider wave propagation at the interface of two media, where the medium on the left is denoted "1" and has impedance  $Z_1$ , and the medium on the right is denoted "2" and has impedance  $Z_2$ , as shown below. The the horizontal direction is the rectangular  $x$ -coordinate and the vertical direction is the rectangular  $y$ -coordinate.



$$p_i = A_i \exp(-jk_1 x \cos \theta_i - jk_1 y \sin \theta_i) \quad (5)$$

$$p_r = A_r \exp(jk_1 x \cos \theta_r - jk_1 y \sin \theta_r) \quad (6)$$

$$p_t = A_t \exp(-jk_2 x \cos \theta_t - jk_2 y \sin \theta_t) \quad (7)$$

What is the condition on the pressure at the boundary? Use this condition to derive the law of reflection and Snell's law. *[ANSWER]*

At  $x = 0$ , there is no force on the boundary; thus  $p_i(x = 0) + p_r(x = 0) = p_t(x = 0)$ , or

$$A_i \exp(-jk_1 y \sin \theta_i) + A_r \exp(-jk_1 y \sin \theta_r) = A_t \exp(-jk_2 y \sin \theta_t)$$

For the above equation to hold, the phases must match, i.e.,  $A_i + A_r = A_t$  or  $1 + R = T$ :

$$k_1 \sin \theta_i = k_1 \sin \theta_r = k_2 \sin \theta_t$$

The first equality reveals the law of reflection,

$$\theta_i = \theta_r,$$

and the second equality reveals Snell's law,

$$\frac{\sin \theta_i}{c_1} = \frac{\sin \theta_t}{c_2}.$$

2. There is another condition at the interface. What is it? Use it to derive the reflection and transmission coefficients  $R$  and  $T$ . *[ANSWER]*

The other condition is that the normal part of the particle velocity, namely the  $x$ -component of  $u$ , is continuous across the boundary. Using equations (5)-(7) in conjunction with the momentum

equation  $-\frac{1}{j\omega\rho_0} \frac{\partial p}{\partial x}$  and the results of the previous question yields

$$\frac{1}{\omega\rho_1} \frac{\partial}{\partial x} (A_i e^{-jk_1 x \cos \theta_i}) + \frac{1}{\omega\rho_1} \frac{\partial}{\partial x} (A_r e^{jk_1 x \cos \theta_r}) = \frac{1}{\omega\rho_2} \frac{\partial}{\partial x} (A_t e^{-jk_2 x \cos \theta_t})$$

Taking the derivatives above and matching the phases gives

$$\begin{aligned} \frac{-k_1}{\rho_1 \omega} A_i \cos \theta_i + \frac{k_1}{\rho_1 \omega} A_r \cos \theta_r &= -\frac{k_2}{\rho_2 \omega} A_t \cos \theta_t \\ \frac{-1}{\rho_1 c_1} A_i \cos \theta_i + \frac{1}{\rho_1 c_1} A_r \cos \theta_r &= -\frac{1}{\rho_2 c_2} A_t \cos \theta_t \end{aligned}$$

Noting the law of reflection ( $\theta_i = \theta_r$ ), multiplying through by  $-1$ , and dividing by  $A_i$  results in

$$1 - R = \frac{Z_1 \cos \theta_t}{Z_2 \cos \theta_i} T \quad (i)$$

Combining equation (i) with  $1 + R = T$  results in

$$\begin{aligned} R &= \frac{Z_2 \cos \theta_i - Z_1 \cos \theta_t}{Z_2 \cos \theta_i + Z_1 \cos \theta_t} \\ T &= \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_i + Z_1 \cos \theta_t} \end{aligned}$$

3. Derive the power reflection and transmission coefficients  $r$  and  $\tau$ . What is the cheap way of deriving  $\tau$ ? [ANSWER]

The power reflection coefficient is

$$r = \frac{W_r}{W_i} = \frac{S_r I_r}{S_i I_i} = \frac{I_r}{I_i} = \frac{|A_i|^2}{|A_r|^2} = |R|^2$$

where  $S_i$  is the cross-sectional area of the incident ray, and  $S_r$  is the cross-sectional area of the reflected ray. The areas are the same by the law of reflection.

Meanwhile, the cheap way to get  $\tau$  is to invoke the conservation of energy:  $\tau = 1 - r$ . The rigorous way accounts for the fact that the cross sectional area of the transmitted beam  $S_t$  is not the same as  $S_i$  due to Snell's law. In this approach, it is noted that  $S_t/S_i = \cos \theta_t/\cos \theta_i$  (see Blackstock's figure 5.5 on page 192), giving

$$\tau = \frac{W_t}{W_i} = \frac{S_t I_t}{S_i I_i} = \frac{S_t |A_t|^2 / Z_2}{S_i |A_i|^2 / Z_1} = \frac{Z_1 \cos \theta_t}{Z_2 \cos \theta_i} |T|^2.$$

4. What is the fancy name for the angle at which sound transmits perfectly from one medium to the next? Derive an expression for this angle. [ANSWER]

It is called the angle of intromission. See [here](#) for the derivation, which was not rigorously derived in class, nor in Blackstock's text. Its derivation is straightforward, and I have shown the steps that lead to Blackstock's Eq. (B-14) in Sec. 5.B.2.a of *Fundamentals of Physical Acoustics*. Set  $T = 1$ , which corresponds to perfect transmission. Solve for  $\cos \theta_t$  and square the result.

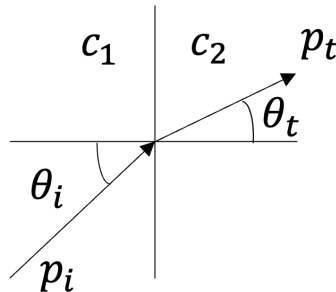
Meanwhile, solve Snell's law for  $\sin \theta_t$  and square it. Add these two equations and solve for  $\sin^2 \theta_i$ .

5. What is the condition for total internal reflection? What is name of the minimum angle  $\theta_i$  at which total internal reflection occurs? Find that angle. [ANSWER]

The condition for total internal reflection is that the transmitted angle is  $\theta_t = 90^\circ$ , and the angle  $\theta_i$  at which this occurs is called the *critical angle*,  $\theta_{\text{crit}}$ .  $\theta_{\text{crit}}$  is given by Snell's law:

$$\theta_{\text{crit}} = \arcsin \frac{c_1}{c_2}$$

6. Based on the figure below, which is larger:  $c_1$  or  $c_2$ ?



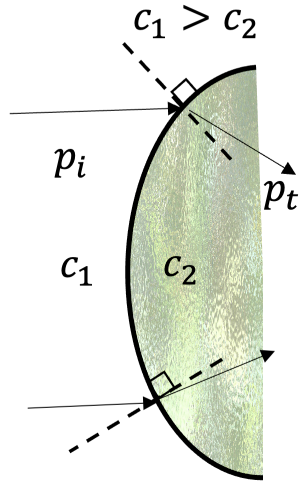
And from which direction of wave propagation does there exist a critical angle:  $c_1$  to  $c_2$ , or  $c_2$  to  $c_1$ ? [ANSWER]

Sound (and light) traveling from a medium of higher wave speed into lower wave speed bend *toward* the normal. Thus  $c_1 > c_2$ . There can exist a critical angle for wave propagation from  $c_2$  to  $c_1$ .

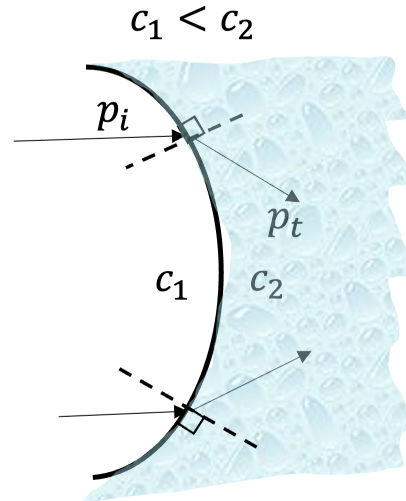
7. In optics, a converging lens generally has a convex shape ( ), while a diverging lens generally has a concave shape ( ). But in acoustics, a converging lens generally has a *concave* shape ( ), while a diverging lens generally has a *convex* shape ( ). Why is this? [ANSWER]

This is because in optics, light travels faster in air than in the material of the lens (e.g., a glass lens in air), while in acoustics, sound travels *slower* in air than the material of the lens (e.g., a water lens in air). For example, a converging lens bends light towards the local normal in the lens, so a convex shape is used in optics, while a concave shape is used in acoustics:

### Converging lens in optics



### Converging lens in acoustics



8. For  $\theta_i > \theta_{\text{crit}}$ , what happens to equation (7)? What is the term for the transmitted waves in the  $x$ -direction? [ANSWER]

For  $\theta_i > \theta_{\text{crit}}$ ,  $\cos \theta_t$  is negative, because for  $\pi/2 < \psi < 3\pi/2$ ,  $-1 < \cos \psi < 0$ . Therefore, equation (7) becomes

$$\begin{aligned} p_t &= A_t \exp(\mp jk_2 x \sqrt{1 - \sin^2 \theta_t} - jk_2 y \sin \theta_t) \\ &= A_t \exp\left(\mp jk_2 x \sqrt{1 - (c_1/c_2)^2 \sin^2 \theta_i} - jk_2 y \sin \theta_t\right) \\ &= A_t \exp\left(\pm k_2 x \sqrt{(c_1/c_2)^2 \sin^2(\theta_i) - 1} - jk_2 y \sin \theta_t\right) \\ &= A_t \exp\left(-k_2 x \sqrt{(c_1/c_2)^2 \sin^2(\theta_i) - 1} - jk_2 y \sin \theta_t\right). \end{aligned}$$

Thus  $p_t$  decays exponentially in the  $x$  direction. Note that in the last line above, the  $-$  sign in the exponential is chosen because it is physical, i.e., exponential growth is not physical. These are called *evanescent waves*. The attenuation coefficient as identified as

$$\alpha = k_2 \sqrt{(c_2/c_1)^2 \sin^2 \theta_i - 1}.$$

For convenience, Dr. Blackstock also denotes the dimensionless quantity  $\sqrt{(c_2/c_1)^2 \sin^2 \theta_i - 1} \equiv b$ .

9. ✪ For  $\theta_i > \theta_{\text{crit}}$ , what happens to the reflection coefficient

$$R = \frac{Z_2 \cos \theta_i - Z_1 \cos \theta_t}{Z_2 \cos \theta_i + Z_1 \cos \theta_t} ?$$

Use the force-free boundary condition on the interface  $1 + R = T$  to also determine what happens to the transmission coefficient for  $\theta_i > \theta_{\text{crit}}$ . Find  $|T|$  as well. Use Dr. Blackstock's notation  $b \equiv \sqrt{(c_2/c_1)^2 \sin^2 \theta_i - 1}$ . [ANSWER]

As seen in the previous part, the quantity  $\cos \theta_t$  can be written as  $-j\sqrt{(c_2/c_1)^2 \sin^2 \theta_i - 1} = -jb$ . Therefore, the reflection coefficient becomes

$$\begin{aligned} R &= \frac{Z_2 \cos \theta_i + jbZ_1}{Z_2 \cos \theta_i - jbZ_1} \\ &= \frac{\sqrt{Z_2^2 \cos^2 \theta_i + b^2 Z_1^2} e^{j \arctan bZ_1/Z_2 \cos \theta_i}}{\sqrt{Z_2^2 \cos^2 \theta_i + b^2 Z_1^2} e^{-j \arctan bZ_1/Z_2 \cos \theta_i}} \\ &= \exp \left[ 2j \arctan \left( \frac{b}{\cos \theta_i} \frac{Z_1}{Z_2} \right) \right] \\ &= \exp(2j\psi) \end{aligned}$$

where  $\psi = \arctan \left( \frac{b}{\cos \theta_i} \frac{Z_1}{Z_2} \right)$ . The transmission coefficient  $T$  is therefore

$$T = 1 + \exp(2j\psi),$$

and

$$|T| = \sqrt{2 + 2 \cos 2\psi}.$$

10. The mass law for oblique incidence was covered in class by considering the wall in two different ways. Qualitatively describe the boundary conditions at the wall and how one would obtain the reflection and transmission coefficients in both cases. [ANSWER]

The first way was by considering the wall to be mass-like (i.e., only compliance, no stiffness). In that case, the pressure at the boundary is given by Newton's second law, i.e.,  $p_i + p_r - p_t = m \partial u_x / \partial t$ , where  $u_x$  is the normal component of the velocity. The other boundary condition is that the normal component of the velocities must match on either side of the boundary, i.e.,  $u_i \cos \theta + u_r \cos \theta = u_t \cos \theta$  which gives  $1 - R = T$  upon division by  $u_i \cos(\theta) \times$  the impedance of the propagation medium. These relations can be combined to get  $T$ ,

$$T = \frac{1}{1 + j\omega m \cos(\theta) / 2\rho_0 c_0},$$

which recovers the special case of the three-medium problem considered in chapter 4. What is remarkable about  $T$  is that it equals 1 for  $\theta = 90^\circ$ .

The second way is more refined, in which the wall's *stiffness* is incorporated. See [here](#) for the derivation. In short, the compliance of the wall makes the right-hand side of Newton's second law differ from the rigid case, but the normal component of the particle velocity condition is identical to that in the rigid case. The result is

$$T = \frac{1}{1 + (j\omega m \cos(\theta) / 2\rho_0 c_0) [1 - (f/f_0)^2 \sin^4 \theta]}.$$

For a given angle, the coincidence frequency is the frequency at which  $T = 1$ . Physically, coincidence is the matching of the trace speed to the natural flexural wave speed of the panel. Coincidence takes a massive hit to the transmission loss of a panel.

11. How does one calculate the power transmission coefficient  $\bar{\tau}$  for a composite wall, i.e., a wall with windows, doors, etc. How does one then calculate the composite transmission loss  $TL_{\text{comp}}$ .

*[ANSWER]*

For composite walls, the power transmission coefficients are averaged by the area they occupy:

$$\bar{\tau} = \frac{\sum_n S_n \tau_n}{\sum_n S_n}.$$

Note that  $S$  can be in any unit of area. The composite transmission loss is

$$TL_{\text{comp}} = -10 \log_{10} \bar{\tau}.$$

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## Chapter 6: Normal modes in Cartesian coordinates

1. Solve  $\xi_{xx} - \frac{1}{c^2} \xi_{tt} = 0$  for a fixed-fixed string for an arbitrary initial displacement  $\xi(x, 0)$ . How does the answer change for an initial velocity condition  $\dot{\xi}(x, 0)$ ? [ANSWER]

Separation of variables and application of the fixed-fixed boundary condition leads to

$$\xi(x, t) = \sum_{n=1}^{\infty} \sin(n\pi x/L) (a_n \cos \omega t + b_n \sin \omega t) \quad (i)$$

To find the expansion coefficients for the initial condition  $\xi(x, 0)$ , first note that  $b_n = 0$  in equation (i):

$$\xi(x, 0) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)$$

Invoking the orthogonality of sines gives

$$\begin{aligned} \int_0^L \xi(x, 0) \sin(m\pi x/L) dx &= \sum_{n=1}^{\infty} a_n \int_0^L \sin(n\pi x/L) \sin(m\pi x/L) dx \\ \int_0^L \xi(x, 0) \sin(m\pi x/L) dx &= \frac{L}{2} \sum_{n=1}^{\infty} a_n \delta_{nm} \\ \Rightarrow a_n &= \frac{2}{L} \int_0^L \xi(x, 0) \sin(n\pi x/L) dx \end{aligned}$$

Thus the solution is  $\xi(x, t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)$ , where  $a_n$  is given above.

To find the expansion coefficients for the initial condition  $\dot{\xi}(x, 0)$ , first note that the time derivative of equation (i) must be taken,

$$\dot{\xi}(x, t) = \sum_{n=1}^{\infty} \sin(n\pi x/L) (-a_n \omega \sin \omega t + \omega b_n \cos \omega t),$$

from which evaluation at  $t = 0$  shows that  $a_n = 0$ :

$$\dot{\xi}(x, 0) = \omega \sum_{n=1}^{\infty} b_n \sin(n\pi x/L).$$

Invoking the orthogonality of sines gives

$$\int_0^L \dot{\xi}(x, 0) \sin(m\pi x/L) dx = \omega \sum_{n=1}^{\infty} b_n \int_0^L \sin(n\pi x/L) \sin(m\pi x/L) dx$$

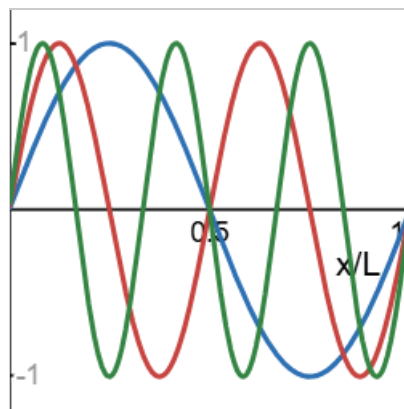
$$\int_0^L \dot{\xi}(x, 0) \sin(m\pi x/L) dx = \frac{L\omega}{2} \sum_{n=1}^{\infty} b_n \delta_{nm}$$

$$\Rightarrow b_n = \frac{2}{L\omega} \int_0^L \dot{\xi}(x, 0) \sin(n\pi x/L) dx$$

Thus the solution is  $\xi(x, t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)$ , where  $b_n$  is given above.

2. ✨ Without performing a calculation, which eigenmode(s) of a 1D string will contribute to the solution if the initial velocity condition is  $-u_0$  for  $0 < x \leq L/2$  and  $u_0$  for  $L/2 < x < L$ ? [ANSWER]

First of all, the  $n = 1, 3, 5 \dots$  modes will not contribute because they violate the symmetry of the initial condition: these modes are even about  $L/2$ , while the initial condition is odd about  $L/2$ . Of the remaining  $n = 2, 4, 6 \dots$  modes, only  $n = 2, 6, 10 \dots$  contribute. While the  $n = 4, 8, 12 \dots$  modes are odd about  $L/2$ , their derivative is positive at  $x = L/2$ , while the derivative of the initial condition is  $-\infty$  at  $x = L/2$ . This can be seen in the plots of the normalized standing waves vs.  $x/L$  below, where the red curve is the  $n = 4$  mode.



3. Given the expression for  $b_n$  from the first problem, calculate the expansion coefficients for a string that has an initial velocity of  $-u_0$  for  $0 < x \leq L/2$  and  $u_0$  for  $L/2 < x < L$  and thereby verify the result above. [ANSWER]

The equation for  $b_n$  is

$$b_n = \frac{2}{L\omega_n} \int_0^L \dot{\xi}(x, 0) \sin \frac{n\pi x}{L} dx$$

so for the initial velocity condition one obtains

$$b_n = -\frac{2}{L\omega_n} \int_0^{L/2} \sin \frac{n\pi x}{L} dx + \frac{2}{L\omega_n} \int_{L/2}^L \sin \frac{n\pi x}{L} dx$$

$$= \frac{2Lu_0}{(n\pi)^2 c} \left( 2 \cos \frac{n\pi}{2} - \cos n\pi - 1 \right)$$

$$= \begin{cases} 0 & n = \text{odd} \\ 0 & n = 4, 8, 12 \dots \\ -4 & n = 2, 6, 10, 14 \dots \end{cases}$$

4. ✪ What is the boundary condition at a free end of a string? Provide some rationale. Of which boundary condition of acoustics is the free-end boundary condition of a string reminiscent? And of which boundary condition in acoustics is the fixed-end boundary condition of a string reminiscent? [ANSWER]

The boundary condition at a free end of a string is  $\partial \xi / \partial x = 0$ . The reason is that the restoring force on the string is  $f = -\mathcal{T} \sin \theta \simeq -\mathcal{T} \theta$ . At the free end of the string,  $x = \ell$ , there is no element at  $\ell + \Delta x$  to pull the string up or down. Thus  $\theta = 0 \simeq \tan \theta \simeq \partial \xi / \partial x$ . This condition is reminiscent of a rigid surface in acoustics because the derivative of the wave variable vanishes at that location.

The fixed-end boundary condition is  $\xi = 0$ , which is reminiscent of a pressure-release boundary in acoustics.

5. ✪ Solve  $\partial^2 p / \partial x^2 + k^2 p = 0$  for the boundary conditions  $dp/dx = 0$  and  $j\omega \rho_0 p + Z_n dp/dx = 0$  where  $Z_n = j\omega m$  for  $x = L$ .  $dp/dx = 0$  is a rigid boundary condition, but what is  $j\omega \rho_0 p + Z_n dp/dx = 0$  called? Would this tube with a mass at one end sound good? [ANSWER]

The general solution to the Helmholtz equation is

$$p = \sum_n (a_n \cos kx + b_n \sin kx),$$

the gradient of which is

$$\frac{dp}{dx} = \sum_n k(-a_n \sin kx + b_n \cos kx).$$

Since  $\frac{dp}{dx}|_{x=0} = 0$ ,  $b_n = 0$ , leaving

$$p = \sum_n a_n \cos kx. \quad (i)$$

Substituting equation (i) into the boundary condition at  $x = L$ ,  $\rho_0 p + m \frac{dp}{dx} = 0$ , gives

$$\begin{aligned} \rho_0 p|_{x=L} + m \frac{dp}{dx}|_{x=L} &= 0 \\ \rho_0 \sum_n a_n \cos kL - m \sum_n k a_n \sin kL &= 0 \end{aligned} \quad (ii)$$

Each term in the summation of equation (ii) must hold, giving

$$\rho_0 \cos kL - mk \sin kL = 0,$$

which in turn gives the transcendental equation  $kL$ ,

$$\cot kL = \frac{mk}{\rho_0}.$$

This tube with a mass at one end would not sound "good," in the sense that its eigenfrequencies are not evenly spaced, i.e., anharmonic, like the gamelan.

$j\omega\rho_0p + Z_n\partial p/\partial x = 0$  at  $x = L$  is called an *impedance boundary condition*. This boundary condition is a linear combination of the field variable and its gradient vanishing at the boundary.

6. What types of boundary conditions usually give rise to harmonic eigenfrequencies? [ANSWER]

Boundary conditions in which either the field variable or its gradient vanishes generally give rise to harmonic eigenfrequencies. The previous example showed that anharmonic eigenfrequencies arise due to impedance boundary conditions.

7. What is the difference between resonance frequencies, eigenfrequencies, and natural frequencies? [ANSWER]

Resonance is the infinite response (assuming no loss) of a system due to an external excitation. The frequency of that external excitation is called the resonance frequency. In contrast, eigenfrequencies are the frequencies at which a system vibrates, and they are found by solving the wave equation using orthogonal eigenfunctions. Eigenfrequencies depend on the shape, stiffness, and density of a system. Natural frequencies are synonymous with "eigenfrequencies."

In practice, eigenfrequencies coincide with resonance frequencies, and hence the three terms are often used interchangeably. But technically, resonance frequency refers to a driving frequency that coincides with an eigenfrequency, while the eigenfrequency is more closely related to the system.

8. Solve the wave equation for the transverse displacement  $\xi(x, t)$

$$\frac{\partial^2 \xi}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2 \xi}{\partial t^2} = 0$$

for

$$\xi(0, t) = 0 \quad \text{and} \quad \xi(L, t) = \xi_0 e^{j\omega_0 t},$$

the fixed-driven boundary conditions of a string. [ANSWER]

See [here](#) for the solution, as well as some animations.

9. Derive an expression for the phase speed  $c_{\text{ph}}$ , which is the speed of a point of constant phase. Write the phase as  $\Phi = \omega t - kx$  and set a differential element  $d\Phi$  equal to 0. [ANSWER]

Since the phase speed is the speed of a point of constant phase, a differential element of phase  $d\Phi$  is constant:

$$d\Phi = \frac{\partial \Phi}{\partial t} dt + \frac{\partial \Phi}{\partial x} dx = 0.$$

Solving for  $dx/dt$  and taking the derivatives of  $\Phi$  with respect to  $x$  and  $t$  gives the phase speed:

$$c_{\text{ph}} = -\frac{\partial \Phi / \partial t}{\partial \Phi / \partial x} = \frac{\omega}{k}$$

10. ✳ Derive an expression for the group speed. Write a signal in terms of its Fourier components, i.e.,  $p(x, t) = \int_{-\infty}^{\infty} P(k) e^{j(\omega t - kx)} dk$ . Take the first-order Taylor expansion of  $\omega(k)$  about  $k_0$ , and denote  $\omega(k_0) = \omega_0$ . Factor out from the integral  $e^{j(\omega_0 t - k_0 x)}$ , which represents a monochromatic wave with points of constant phase moving at  $\omega_0/k_0$ . The remaining integral is the envelope of the wave packet. What is the speed of the wave packet? [ANSWER]

The procedure above is followed:

$$\begin{aligned}
 p(x, t) &= \int_{-\infty}^{\infty} P(k) e^{j(\omega t - kx)} dk \\
 &= \int_{-\infty}^{\infty} P(k) e^{j\left\{\left[\omega_0 + \left.\frac{\partial \omega}{\partial k}\right|_{k_0} (k - k_0)\right] t - kx\right\}} dk \\
 &= e^{j(\omega_0 t - k_0 x)} \int_{-\infty}^{\infty} P(k) e^{j\left[\left.\frac{\partial \omega}{\partial k}\right|_{k_0} (k - k_0) t - (k - k_0)x\right]} dk \\
 &= e^{j(\omega_0 t - k_0 x)} \int_{-\infty}^{\infty} P(k) e^{j\left[(k - k_0) \left.\frac{\partial \omega}{\partial k}\right|_{k_0} t - (k - k_0)x\right]} dk \\
 &= e^{j(\omega_0 t - k_0 x)} \int_{-\infty}^{\infty} P(k) e^{j(k - k_0) \left(\left.\frac{\partial \omega}{\partial k}\right|_{k_0} t - x\right)} dk
 \end{aligned}$$

The first factor outside the integral is a monochromatic wave with points of constant phase moving at  $\omega_0/k_0$ . Meanwhile, the remaining integral is the envelope of the wave packet, traveling at group speed

$$c_{\text{gr}} = \left. \frac{\partial \omega}{\partial k} \right|_{k_0}.$$

The only assumption made in this derivation is that the wave packet is fairly narrowband, such that the first-order Taylor expansion of  $\omega(k)$  suffices.

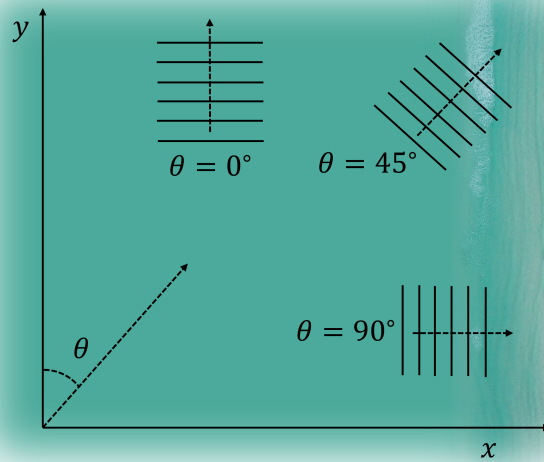
11. At what speed does energy in a wave travel: the phase speed or the group speed? [ANSWER]

Wave energy travels at the group speed. Thus the phase speed can be infinite, while the group speed is bounded by the speed of light.

12. On a plot of  $\omega$  vs  $k$ , the slope of what line gives the phase speed? The slope of what line gives the group speed? [ANSWER]

The slope of the secant line connecting the origin  $(0, 0)$  to a point  $(k, \omega)$  is the phase speed. The slope of the tangent line at  $(k, \omega)$  is the group speed.

13. The illustration below shows ocean waves traveling at various angles from the y-axis, which is parallel to the beach.



For wave propagation toward the beach, i.e., in the  $x$ -direction, qualitatively describe what the phase speed and group speed correspond to. Without performing calculations, rank the group and phase speeds of the three sets of ocean waves. Then quantitatively provide expressions for the phase speed and group speed in the  $x$ -direction in terms of the ambient wave speed  $c$  and the angle  $\theta$ . [ANSWER]

*Qualitatively:* The phase speed is the apparent speed with which one wavefront sweeps along the  $x$ -axis, while the group speed is the component of the velocity  $c$  along the  $x$ -axis. For example, the phase speed of the ocean waves traveling at  $90^\circ$  with respect to the  $y$ -axis is the ambient wave speed  $c$  because its wavefronts sweep along the  $x$ -axis at the speed at which the wave travels, and the group speed is also  $c$  because the wave motion is purely in the  $x$ -direction. Meanwhile, the phase speed of the ocean waves traveling at  $0^\circ$  with respect to the  $y$ -axis is infinite because its wavefronts sweep along the  $x$ -axis infinitely fast, while the group speed is also 0 because the wave motion has no component in the  $x$ -direction. The waves traveling at  $45^\circ$  have a phase speed larger than  $c$  but less than  $\infty$ , and a group speed less than  $c$  but greater than 0. In summary,  $c = c_{\text{ph}}^{90^\circ} < c_{\text{ph}}^{45^\circ} < c_{\text{ph}}^{0^\circ} = \infty$ , and  $0 = c_{\text{gr}}^{0^\circ} < c_{\text{gr}}^{45^\circ} < c_{\text{gr}}^{90^\circ} = c$ .

*Quantitatively:* The phase speed is the trace velocity, which is the trace wavelength times the frequency,

$$c_{\text{ph}} = \frac{c}{\sin \theta} ,$$

while the group speed is the component of the velocity along the  $x$ -axis,

$$c_{\text{gr}} = c \sin \theta .$$

More questions on phase speed and group speed follow in [section 12](#).

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## Chapter 7: Horns

1. Of the continuity, momentum, and mass equations, which is/are adjusted to account for the changing cross-sectional area of a horn? *[ANSWER]*

The momentum equation remains the same (though it takes several lines of algebra and the insertion of the continuity equation to show this). The state equation more obviously remains the same. The continuity is the only equation that ends up requiring an extra term. Repeating the derivation for the 1D continuity equation presented in chapter 1 while letting  $S = S(x)$  gives

$$\frac{\partial \rho}{\partial t} + \frac{S'}{S} \rho u + \frac{\partial(\rho u)}{\partial x} = 0.$$

Linearizing and combining the equations of continuity, momentum, and state results in the Webster horn equation,

$$\frac{\partial^2 p}{\partial x^2} + \frac{dS/dx}{S} \frac{\partial p}{\partial x} = \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2}.$$

2. Obtain the phase speed for the exponential horn, in which  $S = S_0 e^{mx}$ . Identify the cutoff frequency. How does the magnitude of the solution grow? How could this growth be alternatively predicted from the conservation of energy? *[ANSWER]*

Inserting  $S = S_0 e^{mx}$  into the Webster horn equation gives

$$\frac{\partial^2 p}{\partial x^2} + m \frac{\partial p}{\partial x} = \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2}.$$

A solution proportional to  $e^{j(\omega t - kx)}$  is sought, giving

$$\begin{aligned} -k^2 - jkm &= -\frac{\omega^2}{c_0^2} \\ k^2 + jkm - k_0^2 &= 0 \\ \Rightarrow k &= -j\frac{m}{2} \pm \frac{\omega}{c_0} \sqrt{1 - (m/2k_0)^2} \end{aligned}$$

The phase speed is found by taking the real part of the above and solving for  $\omega/k$ :

$$c_{\text{ph}} = \frac{\omega}{k} = \frac{c_0}{\sqrt{1 - (m/2k_0)^2}}$$

Writing  $k_0 = \omega/c_0$  gives  $c_{\text{ph}} = \frac{c_0}{\sqrt{1 - (mc_0/2\omega)^2}} = \frac{c_0}{\sqrt{1 - (mc_0/4\pi f)^2}}$ . The cutoff frequency is identified to be  $f^{(c)} = mc_0/4\pi$ , and the phase speed can therefore be written as

$$c_{\text{ph}} = \frac{c_0}{\sqrt{1 - (f^{(c)}/f)^2}}.$$

By substituting the expression for  $k$  into the form of solution  $e^{j(\omega t - kx)}$ , it is seen that the magnitude of the solution grows as

$$|p| \propto e^{-mx/2}.$$

This result could be derived by noting that the power, remains constant as the wave propagates through the horn:

$$\begin{aligned} W = IS &\propto p_{\text{rms}}^2 S = \text{constant} \\ \Rightarrow p_{\text{rms}} &\propto 1/\sqrt{S} \\ &\propto 1/\sqrt{e^{mx}} \\ |p| &\propto e^{-mx/2} \end{aligned}$$

3. Using the expression for  $k$  found above, find the impedance  $Z = p/u = \omega \rho_0/k$  of the exponential horn. *Where does  $p/u = \omega \rho_0/k$  come from?* [ANSWER]

Inserting  $k = -j\frac{m}{2} \pm \frac{\omega}{c_0} \sqrt{1 - (m/2k_0)^2}$  into this expression for the impedance gives

$$\begin{aligned} Z &= \frac{\omega \rho_0}{-j\frac{m}{2} \pm \frac{\omega}{c_0} \sqrt{1 - (mc_0/4\pi f)^2}} \\ &= \frac{2\pi f \rho_0}{-j\frac{m}{2} \pm \frac{\omega}{c_0} \sqrt{1 - (f^{(c)}/f)^2}} \\ &= \frac{\rho_0}{-j\frac{m}{4\pi f} \pm \frac{1}{c_0} \sqrt{1 - (f^{(c)}/f)^2}} \\ &= \frac{\rho_0 c_0}{-j\frac{mc_0}{4\pi f} \pm \sqrt{1 - (f^{(c)}/f)^2}} \\ &= \frac{\rho_0 c_0}{-jf^{(c)}/f \pm \sqrt{1 - (f^{(c)}/f)^2}} \end{aligned}$$

The rectangular form of the impedance is

$$\begin{aligned} Z &= \frac{\rho_0 c_0}{-jf^{(c)}/f \pm \sqrt{1 - (f^{(c)}/f)^2}} \frac{jf^{(c)}/f \pm \sqrt{1 - (f^{(c)}/f)^2}}{jf^{(c)}/f \pm \sqrt{1 - (f^{(c)}/f)^2}} \\ &= \rho_0 c_0 \left[ jf^{(c)}/f \pm \sqrt{1 - (f^{(c)}/f)^2} \right] \end{aligned}$$

4. ✳ Calculate the input power  $W_{\text{in}} = S_0 I = S_0 \langle pu \rangle = \frac{1}{2} S_0 \text{Re}(pu^*)$  from a velocity source of an exponential horn, and write the result in terms of the power in a plane waveguide of constant cross section  $W_0 = \frac{1}{2} \rho_0 c_0 S_0 u_0^2$ . How does this result change for a *pressure* source? [ANSWER]

The power for the velocity source is

$$\begin{aligned}
W_{\text{in}} &= \frac{1}{2} S_0 \text{Re}(pu^*) \\
&= \frac{1}{2} S_0 |u|^2 \text{Re}(Z_{\text{in}}) \\
&= \frac{W_0}{\rho_0 c_0} \text{Re}(Z_{\text{in}}) \\
&= W_0 \sqrt{1 - (f^{(c)}/f)^2}
\end{aligned}$$

where  $|u|^2 = u_0^2$ .

Meanwhile, for a pressure source,  $W_0 = \frac{1}{2\rho_0 c_0} S_0 p_0^2$ . The power for the pressure source is

$$\begin{aligned}
W_{\text{in}} &= \frac{1}{2} S_0 \text{Re}(pu^*) \\
&= \frac{1}{2} S_0 |p|^2 \text{Re}(1/Z_{\text{in}}) \\
&= W_0 \rho_0 c_0 \text{Re}(1/Z_{\text{in}}) \\
&= W_0 \sqrt{1 - (f^{(c)}/f)^2}
\end{aligned}$$

where  $|p|^2 = p_0^2$ . It is rather remarkable that the power exerted by an exponential horn driven by a pressure source equals the power exerted by the horn driven by a velocity source. This invariance is unique to exponential horns.

5. What is the definition of the transmission factor TF? Calculate TF for the exponential horn for a velocity source and a pressure source. *[ANSWER]*

The transmission factor is defined as

$$\text{TF} = \frac{W_{\text{in}}}{W_0}$$

For the exponential horn, the previous result is used, giving  $\text{TF} = \sqrt{1 - (f^{(c)}/f)^2}$  for above the cutoff frequency, and 0 for below the cutoff frequency. This is the transmission factor for both the pressure source and the velocity source.

6. For a conical horn, the horn height grows linearly with distance, i.e.,  $y = mx$ . Write the Webster horn equation for the conical horn. Read off the solution to this partial differential equation. What is the impedance of this horn in rectangular form? *[ANSWER]*

The cross-sectional area is  $S(x) = \pi x^2$ , and thus  $dS/dx = 2\pi x$ . Thus the Webster horn equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{2}{x} \frac{\partial p}{\partial x} = \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2}$$

The solution to this PDE is immediately read off as spherical waves in  $x$ .

The input impedance of this horn is given by the spherical wave impedance in rectangular form found in chapter 1, where  $x = x_0$  corresponds to the location of the mouth:

$$Z_{\text{in}}/\rho_0 c_0 = \frac{jk_0 x_0 + (k_0 x_0)^2}{1 + (k_0 x_0)^2}$$

7. Find the input power  $W_{\text{in}}$  from a velocity source of a conical horn, and write the result in terms of the power in a plane waveguide of constant cross section  $W_0 = \frac{1}{2} \rho_0 c_0 S_0 u_0^2$ . How does this result change for a *pressure* source? Also find the transmission factor for both velocity and pressure sources. [ANSWER]

The power for the velocity source is

$$\begin{aligned} W_{\text{in}} &= \frac{W_0}{\rho_0 c_0} \text{Re}(Z_{\text{in}}) \\ &= W_0 \frac{(k_0 x_0)^2}{1 + (k_0 x_0)^2} \\ \Rightarrow \text{TF} &= \frac{(k_0 x_0)^2}{1 + (k_0 x_0)^2} \end{aligned}$$

Meanwhile, the power for the pressure source is

$$\begin{aligned} W_{\text{in}} &= W_0 \rho_0 c_0 \text{Re}(1/Z_{\text{in}}) \\ &= W_0 \\ \Rightarrow \text{TF} &= 1 \end{aligned}$$

8. Does the conical horn give rise to dispersion? For what frequencies can the conical horn be used? At low frequencies, is it better to use a conical horn driven by a pressure source or a velocity source? [ANSWER]

The conical horn does not give rise to dispersion, because  $d\omega/dk = 1$  for spherical waves. The conical horn can be used for all frequencies. At low frequencies, it is better to use a conical horn driven by a pressure source, because  $\text{TF} = 1$  for all frequencies.

9. What is an advantage and a disadvantage of the catenoidal horn, which has the height profile  $y(x) = y_0 \cosh \frac{x}{h}$ . [ANSWER]

The catenoidal horn, which must be treated using the WKB method (not covered in class) has the advantage of having vanishing slope at  $x = 0$ , which allows for it to be joined smoothly to a tube of radius  $y_0$ . This eliminates reflections from the junction of the tube to the horn. A disadvantage is that the horn, like an exponential horn, has a cutoff frequency, below which the horn does not propagate waves.

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## Chapter 8: Sound in stratified media

1. What is primarily responsible for the stratification of the atmosphere? In which governing equation does this manifest? *[ANSWER]*

Gravity is primarily responsible for the stratification of the atmosphere. It appears on the right-hand side of the momentum equation, as seen in the following problem.

2. Simplify the 1D momentum equation with a body force  $\rho(u_t + uu_z) + P_z = -\rho g$  for a motionless fluid, i.e.,  $u = 0, P = p_0, \rho = \rho_0$ , to obtain

$$\frac{dp_0}{dz} = -\rho_0 g. \quad (8)$$

What are two different assumptions on  $\rho_0$  that allow for simple solutions of equation (8)? *[ANSWER]*

Setting  $u = 0, P = p_0, \rho = \rho_0$  in the 1D momentum equation with a body force gives

$$\begin{aligned} \rho_0(0) + (p_0)_z &= -\rho_0 g \\ \Rightarrow \frac{dp_0}{dz} &= -\rho_0 g. \end{aligned}$$

One simple way to solve equation (8) is to assume  $\rho_0 = \text{constant}$  (see problem 3). The next simplest approach is to consider an atmosphere in which  $T_0$  is constant, and thus by the ideal gas law  $\rho_0 \propto p_0$  (see problem 4).

3. Assume that the density is constant i.e.,  $\rho_0 \neq \rho_0(z)$ , and thus integrate equation (8). Let  $\bar{p}_0 = p_0(z = 0)$  be the sea-level pressure. What is the scale height of the atmosphere? Use the adiabatic relation  $c_0^2 = \gamma \bar{p}_0 / \rho_0$  to eliminate density from the final expression. *[ANSWER]*

Integrating  $dp_0/dz = -\rho_0 g$  over  $z$  gives

$$p_0 = \bar{p}_0 - \rho_0 g z.$$

The scale height is the height  $H$  at which the atmosphere terminates:  $p_0 = 0 = \bar{p}_0 - \rho_0 g H$ , which gives

$$H = \frac{\bar{p}_0}{\rho_0 g} = \frac{c_0^2}{\gamma g}$$

4. Next assume that the temperature is  $T_0$  at all heights. Why is "isothermal" not the best word choice for such an atmosphere? Combine the ideal gas law  $\rho_0 = p_0/RT_0 = \gamma p_0/c_0^2$  with the definition of the scale height  $c_0^2 = H\gamma g$  and eliminate  $\rho_0$  in equation (8), and thus obtain

$$\frac{dp_0}{dz} = -\frac{p_0}{H}. \quad (9)$$

Solve equation (9) for  $p_0$ . *[ANSWER]*

"Isothermal" is poor word choice because the sound propagation is still adiabatic in this model. Combining  $\rho_0 = \gamma p_0 / c_0^2$  with the definition of the scale height  $c_0^2 = H\gamma g$  gives

$$\rho_0 = \frac{p_0}{gH}. \quad (i)$$

Using the above to eliminate  $\rho_0$  in equation (8) gives

$$\frac{dp_0}{dz} = -\frac{p_0}{H},$$

the solution of which is  $p_0 = \bar{p}_0 e^{-z/H}$ . Combination with equation (i) gives the same profile for the ambient density,  $\rho_0 = \bar{\rho}_0 e^{-z/H}$ .

5. Linearize the 1D continuity equation,  $\partial \rho / \partial t + (\rho u)_z = 0$ , for the case of a variable ambient density  $\rho_0 = \rho_0(z)$ . [ANSWER]

Writing  $\rho = \rho_0(z) + \rho'$  gives

$$\begin{aligned} \frac{\partial}{\partial t} [\rho_0(z) + \rho'] + \frac{\partial}{\partial z} [(\rho_0(z) + \rho')u] &= 0 \\ \frac{\partial \rho'}{\partial t} + u \frac{\partial \rho_0(z)}{\partial z} + \rho_0(z) \frac{\partial u}{\partial z} &= 0 \end{aligned} \quad (ii)$$

6. Linearize the 1D momentum equation with a body force,  $\rho(\partial u / \partial t + u \partial u / \partial z) + \partial P / \partial z = -\rho g$ , where  $\rho_0 = \rho_0(z)$ . Simplify the result by invoking equation (8) [ANSWER]

Writing  $P = p_0(z) + p$  gives

$$\begin{aligned} [\rho_0(z) + \rho'] \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} \right] + \frac{\partial}{\partial z} [p_0(z) + p] &= -[\rho_0(z) + \rho']g \\ [\rho_0(z) + \rho'] \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} \right] + \frac{\partial p_0(z)}{\partial z} + \frac{\partial p}{\partial z} &= -\rho_0(z)g - \rho'g \end{aligned}$$

Since by equation (8)  $dp_0/dz = -\rho_0(z)g$ , the above reduces to

$$[\rho_0(z) + \rho'] \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} \right] + \frac{\partial p}{\partial z} = -\rho'g$$

Linearization gives

$$\rho_0(z) \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = -\rho'g. \quad (iii)$$

7. In a stratified medium, what assumption is made about the local entropy? Use this assumption to simplify the state equation

$$\frac{DP}{Dt} = \left( \frac{\partial P}{\partial \rho} \right)_s \frac{D\rho}{Dt} + \left( \frac{\partial P}{\partial s} \right)_\rho \frac{Ds}{Dt},$$

and recall that  $Df/Dt = \partial f/\partial t + u\partial f/\partial z$  for an arbitrary quantity  $f$ . [ANSWER]

In a stratified medium, the entropy of a given particle is assumed to remain constant even though the entropy may vary from particle to particle, i.e.,  $Ds/Dt = 0$ . The state equation becomes

$$\begin{aligned}\frac{DP}{Dt} &= c^2 \frac{D\rho}{Dt} \\ \frac{\partial P}{\partial t} + u \frac{\partial P}{\partial z} &= c^2 \frac{\partial \rho}{\partial t} + c^2 u \frac{\partial \rho}{\partial z} \\ \frac{\partial}{\partial t} [p_0(z) + p] + u \frac{\partial}{\partial z} [p_0(z) + p] &= c_0^2 \frac{\partial}{\partial t} [\rho_0(z) + \rho'] + c_0^2 u \frac{\partial}{\partial z} [\rho_0(z) + \rho'] \\ \frac{\partial p}{\partial t} + u \frac{\partial p_0(z)}{\partial z} &= c_0^2 \frac{\partial \rho'}{\partial t} + c_0^2 u \frac{\partial \rho_0(z)}{\partial z}\end{aligned}\quad (\text{iv})$$

8. Equation (ii) is used to eliminate density from the right-hand side of equation (iv), giving  $\partial p/\partial t + u\partial p_0/\partial z = -c_0^2 \rho_0 \partial u/\partial z$ . Using equation (8) to eliminate the static pressure gradient on the left-hand side gives  $\partial p/\partial t - g\rho_0 u = -c_0^2 \rho_0 \partial u/\partial z$ . Taking the  $z$  derivative with respect to this result gives

$$\frac{\partial^2 p}{\partial z \partial t} - g \frac{\partial}{\partial z} \left[ \rho_0 \frac{\partial u}{\partial z} \right] = - \frac{\partial}{\partial z} \left[ c_0^2 \rho_0 \frac{\partial u}{\partial z} \right]. \quad (\text{v})$$

Meanwhile, rearranging equation (iii) and taking the time derivative gives  $\partial^2 p/\partial z \partial t + g \partial \rho'/\partial t = -\rho_0 \partial^2 u/\partial t^2$ . Combining this equation with equation (ii) to eliminate  $\partial \rho'/\partial t$  gives

$$\frac{\partial^2 p}{\partial z \partial t} - g \frac{\partial}{\partial z} (\rho_0 u) = -\rho_0 \frac{\partial^2 u}{\partial t^2}. \quad (\text{vi})$$

Subtract equation (v) from equation (vi) to obtain

$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial(\rho_0 c_0^2)/\partial z}{\rho_0 c_0^2} \frac{\partial u}{\partial z} - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} = 0, \quad (10)$$

the wave equation for plane waves in an inhomogeneous medium. [ANSWER]

Using subscripts to denote partial derivatives, the subtraction yields

$$-\rho_0 u_{tt} + \rho_0 c_0^2 u_{zz} - (\rho_0 c_0^2)_z u_z = 0.$$

Rearranging and dividing by  $\rho_0 c_0^2$  gives

$$u_{zz} + \frac{(\rho_0 c_0^2)_z}{\rho_0 c_0^2} u_z - c_0^{-2} u_{tt} = 0$$

9. In what limit does equation (10) recover  $\frac{\partial^2 u}{\partial z^2} - \frac{1}{c_0} \frac{\partial^2 u}{\partial t^2}$ ? How is  $\frac{\partial^2 u}{\partial z^2} - \frac{1}{c_0} \frac{\partial^2 u}{\partial t^2}$  different from the "normal" 1D wave equation? [ANSWER]

The correct limit is the slowly varying or quasistatic limit  $\lambda \ll L$ , i.e., the wave doesn't care about the variation in ambient density of characteristic length  $L$  over one wavelength  $\lambda$ .

It is good to be able to show this. The following sort of argument is emphasized by Dr. Hamilton in all his classes. Let  $u$  be spatially and temporally harmonic. Then the orders of the quantities in equation (10) are

$$\begin{aligned}\frac{\partial u}{\partial z} &\sim -jku = \mathcal{O}(ku) = \mathcal{O}(u/\lambda) \\ \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} &\sim \frac{\omega^2}{c_0^2} u = \mathcal{O}(k^2 u) = \mathcal{O}(u/\lambda^2) \\ \frac{d(\rho_0 c_0^2)}{dz} &= \mathcal{O}(\rho_0 c_0^2/L)\end{aligned}$$

So in terms of orders, equation (10) is

$$\mathcal{O}(u/\lambda^2) + \mathcal{O}(u/\lambda L) - \mathcal{O}(u/\lambda^2) = 0$$

Dividing through by  $\mathcal{O}(u/\lambda^2)$  gives

$$\mathcal{O}(1) + \mathcal{O}(\lambda/L) - \mathcal{O}(1) = 0$$

But  $\mathcal{O}(\lambda/L) = 0$ , so the middle term of equation (10) is much smaller than the first and third terms, resulting in

$$\frac{\partial^2 u}{\partial z^2} - \frac{1}{c_0} \frac{\partial^2 u}{\partial t^2}.$$

In the above equation, note that  $c_0 = c_0(z)$ , whereas in the "normal" 1D wave equation for isotropic medium,  $c_0$  is a constant.

10. What is striking about equation (10)? *[ANSWER]*

Equation (10) is of the same form as the Webster horn equation, in which the bulk modulus  $\rho_0 c_0^2$  plays the role of the cross-sectional area  $S$  of the horn. However, note that  $c_0$  is not necessarily a independent of  $z$ , unlike for the case of the horn.

11. What kind of atmosphere renders equation (10) identical in form to the Webster horn equation for an exponential horn? *[ANSWER]*

In question 4 above, it was found that for a constant temperature  $T_0$  at all heights,  $c_0$  is constant and  $\rho_0(z) = \bar{\rho}_0 e^{-z/H}$ . In that case, equation (10) becomes

$$u_{zz} + \frac{1}{H} u_z - \frac{1}{c_0^2} u_{tt} = 0$$

12. Insert the trial solution  $u = \bar{u}_0 \exp[j(\omega t - kz)]$  into  $u_{zz} + \frac{1}{H} u_z - \frac{1}{c_0^2} u_{tt} = 0$  to obtain  $|u(z)|$ , where  $\bar{u}_0$  is the particle velocity amplitude at sea level. What is remarkable about this result? Derive this result directly from the conservation of energy. *[ANSWER]*

Inserting  $u = u_0 \exp[j(\omega t - kz)]$  into

$$u_{zz} + \frac{1}{H}u_z - \frac{1}{c_0^2}u_{tt} = 0$$

gives

$$-k^2 + \frac{jk}{H} + k_0^2 = 0.$$

Solving for  $k$  gives

$$k = \frac{j}{2H} \mp \frac{1}{2H} \sqrt{1 - (2kH)^2},$$

and the magnitude of the trial solution becomes

$$|u| = \bar{u}_0 \exp(z/2H).$$

What is remarkable about this result is that it predicts the *exponential growth of sound without violating the conservation of energy*.

To derive this result directly from the conservation of energy, note that the intensity is constant, so

$$\begin{aligned} \langle I \rangle = \text{constant} &= \rho_0 c_0 u_{\text{rms}}^2 \\ &= \bar{\rho}_0 e^{-z/H} c_0 u_{\text{rms}}^2. \end{aligned}$$

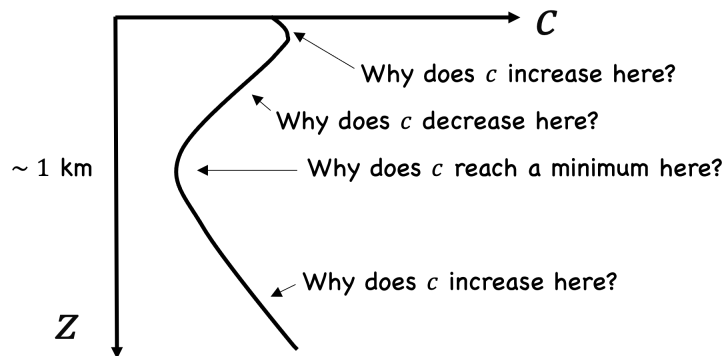
Since  $c_0$  and  $\bar{\rho}_0$  are constants,  $u_{\text{rms}}^2 \propto e^{z/H}$  so  $u_{\text{rms}} \propto e^{z/2H}$ .

13. Water is most dense at 4° C. What is the temperature of the deep ocean? *[ANSWER]*

4° C, which is not at all an intuitive result to me.

In fact this reminds me of questions like, "Mary has three apples and John has two oranges. What is the mass of the sun?"

14. Explain the various features in the ocean sound speed profile.



Adapted from figure 8.2(b) from *Fundamentals of Physical Acoustics* by David T. Blackstock

List any names for the features labeled in the figure. *[ANSWER]*

Starting at the surface, the sound speed profile is fairly constant. This is because the surface layer of the ocean is fairly mixed; thus the temperature is constant throughout that layer. The slight positive slope in this layer is due to the increasing depth.

Below the surface layer is the so-called "thermocline." In this layer, the sound speed gradient is negative due to the drop in temperature, which is due to the attenuation of sunlight (which warms the water) in the  $z$  direction. However, the sound speed's dependence on temperature weakens as depth increases, and its dependence on depth begins to be felt.

At 1 km, the effect of dropping temperature and increasing depth strike a balance (which depends on salinity), and the sound speed attains a minimum. This minimum establishes the SOFAR channel.

As found in the previous question, water is  $4^\circ\text{C}$  for the deep ocean, i.e., temperature is constant. Therefore, going deeper than this minimum, the sound speed profile is dominated by increasing depth.

15. For what frequencies is ray theory a good approximation? What are some other assumptions made when treating sound as rays? How does the amplitude of sound in a ray depend on the cross-sectional area of a ray? [ANSWER]

Ray theory works for  $ka \rightarrow \infty$ . The main assumption (really, the same thing as  $ka \rightarrow \infty$ ), is that diffraction is neglected. Also, since the acoustic power in a ray is assumed to be constant, caustics (the intersection of rays) cannot be handled.

Rays are assumed to conserve power as they propagate. That is,  $W = IS \propto p_{\text{rms}}^2 S = \text{constant}$ . Thus  $p_{\text{rms}} \propto |p| \propto 1/\sqrt{S}$ , as is the case for horns.

16. How are angles defined in ray theory? Write Snell's law in this convention. [ANSWER]

Angles are measured from grazing incidence, i.e.,  $\theta = 0^\circ$  for grazing incidence and  $\theta = 90^\circ$  for normal incidence. Snell's law therefore reads

$$\frac{\cos \theta_1}{c_1} = \frac{\cos \theta_2}{c_2}.$$

17. Since Snell's law applies everywhere in a stratified medium,  $\cos(\theta)/c(\theta)$  is a constant. Find an expression for  $dc/d\theta$ . [ANSWER]

Differentiating  $\cos(\theta)/c(\theta) = \text{constant}$  gives

$$\begin{aligned} \frac{d}{d\theta} \frac{\cos \theta}{c(\theta)} &= 0 \\ -\frac{\sin \theta}{c(\theta)} - \frac{\cos \theta}{c^2(\theta)} \frac{dc}{d\theta} &= 0 \\ \Rightarrow \frac{dc}{d\theta} &= -\frac{\sin \theta}{c(\theta)} \frac{c^2(\theta)}{\cos \theta} \\ &= -\frac{c \sin(\theta)}{\cos \theta} \end{aligned}$$

18. ✨ Define the sound speed gradient to be  $g \equiv dc/dz$ . Define the radius of curvature  $R_c \equiv ds/d\theta$ . Use the chain rule, the definition of the sound speed gradient, and the differential form of Snell's law  $dc/d\theta$  from the previous problem, to express  $R_c$  in terms of  $\theta$ ,  $g$ , and  $c$ . Note that  $dz/ds = \sin(\theta)$ . [ANSWER]

$$\begin{aligned} R_c &= \frac{ds}{d\theta} = \frac{ds}{dz} \frac{dz}{dc} \frac{dc}{d\theta} \\ &= -\frac{1}{\sin \theta} \frac{1}{dc/dz} \frac{c \sin(\theta)}{\cos \theta} \\ &= -\frac{1}{g} \frac{c}{\cos \theta} \end{aligned}$$

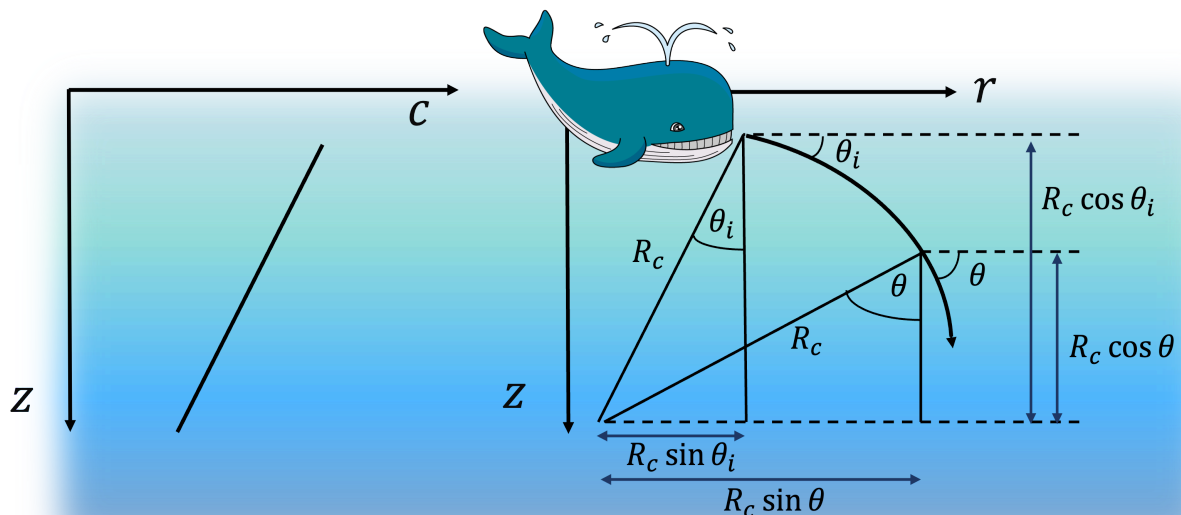
19. If  $g < 0$ , what is the sign of  $R_c$ ? In that case, does the ray bend toward or away from the surface? What about for  $g > 0$ ? [ANSWER]

If  $g < 0$ ,  $R_c$  is positive, so the ray bends away from the surface. The opposite behaviour occurs for  $g > 0$ . The way I remember this is by noting that sound bends in the direction of the slower medium (see chapter 5, problem 6 above).

20. The figure below is adapted from Blackstock's figure 8.8. It shows that a ray in a linear sound speed profile traverses a depth and range of

$$\Delta z = R_c (\cos \theta_i - \cos \theta) \quad (11)$$

$$\Delta r = R_c (\sin \theta - \sin \theta_i) . \quad (12)$$

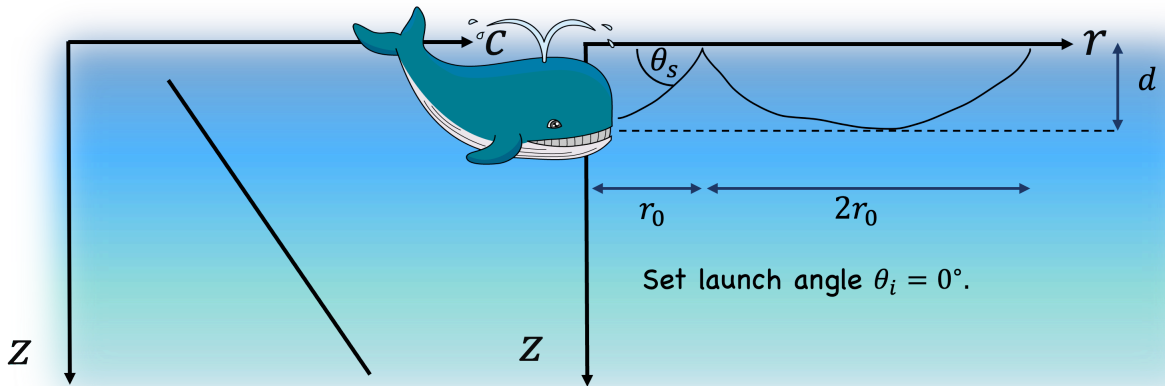


Find the distance from the source to the vertex by setting  $\theta = 0$  and using the expression for  $R_c$  found in problem 18. [ANSWER]

$$\Delta z_v = R_c(\cos \theta_i - 1) = -\frac{1}{g} \frac{c}{\cos \theta_i} (\cos \theta_i - 1) = \frac{c}{g} \frac{1 - \cos \theta_i}{\cos \theta_i}$$

$$\Delta r_v = -R_c \sin \theta_i = \frac{c}{g} \frac{\sin \theta_i}{\cos \theta_i} = \frac{c}{g} \tan \theta_i$$

21. For  $dc/dz = g > 0$ , it is possible for the sound rays to bounce (or "skip", hence  $\theta_s$ ) off the the surface, as shown in the figure below adapted from Blackstock's figure 8.10:



In this case, setting  $\theta_i = 0$  and using equations (11) and (12),

$$-d = R_c(1 - \cos \theta_s)$$

$$r_0 = R_c \sin \theta_s.$$

Solve for  $r_0$ , which is called the cycle distance. [ANSWER]

First solve for  $\cos \theta_s$  and  $\sin \theta_s$ :

$$1 + \frac{d}{R_c} = \cos \theta_s$$

$$-\frac{r_0}{R_c} = \sin \theta_s.$$

Square the above equations and add:

$$(1 + d/R_c)^2 + (r_0/R_c)^2 = 1$$

$$1 + 2d/R_c + (d/R_c)^2 + (r_0/R_c)^2 = 1$$

$$2d/R_c + (d/R_c)^2 + (r_0/R_c)^2 = 0$$

Solve for  $r_0$ :

$$r_0 = R_c \sqrt{-2d/R_c - (d/R_c)^2}$$

$$= \sqrt{-2dR_c - d^2}$$

22. Is it possible for a ray to execute a closed trajectory, i.e., make a loop? Why are or why not?

[ANSWER]

It is not possible for a ray to execute a closed trajectory. While mathematically a ray follows a circular trajectory in a constant sound speed gradient, the model falls apart by the time the ray is traveling vertically (in the  $z$  direction, for the examples considered in the above problems). This is because Snell's law for normal incidence results in refraction at normal incidence. Therefore, once the ray is headed in the  $z$  direction, it will go down until it hits the ocean floor.

23. Obtain the phase speed and group speed given the dispersion relation  $k = \omega n(\omega)/c_0$ . [ANSWER]

The phase speed is simply

$$c_{\text{ph}} = \frac{\omega}{k} = \frac{c_0}{n(\omega)} .$$

To find the group speed, take  $dk/d\omega$

$$\frac{dk}{d\omega} = \frac{1}{c_0} [n(\omega) + \omega n'(\omega)]$$

and invert for  $d\omega/dk$ :

$$\frac{d\omega}{dk} = \frac{c_0}{n(\omega) + \omega n'(\omega)} .$$

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## Chapter 9: Absorption and dispersion

1. Acoustics is the continuum approximation of sound in fluid media, which in reality consists of the compressions and rarefactions of vast ensembles of molecules. What must the mean free path of these molecules be for sound to propagate? Note that the mean free path is the average distance traveled by a molecule before hitting another molecule. *[ANSWER]*

This is a point made by R. P. Feynman in his lecture "Sound and the Wave Equation." For sound to exist, the mean free path must be much less than a wavelength. Otherwise, the molecular behaviour is diffusive.

2. What are the units of the absorption coefficient  $\alpha$ ? How does one convert between SPL and  $\alpha$ ? *[ANSWER]*

$\alpha$  has units of nepers meters<sup>-1</sup>. In fact, nepers are dimensionless! (How ridiculous). To convert from SPL to  $\alpha$ , consider the form of an attenuating wave,  $|p| = Ae^{-\alpha x}$ . Then,

$$\begin{aligned} \text{SPL} &= 20 \log_{10} \frac{A/\sqrt{2}}{p_{\text{ref}}} \\ &= 20 \log_{10} \frac{A e^{-\alpha x}/\sqrt{2}}{p_{\text{ref}}} \\ &= 20 \log_{10} \frac{A/\sqrt{2}}{p_{\text{ref}}} + 20 \log_{10} e^{-\alpha x} \\ &= \text{SPL}_0 + 20 \log_{10} e^{-\alpha x} \\ &= \text{SPL}_0 - 8.686\alpha x, \end{aligned}$$

where  $\text{SPL}_0$  is the sound pressure level at the source  $x = 0$ .

3. The linearized viscous wave equation in 1D is found by combining the linearized continuity, momentum, and state equations (the momentum equation being the only one with modification from the "classic" wave equation). It is given by

$$\frac{\nu \tilde{V}}{c_0^2} u_{xxt} + u_{xx} - \frac{1}{c_0^2} u_{tt} = 0,$$

where  $\nu$  is the kinematic viscosity coefficient and  $\tilde{V}$  is the viscosity number. Derive the attenuation coefficient, defining  $\delta_\nu = \omega \nu / c_0^2 \ll 1$ . Is the medium dispersive? *[ANSWER]*

Let  $u \propto e^{j(\omega t - k(\omega)x)}$ . Substitution into the viscous wave equation gives

$$-j\omega k^2 \frac{\nu \tilde{V}}{c_0^2} - k^2 + \frac{\omega^2}{c_0^2} = 0$$

Solve for  $k$ :

$$\begin{aligned}
k^2(1 + j\omega\nu\tilde{V}/c_0^2) &= \omega^2/c_0^2 \\
k^2 &= \frac{\omega^2/c_0^2}{1 + j\omega\nu\tilde{V}/c_0^2} \\
k &= \frac{\omega}{c_0}(1 + j\omega\nu\tilde{V}/c_0^2)^{-1/2} \\
k &= \frac{\omega}{c_0}(1 + j\delta_\nu\tilde{V})^{-1/2}
\end{aligned}$$

Since  $\delta_\nu \ll 1$ , the binomial expansion is used, giving

$$\begin{aligned}
k &= \frac{\omega}{c_0}(1 - j\delta_\nu\tilde{V}/2) \\
&= \frac{\omega}{c_0} - \frac{j\omega\delta_\nu\tilde{V}}{2}
\end{aligned}$$

The attenuation coefficient is identified as  $\alpha = -\text{Im}(k) = \omega\delta_\nu\tilde{V}/2c_0$ , or reinstating  $\delta_\nu = \omega\nu/c_0^2$ ,

$$\alpha_\nu = \frac{\omega^2\nu\tilde{V}}{2c_0^3}.$$

Meanwhile, note that the phase speed is  $c^{\text{ph}} = \omega/\text{Re}(k) = c_0$ . Thus the medium is non-dispersive (because  $\delta_\nu \ll 1$ ).

4. The linearized wave equation for a thermally conducting fluid is found by combining the linearized continuity, momentum, and state equations (the state equation this time being the one with modifications). It is given by

$$\frac{\kappa}{\rho_0 C_p} \left[ u_{xx} - \frac{\gamma}{c_0^2} u_{tt} \right]_{xx} - \left[ u_{xx} - \frac{1}{c_0^2} u_{tt} \right]_t = 0,$$

where  $\kappa$  is the heat conduction coefficient and  $C_p$  is the specific heat at constant pressure. As a zeroth-order approximation, show that this equation reduces to the "ordinary wave equation" when the thermal conduction of the fluid is negligible. Then as a first-order approximation, approximate  $u_{xx}$  in the first term as  $-u_{tt}/c_0^2$  and integrate over time. Compare the above approximation of the thermally conducting wave equation to the viscous wave equation and read off the attenuation coefficient. [ANSWER]

**Zeroth order approximation.** For  $\kappa/C_p \rightarrow 0$ , the given equation (upon integration over  $t$ ) gives the "ordinary wave equation."

**First order approximation.** Replacing  $u_{xx}$  in the first term as  $-u_{tt}/c_0^2$  gives

$$\frac{\kappa}{\rho_0 C_p} \left[ -\frac{u_{tt}}{c_0^2} - \frac{\gamma}{c_0^2} u_{tt} \right]_{xx} - \left[ u_{xx} - \frac{1}{c_0^2} u_{tt} \right]_t = 0$$

Integrating over time gives

$$\frac{(\gamma - 1)\kappa}{\rho_0 c_0^2 C_p} u_{xxt} + u_{xx} - \frac{1}{c_0^2} u_{tt} = 0.$$

This has the same form as the viscous wave equation which had the attenuation coefficient of  $\alpha = \frac{\omega^2 \nu \tilde{\nu}}{2c_0^3}$ . Thus the attenuation coefficient for the thermally conducting wave equation is

$$\alpha_{\text{th}} = \frac{\omega^2 (\gamma - 1) \kappa}{2\rho_0 c_0^3 C_p}.$$

Similarly, it can be concluded that the thermally conducting wave equation describes non-dispersive wave motion.

See Blackstock's chapter 9, section B for a much more detailed derivation.

5. Having derived the attenuation coefficient for viscous fluids and thermally conducting fluids, derive the attenuation coefficient for thermoviscous fluids. What allows for the simple addition of these attenuation coefficients? [ANSWER]

From Blackstock, footnote of page 300:

The validity of superposing the various contributions, e.g., putting  $\alpha_{\text{total}} = \alpha_1 + \alpha_2 + \alpha_3 + \dots$ , is rarely discussed. In fact, superposition is in general not justified because although the various absorption mechanisms act mainly separately, they do have interactions with each other; see, for example, Sec. B.3. In practice, however, the effect of each mechanism is normally so small that the interactions may be neglected, in which case superposition is justified.

Simply adding the previously found attenuation coefficients together gives

$$\begin{aligned} \alpha_{\text{tv}} &= \alpha_{\text{v}} + \alpha_{\text{th}} \\ &= \frac{\omega^2 \nu \tilde{\nu}}{2c_0^3} + \frac{\omega^2 (\gamma - 1) \kappa}{2\rho_0 c_0^3 C_p} \\ &= \frac{\omega^2}{2c_0^3} \left[ \nu \tilde{\nu} + \frac{(\gamma - 1) \kappa}{\rho_0 C_p} \right] \end{aligned}$$

The above result can be rewritten in a number of ways (see Blackstock page 314); the main result, however, is that the thermoviscous absorption is quadratic in frequency.

6. Derive the 1D linearized wave equation for a relaxing medium, for which the equation of state is  $\tau(p - c_\infty^2 \rho')_t + (p - c_0^2 \rho') = 0$ , where  $\rho'$  is the perturbation density. Assess the behaviour for when the sound's period is much smaller than than the relaxation time ( $\omega\tau \rightarrow 0$ ) and for when the sound's period is much larger than the relaxation time ( $\omega\tau \rightarrow \infty$ ). [ANSWER]

The linearized 1D continuity equation is

$$\rho'_t + \rho_0 u_x = 0, \tag{i}$$

and the linearized 1D momentum equation is

$$p_x + \rho_0 u_t = 0. \quad (\text{ii})$$

The particle velocity is eliminated by taking the time derivative of equation (i) and subtracting from the space derivative of equation (ii):

$$\rho'_{tt} = p_{xx}.$$

Meanwhile the equation of state is differentiated twice w.r.t.  $t$ :

$$\tau p_{ttt} - \tau c_\infty^2 \rho'_{ttt} + p_{tt} - c_0^2 \rho'_{tt} = 0.$$

The density is eliminated between  $\rho'_{tt} = p_{xx}$  and the above equation of state, giving the wave equation for a relaxing fluid:

$$\tau p_{ttt} - \tau c_\infty^2 p_{xxt} + (p_{tt} - c_0^2 p_{xx}) = 0.$$

The term in parentheses is simply the "ordinary linear wave equation."

To assess the low- and high-frequency limits, insert a harmonic test solution i.e.,  $p \propto e^{j\omega t - jkx}$ :

$$\begin{aligned} -j\omega^3 \tau + j\omega \tau k^2 c_\infty^2 + (-\omega^2 + k^2 c_0^2) &= 0 \\ j\omega \tau (-\omega^2 + k^2 c_\infty^2) + (-\omega^2 + k^2 c_0^2) &= 0 \end{aligned}$$

In the limit that  $\omega \tau \rightarrow 0$  (the sound's period is much larger than the relaxation time), the first term vanishes, resulting in wave propagation at  $c_0$ . In this limit, the change in pressure and density is so slow that "equilibrium is reestablished after each infinitesimal pressure change in the acoustical cycle" (Blackstock page 318). That is to say, the relaxation mechanism happens basically instantaneously compared to the period of the wave, and thus the wave propagates oblivious to the reaction mechanism, and the sound travels at the ambient speed of sound propagation,  $c_0$ , with the molecules relaxed for almost the entire time.

In the limit that  $\omega \tau \rightarrow \infty$  (the sound's period is much shorter than relaxation time) the first term dominates, resulting in wave propagation at  $c_\infty$ . In this limit, the frequency of the sound wave is so high that the pressure and density change much faster than the relaxation mechanism. Thus, over one period of sound, the molecules don't get the slightest chance to relax. They stay frozen in their "anxious" state.

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## Chapter 10: Spherical waves

1. Obtain the general solution to the wave equation in spherical coordinates. [ANSWER]

See the first section of Chap. 10 of Blackstock for the general solution of the wave equation in spherical coordinates.

2. Every second-order ordinary differential equation has two linearly independent solutions. For example,  $y'' + k^2y = 0$  has solutions  $\cos kx$  and  $\sin kx$ .  $y'' - k^2y = 0$  has solutions  $e^{kx}$  and  $e^{-kx}$ . Bessel's equation has solutions  $j_n(kr)$  and  $n_n(kr)$ . Why, then, does the solution of the spherical wave equation only contain *one* solution to Legendre's equation,  $P_n(\cos \theta)$  (or  $P_n^m(\cos \theta)$  in the most general case)? [ANSWER]

This was discussed in the "Orthogonality and Special Functions" section of the math page: see problem 2.  $Q_n$  diverges at the poles and is therefore neglected in the solution to the Helmholtz equation, because most problems in acoustics involve  $\theta = 0$  and  $\theta = \pi$ .

3. Is it possible for multiple eigenfunctions to map to the same eigenfrequency? Is it possible for multiple eigenfrequencies to map to the same eigenfunction? Why or why not? [ANSWER]

It is possible for multiple eigenfunctions to map to the same eigenfrequency. This is called degeneracy, as a student of quantum mechanics would know. For example, take the eigenfrequencies of sound in a cube is proportional to  $\sqrt{n^2 + m^2 + l^2}$ , where  $n$ ,  $m$ , and  $l$  are non-negative integers. The eigenfrequency corresponding to  $n = 1, m = 0, l = 0$  equals that corresponding to  $n = 0, m = 1, l = 0$ , which in turn equals that corresponding to  $n = 0, m = 0, l = 1$ . The degeneracy for this lowest energy is thus 3. For the degeneracy of the next highest energy would be 6.

However, it is not possible for multiple eigenfrequencies to map to the same eigenfunction. In other words, it is not possible for one eigenfunction to correspond to multiple eigenfrequencies. One can employ a simple proof by contradiction. Suppose for a nontrivial  $\mathbf{x}$  that  $A\mathbf{x} = \lambda\mathbf{x}$  and  $A\mathbf{x} = \mu\mathbf{x}$ , where  $\lambda \neq \mu$ . Then by subtraction,

$$0 = (\lambda - \mu)\mathbf{x}.$$

Since  $\mathbf{x} \neq 0$ , by the zero product property,  $\lambda - \mu = 0 \implies \lambda = \mu$ . But this contradicts the assumption that  $\lambda \neq \mu$ . Since the assumption led to a contradiction, the assumption must be false.

While the argument above has been written in the form of an eigenvalue problem of eigenvectors, eigenvalues, and matrices, it is identical to the argument that would be made for eigenfunctions, eigenfrequencies, and linear differential operators. Simply replace  $\mathbf{x}$  above with  $p(\mathbf{r})$ ,  $\lambda$  and  $\mu$  with  $\omega_1$  and  $\omega_2$ , and  $A$  with  $\nabla^2$ . The argument then shows that one eigenfunction of the Helmholtz equation cannot map to multiple eigenfrequencies.

4. Let  $n$  be either  $r$ ,  $\theta$ , or  $\psi$ . How does one satisfy the boundary conditions if the  $n$ th surface is pressure-release? How does one satisfy the boundary conditions if the  $n$ th surface is rigid?

[ANSWER]

**Pressure-release:** The condition is simply that  $p$  evaluated at the surface  $n$  vanishes. That is, if the pressure-release surface is specified at a particular radius,  $p(r = a) = 0$ ; if the pressure-release surface is specified at a particular polar angle  $\theta_0$ ,  $p(\theta = \theta_0) = 0$ ; if the pressure-release surface is specified at a particular azimuthal angle  $\psi_0$ ,  $p(\psi = \psi_0) = 0$ .

**Rigid:** The condition is that  $\partial p^{(n)} / \partial n = 0$ . For example, if the rigid surface is specified at a particular radius, the condition is  $\partial p^{(r)} / \partial r = 0$ ; if the rigid surface is specified at a particular polar angle, the condition is  $\partial p^{(\theta)} / \partial \theta = 0$ ; if the rigid surface is specified at a particular azimuthal angle, the condition is  $\partial p^{(\psi)} / \partial \psi = 0$ .

5. ✨ For a boundary condition on a Cartesian coordinate, show that the particle velocity attains an extremum at a pressure-release surface. Also show that the particle velocity is zero at a rigid surface. For a boundary condition on a spherical coordinate, does the particle velocity attain an extremum at a pressure-release surface? Is the particle velocity zero at a rigid surface? [ANSWER]

**Boundary condition on Cartesian coordinate.** The boundary condition at a pressure-release surface is that  $p = 0$ . Therefore, when the continuity equation  $\nabla \cdot (\rho \mathbf{u}) + \partial \rho / \partial t = 0$  is linearized and applied to a time-harmonic pressure, one obtains the condition

$$j\omega p + \rho_0 c_0^2 \nabla \cdot \mathbf{u} = 0, \quad (\text{i})$$

where the linear relation  $\rho' c_0^2 = p$  has been used to eliminate density. In Cartesian coordinates, equation (i) applied at the  $n$ th surface reads

$$j\omega p + \rho_0 c_0^2 \frac{\partial u^{(n)}}{\partial n} = 0,$$

from which it can be seen that at the pressure release surface,  $\frac{\partial u^{(n)}}{\partial n} = 0$ , i.e., the particle velocity is extremized.

Meanwhile the boundary condition at the  $n$ th rigid surface is that  $\partial p / \partial n = 0$ . Therefore, by the the linearized momentum equation for a time-harmonic particle velocity,

$$\nabla p + j\omega \rho_0 \mathbf{u} = 0, \quad (\text{i})$$

from which it can be see that at the rigid surface defined along coordinate  $n$ ,  $u^{(n)} = 0$ , i.e., the particle velocity vanishes.

**Boundary condition on a spherical coordinate.** The story is different for a boundary condition on a spherical coordinate. For example, at a pressure release surface defined at a particular radius, equation (i) reads

$$j\omega p + \rho_0 c_0^2 \left[ u_r^{(r)} + \frac{2}{r} u^{(r)} \right] = 0,$$

from which it can be seen that the radial component of the particle velocity does not go to zero when  $p = 0$  (Only in the limit  $r \rightarrow \infty$  does the particle velocity go to 0). This is a distinction between Cartesian boundary conditions and spherical boundary conditions.

However, for rigid surfaces defined along a spherical coordinate, equation (i) guarantees that the particle velocity vanishes at the boundary.

6. ✳ Do Blackstock problem 10-13 two different ways, first in the "conventional" orientation with spherical coordinates defined as the original figure is drawn, and second in a  $90^\circ$ -rotated coordinate system in which the flat part of the hemisphere is in the plane defined by  $\psi = 0$  and  $\psi = \pi$ . [ANSWER]

See [here](#) for the solution. Note that Blackstock's associated Legendre functions  $P_1^1, P_2^1, P_3^1$ , and  $P_3^3$  on page 348 follow a different convention than what is listed on Wikipedia, Wolfram, and most other references. These polynomials (and in general, those with odd  $m$ , are off by a factor of  $-1$  from the "normal" convention).

7. What is the degeneracy of modes in a spherical enclosure? *Bonus: how does this compare to the degeneracy in the hydrogen atom? Careful! I am using Blackstock's convention in which  $l$ , the orbital angular momentum number of quantum mechanics, is interchanged with  $m$ .* [ANSWER]

Note that the eigenfrequencies are independent of  $m$ . They only depend on  $n$  (the principal quantum number, if you will) and  $l$ , the index of the root of the Bessel function. Therefore, all the eigenfunctions with the same  $n$  and  $l$  indices but different values of  $m$  correspond to the same eigenfrequency. Recall that  $n = 0, 1, 2, \dots$  and  $m = 0, 1, 2, \dots, n$ . There are  $n + 1$  values of  $m$  for every  $n$ . The degeneracy is therefore  $n + 1$ . (If negative values of  $m$  were included, the degeneracy would be  $2m + 1$ ).

In the hydrogen atom, the degeneracy is  $n^2$ , and if spin is included,  $2n^2$ .

For more on parallels between acoustics and quantum mechanics related to spherical coordinates, see [these notes](#).

8. Solve the wave equation for the air trapped between two concentric pressure-release spheres, where the inner radius is  $a$  and the outer radius is  $b$ . Numerically obtain the eigenfrequencies. [ANSWER]

This is Blackstock's problem 10-10 of *Fundamentals of Physical Acoustics*, so I cannot post the full solution, but [displayed here](#) are numerically determined combinations of  $ka$  and  $kb$  that correspond to the eigenfrequencies for sound enclosed between two concentric pressure-release spheres. Bright spots on the surface plots correspond to eigenfrequencies.

9. Solve the wave equation in an axisymmetric spherical enclosure of radius  $a$  and angular distribution  $f(\theta)$ , such that the field at  $r = a$  is  $p(a, \theta, t) = p_0 f(\theta) e^{j\omega t}$ . [ANSWER]

See [here](#) for the solution.

10. ✳ Why do eigenfrequencies of sound spherical coordinates not depend on the azimuthal index  $m$ ? How do the eigenfrequencies of acoustic wave equation relate to the eigenfrequencies of the quantum mechanical equation? *Note: the letters conventionally used for the indices corresponding to spherical harmonics in acoustics are flipped from those used in quantum mechanics!* [ANSWER]

See [here](#) for a discussion. Thanks to J. S. Hallvold for his insights.

11. Attention is now turned to radiation problems. What limit on the dimensionless parameter  $kr$  of spherical radiation corresponds to far field radiation? What limit corresponds to near field radiation? [ANSWER]

$kr \gg 1$  corresponds to far-field radiation, and  $kr \ll 1$  corresponds to near-field radiation.

12. What does the combination of limits  $kr \gg 1$  and  $ka \ll 1$  correspond to physically? What kind of directivity does this limit reduce to? [ANSWER]

This is the subwavelength limit of the far field. The radiation is governed purely by diffraction and is in the form of a monopole, which is proportional to the volume velocity  $Q_0$ . The directivity of a monopole is 1.

13. And what does the combination of limits  $kr \gg 1$  and  $ka \gg 1$  correspond to? What kind of directivity does this limit reduce to? *Bonus: How do these limits relate to the structure of the cosmic microwave background radiation, and hence the distribution of galaxies in the universe today?* [ANSWER]

This is the geometric acoustics limit of the far field. The radiation is non-diffractive, and is simply a projection of the original source condition of rays to  $r = \infty$ .

The early universe was a plasma in which there existed an outward pressure due to photon-matter interactions, as well as an inward force due to gravity. This balance of attractive and repulsive force created a harmonic potential, giving rise to acoustic waves. The acoustic waves were spherical traveling waves propagating outward over vast distances. Thus the far field limit  $kr \gg 1$  appropriately describes these spherical waves (i.e., The propagation distance was much greater than the wavelength). Further, the sources of sound were much larger than a wavelength, i.e.,  $ka \gg 1$ , and thus the structure of the initial conditions was projected into the far field. When the universe cooled sufficiently such that the protons and neutrons combined to form hydrogen atoms, these atoms were arranged in the structure of the far-field geometric acoustic limit of the spherical acoustic oscillations. This so-called recombination (the formation of hydrogen) released photons that have redshifted (due to the expansion of the universe) into the microwave spectrum: this is the cosmic microwave background radiation. This initial distribution of matter, preserved as microwave radiation, explains the structure of the universe today, with denser regions giving rise to galaxies. See [this video](#) for a nice animation of these oscillations.

*Note that before recombination, photons could not travel freely; thus acoustic waves could propagate while electromagnetic waves could not. The old joke that sound came before light because "God said, 'Let there be light'" is actually true!*

14. Obtain the field radiated by a general axisymmetric spherical velocity source of radius  $a$ . Suppose the field at  $r = a$  has the form  $u_r(a, \theta, t) = u_0 f(\theta) e^{j\omega t}$ . Take the limit  $kr \rightarrow \infty$ . Then further apply the limits  $ka \ll 1$  and  $ka \gg 1$ . [ANSWER]

See [here](#) for the solution.

15. Solve the spherical radiation problem for the boundary conditions

$$u^{(r)}(a, \theta, t) = \begin{cases} u_0 e^{j\omega t}, & \text{for } \theta \in [0, \pi/2) \\ -u_0 e^{j\omega t}, & \text{for } \theta \in [\pi/2, \pi], \end{cases}$$

i.e., the bipolar pulsating sphere. Once the solution is obtained, take the limit as  $kr \gg 1$  and  $ka \ll 1$ . It will be helpful to note that  $h_n^{(2)}(kr) \rightarrow e^{j(\omega t - kr)}/r$  as  $kr \rightarrow \infty$ , and that  $1/h_n'(ka) \rightarrow j(ka)^3/2$  as  $ka \rightarrow 0$ . Also note that in this limit, the leading term corresponds to  $n = 1$  (because the  $n = 0$  term is 0). [ANSWER]

Noting that the solution must be time harmonic, suppress the linear combination

$$\begin{cases} e^{j\omega t} \\ e^{-j\omega t} \end{cases}$$

i.e., solve the Helmholtz equation. Note that there is no dependence on  $\psi$ , the azimuthal angle. Therefore  $m = 0$ . Also, the Hankel functions of the first kind are tossed since waves are outgoing. Let  $h_n^{(2)} \equiv h_n$ . Then general solution reads

$$p(r, \theta) = \sum_{n=0}^{\infty} A_n h_n(kr) P_n(\cos \theta). \quad (i)$$

To satisfy the velocity boundary conditions, the linearized momentum equation must be invoked for a time harmonic velocity:

$$\begin{aligned} u^{(r)} &= -\frac{1}{j\omega \rho_0} \frac{\partial p}{\partial r} \\ &= -\frac{A_n k}{j\omega \rho_0} h_n'(kr) P_n(\cos \theta) \\ &= -\frac{1}{j\rho_0 c_0} h_n'(kr) P_n(\cos \theta) \end{aligned}$$

Therefore, at  $r = a$ ,

$$-\frac{A_n}{j\rho_0 c_0} h_n'(ka) P_n(\cos \theta) = \begin{cases} u_0, & \text{for } \theta \in [0, \pi/2) \\ -u_0, & \text{for } \theta \in [\pi/2, \pi] \end{cases}$$

Thus

$$A_n P_n(\cos \theta) = -\frac{j\rho_0 c_0}{h_n'(ka)} \begin{cases} u_0, & \text{for } \theta \in [0, \pi/2) \\ -u_0, & \text{for } \theta \in [\pi/2, \pi] \end{cases}$$

To solve for the expansion coefficients  $A_n$ , multiply both sides by  $P_m(\cos \theta) \sin \theta d\theta$  and integrate:

$$\int_0^\pi A_n P_n(\cos \theta) P_m(\cos \theta) \sin \theta d\theta = -\frac{j\rho_0 c_0 u_0}{h'_n(ka)} \left[ \int_0^{\pi/2} P_m(\cos \theta) \sin \theta d\theta - \int_{\pi/2}^\pi P_m(\cos \theta) \sin \theta d\theta \right]$$

$$A_n \frac{2}{2n+1} = \frac{j\rho_0 c_0 u_0}{h'_n(ka)} \left[ \int_0^1 P_n(z) dz - \int_{-1}^0 P_n(z) dz \right]$$

$P_n(z) = P_n(-z)$  is for even  $n$ , so the term in the square brackets is 0 in that case. Meanwhile,  $P_n(-z) = -P_n(z)$  for odd  $n$ , so

$$A_n = (2n+1) \frac{j\rho_0 c_0 u_0}{h'_n(ka)} \int_0^1 P_n(z) dz \quad (\text{ii})$$

To evaluate the integral, use the property  $(2n+1)P_n(z) = P'_{n+1}(z) - P'_{n-1}(z)$ , which is integrated trivially using the fundamental theorem of calculus:

$$\int_0^1 P_n(z) dz = \frac{1}{2n+1} \int_0^1 [P'_{n+1}(z) - P'_{n-1}(z)] dz$$

$$= \frac{1}{2n+1} [P_{n+1}(1) - P_{n-1}(1) - P_{n+1}(0) + P_{n-1}(0)].$$

Since the Legendre polynomials are normalized such that  $P_n(1) = 1$ , the first two terms above cancel, giving

$$\int_0^1 P_n(z) dz = \frac{1}{2n+1} [-P_{n+1}(0) + P_{n-1}(0)].$$

The two Legendre polynomials above can be combined into one by noting that  $(2n+1)zP_n(z) = (n+1)P_{n+1}(z) + nP_{n-1}(z)$ , or for  $z = 0$ ,

$$-P_{n+1}(0) = \frac{n}{n+1} P_{n-1}(0)$$

Thus the integral becomes

$$\int_0^1 P_n(z) dz = \frac{1}{2n+1} \left[ \frac{2n+1}{n+1} P_{n-1}(0) \right] = (n+1)^{-1} P_{n-1}(0).$$

and equation (ii) becomes

$$A_n = \frac{2n+1}{n+1} \frac{j\rho_0 c_0 u_0}{h'_n(ka)} P_{n-1}(0)$$

The problem is solved by inserting the coefficients above into equation (i):

$$p(r, \theta) = j\rho_0 c_0 u_0 \sum_{n=0}^{\infty} \frac{2n+1}{n+1} \frac{h_n(kr)}{h'_n(ka)} P_{n-1}(0) P_n(\cos \theta) \quad (\text{iii})$$

Taking the limits of equation (iii) for  $n = 1$  (the leading order term) gives

$$p = -\frac{3}{4} \rho_0 c_0 u_0 (ka)^3 \frac{e^{j(\omega t - kr)}}{r} \cos \theta,$$

i.e., the dipole directivity pattern.

16. Write the pressure field due to a pair of antiphase monopole separated by distance  $2h$  in the far field limit,  $r \gg h$ . [ANSWER]

See [here](#) for the solution.

17. Calculate the pressure field due to a translating sphere, for which the boundary condition is

$$u^{(r)}(r = a) = u_0 e^{j\omega t} \cos \theta.$$

Note that  $h_1(kr) = e^{-jkr}(1 + 1/jkr)/kr$ . Evaluate the pressure field in the subwavelength limit, noting that  $h'_1(ka) \rightarrow 2/jka^3$  for  $ka \ll 1$ . Identify the directivity in this limit. [ANSWER]

See [here](#) for the solution. Writing  $h_1(kr) = e^{-jkr}(1 + 1/jkr)/kr$  and evaluating the solution in the limit that  $ka \ll 1$ , the solution becomes

$$p = \frac{1}{2} k^2 a^3 \rho_0 c_0 u_0 \cos \theta (1 + 1/jkr) e^{j(\omega t - kr)} / r.$$

The directivity in this limit is that of a dipole:  $\cos \theta$ .

18. An acoustic monopole is a \_\_\_\_\_ source, while an acoustic dipole is a \_\_\_\_\_ source. [ANSWER]

volume velocity (or mass); force. Note that the force exerted by the dipole is proportional to the so-called "entrained mass" of the fluid.

19. Create a dipole by positioning two out-of-phase monopoles a distance  $2h$  apart along the line  $\theta = 0$ . Take the  $r \gg h$  limit and thus derive the dipole directivity pattern. [ANSWER]

The sum of the monopoles is given (exactly) by

$$p = \frac{A}{r_1} e^{j(\omega t - kr_1)} - \frac{A}{r_2} e^{j(\omega t - kr_2)}.$$

In the far field, the denominators of the amplitudes are approximated  $r$ , and the phases are approximated by  $r_1 \simeq r - h \cos \theta$  and  $r_2 \simeq r + h \cos \theta$ . This leads to

$$p = \frac{2Aj}{r} e^{j(\omega t - kr)} \sin(kh \cos \theta). \quad \text{pair of monopoles}$$

We know that this result is in the far field because the angular and radial dependence is separated. Now letting  $kh \ll 1$ , the pressure approaches that of a point dipole:

$$p = \frac{2Aj}{r} kh \cos \theta e^{j(\omega t - kr)}.$$

20. Calculate the power  $W$  radiated by a pair of out-of-phase monopoles. Using the expression for the [pair of monopoles](#) derived in the previous problem, note that  $p/p_{\text{free}} = 2j \sin(kh \cos \theta)$ , since  $p_{\text{free}} = A e^{j(\omega t - kr)} / r$ . [ANSWER]

The power is given by the surface integral of the intensity. The intensity is calculated by noting that  $\langle I \rangle / \langle I_{\text{free}} \rangle = |p/p_{\text{free}}|^2 = 4 \sin^2(kh \cos \theta)$ . Thus  $\langle I \rangle = \langle I_{\text{free}} \rangle 4 \sin^2(kh \cos \theta)$ .

$$\begin{aligned} W &= 4 \langle I_{\text{free}} \rangle \oint \sin^2(kh \cos \theta) \mathbf{e}_r \cdot d\mathbf{A} \\ &= 4 \langle I_{\text{free}} \rangle \int_0^{\pi/2} \int_0^{2\pi} \sin^2(kh \cos \theta) r^2 \sin \theta d\theta d\psi \\ &= 8 \langle I_{\text{free}} \rangle \pi r^2 \int_0^{\pi/2} \sin^2(kh \cos \theta) \sin \theta d\theta \end{aligned}$$

The integral above is taken by letting  $q = \cos \theta$ . The result is

$$W = 4 \pi r^2 I_{\text{free}} [1 - \sin(2kh)/2kh].$$

Note that  $4 \pi r^2 \langle I_{\text{free}} \rangle = W_{\text{free}}$ , the power radiated by a monopole, so the above result can be written as

$$W = W_{\text{free}} [1 - \sin(2kh)/2kh].$$

Note that Blackstock does not use the  $\langle \dots \rangle$  notation to denote "time average of ...." Blackstock's " $I$ " corresponds to  $\langle I \rangle$  on this site.

21. Create a longitudinal quadrupole by positioning two out-of-phase dipoles along the line  $\theta = 0$ , separated by distance  $h$ . Recall that the pressure field due to each dipole is given by the equation for the [pair of monopoles](#). Take the  $kh \gg 1$  limit to obtain the directivity of a point quadrupole.

[ANSWER]

The pressure field is given by

$$p = p_d [e^{jkh \cos(\theta)/2} - e^{-jkh \cos(\theta)/2}]$$

where

$$p_d = \frac{2A_j}{r} kh \cos \theta e^{j(\omega t - kr)}.$$

Putting the equations together gives

$$p = \frac{2A_j}{r} kh \cos \theta e^{j(\omega t - kr)} \sin[kh \cos(\theta)/2] \quad (i)$$

Taking the  $kh \ll 1$  limit of equation (i) gives the directivity a point quadrupole:

$$p = \frac{A_j}{r} (kh)^2 \cos^2 \theta e^{j(\omega t - kr)}.$$

## Chapter 11: Cylindrical waves

1. Obtain the general solution to the wave equation in cylindrical coordinates. *[ANSWER]*

See the first section of Chap. 11 of Blackstock for the general solution of the wave equation in cylindrical coordinates.

2. ✳ In *spherical* solutions to the wave equation, what are the names of the three indices used? What do they correspond to physically? What possible values can the indices equal? In *cylindrical* coordinates, what indices are used, what do they correspond to physically, and what are their possible values? Comment on the similarities and differences. *[ANSWER]*

Note that Blackstock's convention is used in the following discussion.

In *spherical* coordinates,  $n = 0, 1, 2, \dots$  corresponds to the order of the spherical Bessel function.  $n$  also appears as the index of the Legendre polynomial and the lower index of the associated Legendre function. Meanwhile,  $l = 1, 2, 3, \dots$  corresponds to the zero of the spherical Bessel function. Finally,  $m = 0, 1, 2, \dots, n$  appears in the harmonic azimuthal dependence, as well as in the top index of the associated Legendre polynomial. *The eigenfrequencies in spherical coordinates depend only on  $n$  and  $l$  in even the most general case.*

In *cylindrical* coordinates,  $m = 0, 1, 2, \dots$  is the order of the Bessel function, while  $n = 1, 2, 3, \dots$  corresponds to the zero of the Bessel function.  $m$  is also the index corresponding to the harmonic polar dependence.  $l = 1, 2, 3, \dots$  (and sometimes 0) is the index corresponding to the  $z$ -dependence. *The eigenfrequencies in cylindrical coordinates depend on all three indices in the most general case.*

One similarity between spherical and cylindrical solutions is that the Bessel functions and polar functions in either case *share the same index*. In spherical coordinates, that index is  $n$ ; in cylindrical coordinates, that index is  $m$ . *In light of this, I wish Dr. Blackstock had used  $n$  instead of  $m$  for the index in cylindrical coordinates, to preserve this similarity with spherical coordinates.*

One difference between spherical and cylindrical solutions is that  $n$  in spherical coordinates is necessarily an integer, but  $m$  in cylindrical coordinates can taken on non-integer (and in fact irrational) values. This happens in cylindrical wedge problems. In this case, the Latin  $m$  is replaced with the Greek  $\mu$ .

One should not be militant about the names of indices; it is much more important to have the *form of solution* correct. In summary, the general solution in spherical coordinates is given by

$$R(r)\Theta(\theta)\Psi(\psi) = \begin{Bmatrix} j_n(k_n r) \\ n_n(k_n r) \end{Bmatrix} \begin{Bmatrix} P_n^m(\cos \theta) \\ Q_n^m(\cos \theta) \end{Bmatrix} \begin{Bmatrix} \cos m\psi \\ \sin m\psi \end{Bmatrix} \begin{Bmatrix} \cos \omega t \\ \sin \omega t \end{Bmatrix}$$

while that in cylindrical coordinates is given by

$$R(r)\Theta(\theta)Z(z) = \begin{Bmatrix} J_m(k_{mn}r) \\ N_m(k_{mn}r) \end{Bmatrix} \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \begin{Bmatrix} \cos lz \\ \sin lz \end{Bmatrix} \begin{Bmatrix} \cos \omega t \\ \sin \omega t \end{Bmatrix}$$

3. Obtain the complete solution to the 2D wave equation for a circular drumhead that is clamped at radius  $r = a$  and that is struck at  $r = 0$  at  $t = 0$ . Let  $\eta$  be the displacement of the drumhead. The initial conditions are that the drumhead is initially silent,  $\eta = 0$ , and the initial velocity is  $\dot{\eta} = v_0\delta(r)/2\pi r$ . [ANSWER]

Start with the general solution to the wave equation in  $r$  and  $\theta$ . Neglect the Neumann functions since the origin is included.

$$\eta(r, \theta, t) = J_m(kr) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \begin{Bmatrix} \cos \omega t \\ \sin \omega t \end{Bmatrix}$$

Applying the radial boundary condition gives the eigenfrequencies:

$$J_m(ka) = 0 \quad \Rightarrow \quad k_{mn} = \frac{\alpha_{mn}}{a} \quad \Rightarrow \quad f_{mn} = \frac{c_0\alpha_{mn}}{2\pi a},$$

where  $\alpha_{mn}$  is the  $n$ th root of the Bessel function of order  $m$ . The angular dependence can be written more conveniently as  $\cos(m\theta + \psi_m)$ . Thus the general solution becomes

$$\eta(r, \theta, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} J_m(\alpha'_{mn}r/a) \cos(m\theta + \psi_m) (C_{mn} \cos \omega_{mn}t + D_{mn} \sin \omega_{mn}t)$$

The condition that the drumhead is initially silent implies that  $C_{mn} = 0$ . Also, since the initial condition is independent of  $\theta$ ,  $m = 0$ :

$$\eta(r, \theta, t) = \sum_{n=1}^{\infty} D_n J_0(\alpha_{0n}r/a) \sin \omega_n t \quad (\text{i})$$

To determine  $D_n$ , the initial conditions  $\eta = 0$  and  $\dot{\eta} = v_0\delta(r)/2\pi r$  at  $t = 0$  are utilized. Thus,

$$\dot{\eta} = \frac{v_0\delta(r)}{2\pi r} = \sum_{n=1}^{\infty} D_n \omega_n J_0(\alpha_{0n}r/a)$$

Multiplying both sides by  $J_0(\alpha_{0n'}r/a)rdr$  and integrating from  $r = 0$  to  $r = a$  gives

$$\frac{v_0}{2\pi} \int_0^a \delta(r) J_0(\alpha_{0n'}r/a) dr = \sum_{n=1}^{\infty} \omega_n D_n \int_0^a J_0(\alpha_{0n'}r/a) J_0(\alpha_{0n}r/a) r dr \quad (\text{ii})$$

The integral on the left-hand side is trivial by the sifting property of the delta function.

$$\frac{v_0}{2\pi} \int_0^a \delta(r) J_0(\alpha_{0n'}r/a) dr = \frac{v_0}{2\pi} J_0(0) = \frac{v_0}{2\pi}$$

The right-hand side of equation (ii) is the orthogonality relation for the order 0 Bessel function. Making the substitution  $x \equiv r/a$  makes this more clear:

$$\begin{aligned}\int_0^a J_0(\alpha_{0n}r/a)J_0(\alpha_{0n}r/a)rdr &= a^2 \int_0^1 J_0(\alpha_{0n}x)J_0(\alpha_{0n}x)xdx \\ &= \frac{a^2}{2} [J_0'(\alpha_{0n})]^2 \delta_{nn'}\end{aligned}$$

Thus equation (ii) becomes

$$\frac{v_0}{2\pi} = \frac{a^2}{2} \sum_{n=1}^{\infty} \omega_n D_n [J_0'(\alpha_{0n})]^2 \delta_{nn'}$$

Solving for  $D_n$  gives

$$D_n = \frac{v_0}{[J_0'(\alpha_{0n})]^2 \omega_n \pi a^2}.$$

The complete solution is found by substituting the above into equation (i):

$$\eta(r, \theta, t) = \frac{v_0}{\pi a^2} \sum_{n=1}^{\infty} \frac{J_0(\alpha_{0n}r/a)}{[J_0'(\alpha_{0n})]^2 \omega_n} \sin \omega_n t$$

4. A driven circular drumhead obeys the inhomogeneous wave equation

$$\nabla^2 \eta - \frac{1}{c^2} \eta_{tt} = -\frac{p_s}{\mathcal{T}_l},$$

where  $p_s$  is the drive function, considered here to be  $p_s = p_0 e^{j\omega t}$ . Describe a situation in which this is the drive function of the drumhead. Solve the inhomogeneous wave equation. *[ANSWER]*

To solve the inhomogeneous wave equation, first note that the time dependence will be  $e^{j\omega t}$  (that of the drive frequency), and that the only spatial dependence will be radial, i.e.,  $\eta(r, t) = R(r)e^{j\omega t}$ . Thus the inhomogeneous PDE to solved reduces to an inhomogeneous ODE:

$$R'' + \frac{1}{r}R' + k^2R = -\frac{p_0}{\mathcal{T}}.$$

The solution to this ODE is the sum of the homogeneous and inhomogeneous parts. The homogeneous solution is the solution to Bessel's equation of order 0:  $A J_0(kr)$ . For the inhomogeneous solution, it is noted that the right-hand side is constant, so a trial solution  $C$  is used giving  $C = -p_0/k^2\mathcal{T}$ . Thus the solution to the ODE is

$$R(r) = A J_0(kr) - \frac{p_0}{k^2\mathcal{T}}.$$

Applying the boundary condition that  $R = 0$  at  $r = a$  gives

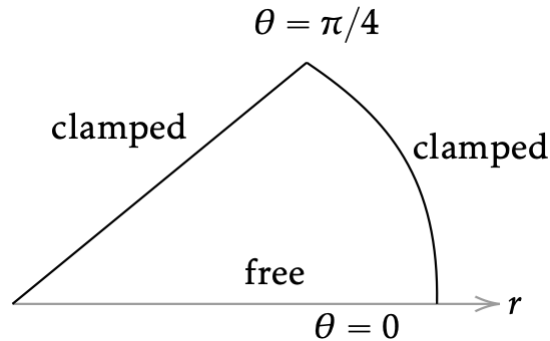
$$A = \frac{p_0}{k^2\mathcal{T}J_0(ka)}$$

The solution is thus

$$\eta(r, t) = \frac{p_0}{k^2\mathcal{T}} \left[ \frac{J_0(kr)}{J_0(ka)} - 1 \right] e^{j\omega t}$$

Note that when  $ka = \alpha_{0n}$ , the response goes to  $\infty$ .

5. **Part 1.** Solve the 2D wave equation in  $\eta(r, \theta)$  for a pie-shaped drumhead subtending  $45^\circ$ , where one edge, say  $\theta = 0$ , is free, while the curved boundary at  $r = a$  and the other edge at  $\theta = 45^\circ$  are clamped:



Find the eigenfrequencies and identify the lowest one. **Part 2.** Suppose there is an initial displacement on the membrane of  $\eta_0$ , which would correspond to a "plucked" initial condition. Also assume there is no initial velocity:

$$\eta(r, \theta, 0) = \begin{cases} \eta_0, & \theta = [0, \pi/4), r = [0, a] \\ 0, & \theta = [\pi/4, 2\pi) \end{cases}$$

$$\dot{\eta}(r, \theta, 0) = 0$$

Apply the initial condition to the eigenfunctions found in part 1 to find  $C_{mn}$  and  $D_{mn}$ . Leave the relevant expansion coefficients in integral form, and evaluate the integral for  $m = 0$  only. Compare the coefficients for  $m = 0$  to Blackstock's equation (11.B-13) on page 400 for a circular drumhead clamped at  $r = a$  subject to the analogous "plucked" initial condition. [ANSWER]

See [here](#) for the solution.

6. What is the general solution to the 3D wave equation for sound in a hollow cylinder of length  $L$  and radius  $a$  with rigid walls? Which eigenfrequencies recover those corresponding purely harmonic axial modes (no propagation in  $r$  and  $\theta$ )? Which eigenfrequencies correspond to purely radial modes (no dependence in  $\theta$  and  $z$ )? Is it possible to have pure spinning modes? [ANSWER]

See [here](#) for the solution. Setting  $m = 0$  and  $n = 1$  results in purely axial modes because  $\alpha'_{01} = 0$ , and thus the radial dependence is eliminated  $J_0(0) = 1$ . Setting  $m = 0$  and  $l = 0$  gives purely radial modes because the angular dependence and axial dependence are both eliminated in that case.

It is not possible to have pure spinning modes. One way to rationalize this is to note that there is no "angular wavenumber." One can also see by inspection that there must exist radial modes if there exist angular modes, because both depend on the same index  $m$ .

7. Solve the wave equation for sound in between two concentric cylinders of length  $L$ , inner radius  $a$ , and outer radius  $b$ . Let all boundaries be pressure-release. Do not attempt to find the eigenfrequencies explicitly. [ANSWER]

See [here](#) for the solution.

8. Calculate the pressure field due to a uniformly pulsating cylinder, for which the velocity boundary condition is  $u^{(r)}(r = a) = u_0 e^{j\omega t}$ . What makes the near field of the impedance of the uniformly pulsating cylinder interesting? [\[ANSWER\]](#)

See [here](#) for the solution. The near field impedance can be calculated given the  $kr \ll 1$  limit of the Hankel function. What makes it interesting is that it depends on  $c_0$ . In contrast, the near field spherical impedance relation is independent of  $c_0$ .

9. Calculate the pressure field due to a translating cylinder, for which the velocity boundary condition is  $u^{(r)}(r = a) = u_0 \cos \theta e^{j\omega t}$ . [\[ANSWER\]](#)

See [here](#) for the solution.

10. Consider a waveguide with pressure-release surfaces at  $z = 0$  and  $z = D$ . A vertically oriented cylindrical source of radius  $a$  extending from  $z = 0$  to  $z = D$  pulses radially, i.e.,  $u(a, \theta, z, t) = u_0 e^{j\omega t}$ . Solve the pressure wave equation for this configuration [\[ANSWER\]](#)

[Here](#) is the solution for (1) a radially pulsating cylindrical source of sound extending between two pressure-release parallel planes. Prof. Blackstock mentions this kind of waveguide on page 432 in *Fundamentals of Physical Acoustics*. He solves for (2) a radially pulsating cylindrical source between two rigid, parallel planes on page 430, and assigns the case of (3) a radially pulsating cylindrical source between one rigid boundary and one pressure-release boundary (a 0th-order model of sound in the ocean) as problem 12-13. Interestingly, case (1) and (3) excite many modes, but case (2) excites only the lowest mode.

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## Chapter 12: Waveguides

1. Find the eigenmodes, cutoff frequency, phase speed, and group speed for an infinite rectangular duct of length  $a$  and height  $b$ , where  $\partial p / \partial x = 0$  from  $x \in [0, a]$  and  $\partial p / \partial y = 0$  from  $y \in [0, b]$ . For what drive frequencies  $\omega$  does the sound propagate? Why does the drive frequency  $\omega$  not bear the indices of the modes? [ANSWER]

The general solution is

$$p(x, y, z, t) = \begin{Bmatrix} \cos k_x x \\ \sin k_x x \end{Bmatrix} \begin{Bmatrix} \cos k_y y \\ \sin k_y y \end{Bmatrix} e^{j(\omega t - k_z z)}.$$

where the relationship between wavenumbers is

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}.$$

Applying the boundary conditions at  $x = 0$  and  $y = 0$  eliminates the  $\sin k_x x$  and  $\sin k_y y$  terms, and applying the boundary conditions at  $x = a$  and  $y = b$  gives

$$\begin{aligned} \sin k_x a = 0 &\quad \Rightarrow \quad k_x = \frac{m\pi}{a} \\ \sin k_y b = 0 &\quad \Rightarrow \quad k_y = \frac{n\pi}{b} \end{aligned}$$

Thus the projection of the wavenumber in the  $z$ -direction is

$$\beta_{mn} = \sqrt{(\omega/c_0)^2 - (m\pi/a)^2 - (n\pi/b)^2}. \quad (i)$$

$\omega$  above does not bear the modal indices  $m$  and  $n$  because it is the frequency at which the system is *driven*. In other words,  $\omega$  is not an *eigenfrequency*. The cutoff frequency, however, *does* bear the indices  $m$  and  $n$  because it depends purely on the system, i.e., it is independent of the nature of the excitation. The cutoff frequency is the frequency below which  $\beta_{mn}$  is evanescent. It is found by setting the radicand ("the thing under the square root") equal to 0 and solving for  $\omega^{(c)}$ :

$$\begin{aligned} 0 &= [\omega^{(c)}/c_0]^2 - (m\pi/a)^2 - (n\pi/b)^2 \\ \omega_{mn}^{(c)} &= c_0 \sqrt{(m\pi/a)^2 + (n\pi/b)^2} \\ f_{mn}^{(c)} &= \frac{c_0}{2} \sqrt{(m/a)^2 + (n/b)^2}. \end{aligned}$$

Thus, since  $[\omega_{mn}^{(c)}]^2/c_0^2 = (m\pi/a)^2 + (n\pi/b)^2$ , equation (i) can be written as

$$\begin{aligned} \beta_{mn} &= \frac{1}{c_0} \sqrt{\omega^2 - [\omega_{mn}^{(c)}]^2} \\ &= \frac{\omega}{c_0} \sqrt{1 - [\omega_{mn}^{(c)}/\omega]^2} \end{aligned}$$

From the expression above, it can be seen that  $\beta_{mn}$  is real and positive if the drive frequency  $\omega$  is greater than the cutoff frequency  $\omega_{mn}^{(c)}$ , i.e., *the  $m$   $n^{\text{th}}$  mode cannot propagate below the  $m$   $n^{\text{th}}$  cutoff frequency.*

Meanwhile, the phase speed is  $c_{\text{ph}} = \omega/k_z = \omega/\beta_{mn}$ :

$$c_{\text{ph}} = \frac{c_0}{\sqrt{1 - [\omega_{mn}^{(c)}/\omega]^2}} .$$

The group speed is  $c_{\text{gr}} = d\omega/d\beta_{mn}$ :

$$\begin{aligned} c_{\text{gr}} &= \left[ \frac{d\beta_{mn}}{d\omega} \right]^{-1} \\ &= \left[ \frac{1}{c_0 \sqrt{1 - (\omega_{mn}^{(c)}/\omega)^2}} \right]^{-1} \\ &= c_0 \sqrt{1 - (\omega_{mn}^{(c)}/\omega)^2} \end{aligned}$$

2. Find the eigenmodes, cutoff frequency, phase speed, and group speed for an infinite cylindrical duct of radius  $a$ , where  $\partial p/\partial r = 0$  at  $r = a$ . For what drive frequencies  $\omega$  does the sound propagate? Why does the drive frequency  $\omega$  not bear the indices of the modes? [\[ANSWER\]](#)

The general solution is

$$p(r, \theta, z, t) = J_m(k_r r) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} e^{j(\omega t - k_z z)} .$$

where the relationship between wavenumbers is

$$k_z = \sqrt{k^2 - k_r^2} .$$

Applying the boundary condition at  $x = a$  gives

$$J'_m(k_r a) = 0 \quad \Rightarrow \quad k_r = \frac{\alpha'_{mn}}{a}$$

Thus the projection of the wavenumber in the  $z$ -direction is

$$\beta_{mn} = \sqrt{(\omega/c_0)^2 - (\alpha'_{mn}/a)^2} . \tag{i}$$

The cutoff frequency is found by setting

$$\begin{aligned} 0 &= [\omega^{(c)}/c_0]^2 - (\alpha'_{mn}/a)^2 \\ \omega_{mn}^{(c)} &= c_0 \alpha'_{mn}/a \\ f_{mn}^{(c)} &= c_0 \alpha'_{mn}/2\pi a . \end{aligned}$$

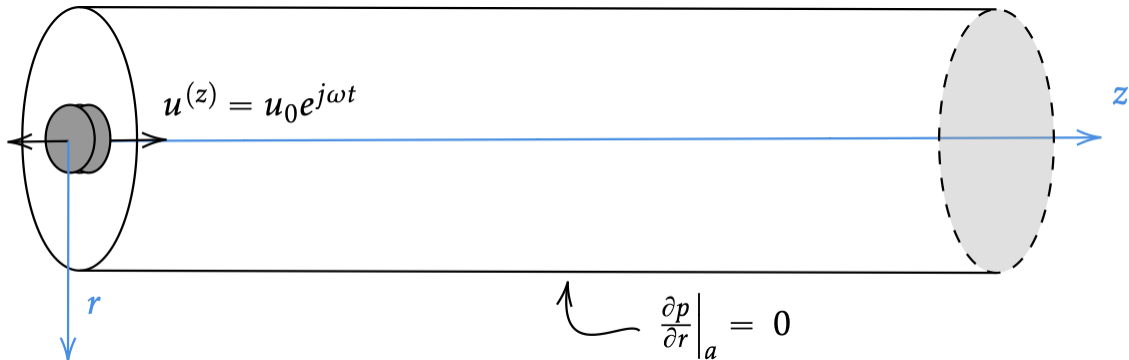
Thus, equation (i) can be written as

$$\begin{aligned}\beta_{mn} &= \frac{1}{c_0} \sqrt{\omega^2 - [\omega_{mn}^{(c)}]^2} \\ &= \frac{\omega}{c_0} \sqrt{1 - [\omega_{mn}^{(c)}/\omega]^2}\end{aligned}$$

As is the case for the rectangular duct, the  $m$   $n^{\text{th}}$  mode cannot propagate below the  $m$   $n^{\text{th}}$  cutoff frequency.

The phase and group speeds have the same functional form as for the rectangular waveguide; the expression for the cutoff frequency is the only difference.

3. ✳ Calculate the pressure field due to a baffled circular piston of radius  $b$  positioned with its center on the axis of a cylindrical tube of radius  $a$ .



The boundary conditions at  $z = 0$  are

$$\begin{aligned}u_z &= u_0 e^{j\omega t}, & r \in [0, b] \\ u_z &= 0, & r \in (b, a]\end{aligned}$$

and  $\partial p / \partial r = 0$  at  $r = a$  for all  $z$ . Note that

$$\int_0^1 J_m(\alpha'_{mn} x) J_m(\alpha'_{mn'} x) x dx = \frac{1}{2} [1 - (m/\alpha'_{mn})^2] J_m^2(\alpha'_{mn'}) \delta_{nn'}$$

is the orthogonality integral. [ANSWER]

First of all, there should be no dependence on  $\theta$ , because the piston is positioned on-axis. Thus  $m = 0$ , and the general solution is

$$p(r, z, t) = \sum_n A_n J_0(k_r r) e^{j(\omega t - k_z z)}$$

Since  $\partial p / \partial r = 0$  at  $r = a$ ,

$$J'_0(k_r a) = 0 \quad \Rightarrow \quad k_r = \alpha'_{0n} / a.$$

Thus,  $k_z = \beta_n = \sqrt{(\omega/c_0)^2 - (\alpha'_{0n}/a)^2}$ . Meanwhile, to satisfy the boundary condition at  $z = 0$ , the momentum equation is invoked and set equal to the boundary condition:

$$\begin{aligned}
u^{(z)} &= -\frac{1}{j\omega\rho_0} \frac{\partial p}{\partial z} \\
&= \frac{1}{\rho_0 c_0 k} \sum_n A_n \beta_n J_0(\alpha'_{0n} r/a) e^{j\omega t} = \begin{cases} u_0 e^{j\omega t}, & r \in [0, b] \\ 0, & r \in (b, a] \end{cases}. \quad (i)
\end{aligned}$$

The orthogonality of the Bessel functions is used to find the expansion coefficient  $A_n$ . Letting  $x = r/a$  and thus  $dx = dr/a$ , the orthogonality integral becomes

$$\int_0^a J_m(\alpha'_{mn} r/a) J_m(\alpha'_{mn'} r/a) r dr = \frac{a^2}{2} [1 - (m/\alpha'_{mn})^2] J_m^2(\alpha'_{mn} x) \delta_{nn'},$$

which for  $m = 0$  reads

$$\int_0^a J_0(\alpha'_{0n} r/a) J_0(\alpha'_{0n'} r/a) r dr = \frac{a^2}{2} J_0^2(\alpha'_{0n}) \delta_{nn'}.$$

Multiplying both sides of equation (i) by  $J_0(\alpha'_{0n'} r/a) r dr$  gives

$$\frac{1}{\rho_0 c_0 k} \sum_n \beta_n A_n J_0(\alpha'_{0n} r/a) J_0(\alpha'_{0n'} r/a) r dr = \begin{cases} u_0 J_0(\alpha'_{0n'} r/a) r dr, & r \in [0, b] \\ 0, & r \in (b, a] \end{cases}$$

Integrate from 0 to  $a$ :

$$\begin{aligned}
\frac{1}{\rho_0 c_0 k} \sum_n \beta_n A_n \int_0^a J_0(\alpha'_{0n} r/a) J_0(\alpha'_{0n'} r/a) r dr &= \begin{cases} u_0 \int_0^a J_0(\alpha'_{0n'} r/a) r dr, & r \in [0, b] \\ 0, & r \in (b, a] \end{cases} \\
\frac{1}{\rho_0 c_0 k} \sum_n \beta_n A_n \int_0^a J_0(\alpha'_{0n} r/a) J_0(\alpha'_{0n'} r/a) r dr &= u_0 \int_0^b J_0(\alpha'_{0n'} r/a) r dr
\end{aligned}$$

The orthogonality relation above for  $m = 0$  is used to integrate the left-hand side, and the recursion relation given by equation (11-A-24c) in Blackstock's book is used to integrate the right-hand side:

$$\begin{aligned}
\frac{a^2}{\rho_0 c_0 k} \sum_n \beta_n A_n \frac{1}{2} J_0^2(\alpha'_{0n}) \delta_{nn'} &= u_0 (a/\alpha'_{0n'})^2 \nu J_1(\nu) \Big|_{\nu=0}^{\nu=\alpha'_{0n'} b/a} \\
\frac{a^2}{2\rho_0 c_0} \frac{\beta_n}{k} A_n J_0^2(\alpha'_{0n}) &= u_0 a b \frac{1}{\alpha'_{0n}} J_1(\alpha'_{0n} b/a)
\end{aligned}$$

Solve for  $A_n$ :

$$A_n = \rho_0 c_0 u_0 \frac{b}{a} \frac{2k}{\beta_n \alpha'_{0n}} \frac{J_1(\alpha'_{0n} b/a)}{[J_0(\alpha'_{0n})]^2}.$$

For more on this problem, see [here](#).

4. ✨ Write the *spinning mode* eigenfunction of the wave equation that has a spiral wavefront. What is the relationship between a spinning mode and a vortex beam? What is the equation for surfaces of constant phase? What is the angular velocity of each mode? What is  $m$  called?

[ANSWER]

This is an unlikely question because it has to do with my research. The spinning mode eigenfunctions are

$$p_{mn} = A_{mn} J_m(\alpha'_{mn} r/a) e^{j(\omega t - \beta_{mn} z \pm m\theta)} .$$

In general, a *beam* refers to the radiation of sound in free space subject to diffraction (see the next section). However, it is possible (theoretically, at least) to consider a beam in free space that does not diffract (although its source condition spans the entire source plane, and although it requires an infinite amount of energy to produce. Such a non-diffracting beam simply consists of an eigenmode of the wave equation, i.e., products of Bessel functions in  $r$  and harmonic functions in  $\theta$  and  $z$ . As such, these beams are called *Bessel beams*. They are attractive to researchers because they remove diffraction from problems and greatly simplify analysis, given the fact that Bessel functions have been studied extensively.

The relationship between a spinning mode in a waveguide and a Bessel vortex beam is that they have identical mathematical descriptions, though the spinning mode in a waveguide is subject to radial boundary conditions while a vortex beam is unconstrained in  $r$ .

The surface of constant phase is given by

$$\omega t - \beta_{mn} z \pm m\theta = \text{const.}$$

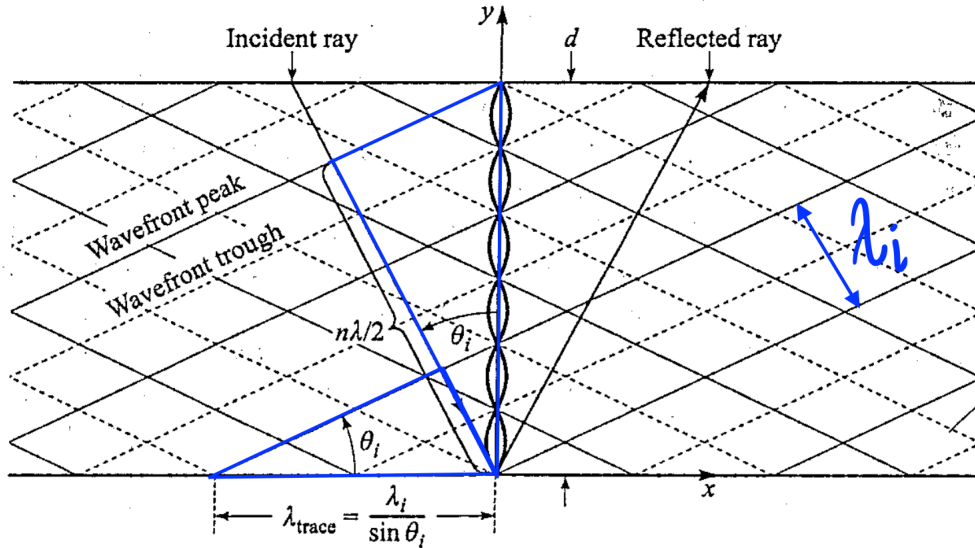
At a given  $z$ , the differential expression is  $\omega dt \pm m d\theta$ , and thus the angular velocity of the  $m^{\text{th}}$  mode is

$$\frac{d\theta}{dt} = -\omega/m.$$

5. ✪ Provide a physical interpretation for the phase speed, cutoff frequency, and group speed in a waveguide. *Hint: It is easiest to think of a two-dimensional waveguide for this discussion.*

[ANSWER]

Think of rectangular drumhead that is clamped at  $y = 0$  and  $y = d$ , and that is infinite in the  $x$  direction. Let  $\theta_i$  be the angle of an incident ray, and let the wavelength of the incident ray be  $\lambda_i$ , as in Blackstock's figure 6.14:



**Figure 6.14** Geometry of the wave field in the waveguide. The  $n$  half wavelengths marked  $n\lambda/2$  along the incident ray project  $n$  loops (on the  $y$  axis) that fit exactly in the waveguide. The fit determines the angle  $\theta_i$  at which the wave bounces along the waveguide.

Then, the component of this wavelength on the  $x$  axis is  $\lambda_{\text{trace}} = \lambda_i / \sin \theta_i$ . The speed at which this projection travels is

$$c_n^{\text{ph}} = f \lambda_{\text{trace}} = \frac{f \lambda}{\sin \theta_i} = \frac{c}{\sin \theta_i} = \frac{c}{\sqrt{1 - \cos^2 \theta_i}}.$$

The superposition of the incident and reflected wave creates a vertical standing wave pattern that has  $n$  loops. From geometry, the  $n$  loops have a projection of  $n\lambda/2$  wavelengths on the incident ray. That is,  $\cos \theta_i = n\lambda/2d = nc/2fd$ . Thus the phase velocity is

$$c_n^{\text{ph}} = \frac{c}{\sqrt{1 - (nc/2fd)^2}}.$$

Therefore, for a given  $n$ , as the wavelength is increased,  $\theta_i$  becomes smaller and smaller so as to maintain that  $n$  loops are fit about the  $y$  axis. (Think about that limit by making the distance between crests larger. Then the fact that  $\theta_i$  decreases to maintain the location of the nodes will be clear).

As  $\theta_i$  approaches 0, lines of constant phase sweep the entire  $x$  axis, and thus the phase speed goes to  $\infty$ . (Qualitatively, this makes sense, because lines of constant phase will reach  $x = \pm\infty$  instantaneously; quantitatively this can be seen from the fact that  $c_n^{\text{ph}} = c/\sin \theta_i$ ). The wavelength for which  $\theta_i = 0$  corresponds to the cutoff frequency. Below this frequency, the sound cannot fit the same  $n$  loops in the vertical direction. Therefore, the sound just cannot live with itself. It kills itself off as an evanescent wave.

Meanwhile, the physical meaning of the group speed is the projection of the wave velocity along the principal axis of the waveguide:  $c_n^{\text{ph}} = c \sin \theta_i$ .

6. In what frequency limit does the phase speed  $c_{\text{ph}}$  in a waveguide approach the speed of sound in free space,  $c$ ? [ANSWER]

In the high-frequency limit, the phase speed approaches the free-space sound speed. This is because at high frequencies, a standing wave of very high spatial frequency (large  $k$ , small  $\lambda$ ) exists in the direction perpendicular to the primary axis of the waveguide. To create this interference pattern, the difference between the trace wavelength and the incident wavelength must be small, and thus  $\theta_i$  must be close to  $90^\circ$  (see figure 6.14 from Blackstock in the answer to the previous question). For  $\theta_i \sim 90^\circ$ , the incident sound must be predominantly *traveling* (as opposed to *standing*). In the limit that the incident wave is purely traveling, the lines of constant phase travel at the speed of the wavefronts along the axis of the waveguide, and thus  $c_{\text{ph}} = c$ .

7. What is the phase speed at a cutoff frequency? [ANSWER]

At a cutoff frequency, the phase speed is infinite. This is because at cutoff, the sound has no traveling wave component; rather, the sound is purely a standing wave perpendicular to the waveguide's primary axis. In this standing wave, the lines of constant phase extend infinitely along the waveguide's primary axis. The speed at which lines (in 2D, or surfaces in 3D) of constant phase travel is the phase speed. Since these lines travel an infinite distance in zero time, the phase speed is infinite at a cutoff frequency.

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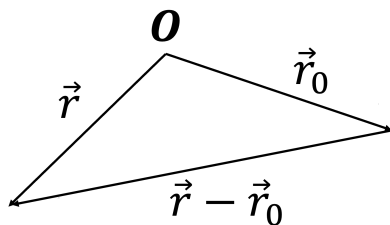
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## Chapters 13-14: Diffraction

This section is almost entirely based on my class notes from Dr. Hamilton's Acoustics II course. The derivations are challenging, so hints/outlines have been provided. It is more important to understand the underlying concepts.

1. How does the quantity  $R = |\mathbf{r} - \mathbf{r}_0|$  relate a sound source to the listener? *[ANSWER]*

If the source is located at  $\mathbf{r}_0$  and the listener is located at  $\mathbf{r}$ ,  $R = |\mathbf{r} - \mathbf{r}_0|$  is the distance between source and listener. Its shorthand when appearing in the argument of a Green's function is  $\mathbf{r}|\mathbf{r}_0$ .



2. ✪ Prove that the free space Green's function  $g = e^{-jkR}/4\pi R$  solves the inhomogeneous Helmholtz equation  $(\nabla^2 + k^2)f = -\delta(\mathbf{r} - \mathbf{r}_0)$ , where  $R = |\mathbf{r} - \mathbf{r}_0|$ . *Hint: Integrate the differential equation over the volume of a sphere of radius  $\epsilon$  and use the divergence theorem to convert the volume integral into a surface integral. Evaluate the surface integral and then take the limit as  $\epsilon \rightarrow 0$ . Also note that  $\int_0^\epsilon e^{-jkR} R dR \rightarrow 0$  as  $\epsilon \rightarrow 0$ .* *[ANSWER]*

Inserting the Green's function into the Helmholtz equation and integrating over space gives

$$\int \nabla^2 g dV + k^2 \int g dV = -1$$

The divergence theorem is applied to the first integral:

$$\begin{aligned} \int \nabla^2 g dV &= \int \nabla \cdot (\nabla g) dV \\ &= \oint \nabla g dS. \end{aligned}$$

Since at radius  $R = \epsilon$  the differential area is  $dS = \epsilon^2 d\theta d\psi$ , the equation above becomes

$$\begin{aligned} \int \nabla^2 g dV &= \int_0^{2\pi} \int_0^\pi \left( \frac{dg}{dR} \right)_{R=\epsilon} \epsilon^2 d\theta d\psi \\ &= 4\pi \epsilon^2 \left[ \frac{d}{dR} \left( \frac{e^{-jkR}}{4\pi R} \right) \right]_{R=\epsilon} \\ &= 4\pi \epsilon^2 \left[ \left( -\frac{1}{4\pi R^2} - \frac{jk}{4\pi R} \right) e^{-jkR} \right]_{R=\epsilon} \\ &= -(1 + jk\epsilon) e^{-jk\epsilon} \end{aligned}$$

Now the limit of the above result is taken:

$$\begin{aligned}
-\lim_{\epsilon \rightarrow 0} (1 + jk\epsilon) e^{-jk\epsilon} &= -\lim_{\epsilon \rightarrow 0} (1 + jk\epsilon)(1 - jk\epsilon - k^2\epsilon^2/2! \dots) \\
&= -1.
\end{aligned}$$

Meanwhile, the second integral is

$$\begin{aligned}
\int \frac{e^{-jkR}}{4\pi R} dV &= \int_0^\epsilon \frac{e^{-jkR}}{4\pi R} 4\pi R^2 dR \\
&= \int_0^\epsilon e^{-jkR} R dR \rightarrow 0 \text{ as } \epsilon \rightarrow 0.
\end{aligned}$$

Thus one obtains the true statement,  $-1 = -1$ . The free space Green's function thus satisfies the inhomogeneous Helmholtz equation.

3. What is reciprocity in acoustics? *[ANSWER]*

Reciprocity is the invariance under the exchange of sound source and listener. That is to say, the listener perceives the same sound when exchanging locations with the source of that sound.

4. To determine the condition that makes a medium reciprocal, suppose there are point sources at positions  $\mathbf{r}_1$  and  $\mathbf{r}_2$ . The free space Green's functions  $G(\mathbf{r}|\mathbf{r}_1)$  and  $G(\mathbf{r}|\mathbf{r}_2)$  satisfy their respective inhomogeneous Helmholtz equations. Combination of the Helmholtz equations, integration over volume, and application of the divergence theorem leads to

$$\int_S \left[ G(\mathbf{r}|\mathbf{r}_2) \frac{\partial G(\mathbf{r}|\mathbf{r}_1)}{\partial n} - G(\mathbf{r}|\mathbf{r}_1) \frac{\partial G(\mathbf{r}|\mathbf{r}_2)}{\partial n} \right] dS = G(\mathbf{r}_2|\mathbf{r}_1) - G(\mathbf{r}_1|\mathbf{r}_2). \quad (13)$$

Why does the right-hand side of equation (13) equal 0? What are the three conditions on the boundary of surface  $S$  in which the left-hand side vanishes, thereby satisfying the equality and giving the conditions for a reciprocal medium? *[ANSWER]*

The right-hand side of equation (13) equals 0 because the free space Green's function is even in the quantity  $\mathbf{r}_2 - \mathbf{r}_1$ .

The left-hand side vanishes for

$$G = 0 \quad (i)$$

$$\frac{\partial G}{\partial n} = 0 \quad (ii)$$

$$\frac{\partial G / \partial n}{G} = \text{constant}. \quad (iii)$$

Boundary condition (i) corresponds to a pressure release surface; (ii) corresponds to a rigid surface; and (iii) corresponds to the Sommerfeld radiation condition or a locally reacting surface.

5. ✪ Starting with the Helmholtz equation for a point-inhomogeneity,  $\nabla^2 G + k^2 G = -\delta(\mathbf{r} - \mathbf{r}_0)$ , and for a function-inhomogeneity,  $\nabla^2 p + k^2 p = -f(\mathbf{r})$ , derive the Helmholtz-Kirchhoff integral,

$$p(\mathbf{r}) = \int_V f(\mathbf{r}_0) G(\mathbf{r}|\mathbf{r}_0) dV_0 + \oint_S \left[ G(\mathbf{r}|\mathbf{r}_0) \frac{\partial p(\mathbf{r}_0)}{\partial n_0} - p(\mathbf{r}_0) \frac{\partial G(\mathbf{r}|\mathbf{r}_0)}{\partial n_0} \right] dS_0 \quad (14)$$

Outline: Relate the two PDEs by multiplying the point-inhomogeneity PDE by  $p$  and the function-inhomogeneity PDE by  $G$ . Subtract, interchange  $\mathbf{r}$  and  $\mathbf{r}_0$ , integrate over volume, and apply the divergence theorem. Why does the surface integral vanish in free space? [ANSWER]

Start with

$$\nabla^2 p(\mathbf{r}) + k^2 p(\mathbf{r}) = -f(\mathbf{r}) \quad (\text{i})$$

$$\nabla^2 G(\mathbf{r}|\mathbf{r}_0) + k^2 G(\mathbf{r}|\mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0). \quad (\text{ii})$$

Multiply equation (i) by  $G$  and equation (ii) by  $p$ , and subtract the two equations. The  $k^2$  terms cancel, giving

$$G(\mathbf{r}|\mathbf{r}_0)\nabla^2 p(\mathbf{r}) - p(\mathbf{r})\nabla^2 G(\mathbf{r}|\mathbf{r}_0) = -f(\mathbf{r})G(\mathbf{r}|\mathbf{r}_0) + \delta(\mathbf{r} - \mathbf{r}_0)p(\mathbf{r})$$

Next, interchange  $\mathbf{r}$  and  $\mathbf{r}_0$  and integrate over volume  $V_0$ . Note that  $G$  cares not about this exchange, nor does the delta function.

$$\begin{aligned} \int_{V_0} [G(\mathbf{r}|\mathbf{r}_0)\nabla_0^2 p(\mathbf{r}_0) - p(\mathbf{r}_0)\nabla_0^2 G(\mathbf{r}|\mathbf{r}_0)]dV_0 &= \int_{V_0} [\delta(\mathbf{r} - \mathbf{r}_0)p(\mathbf{r}_0) - f(\mathbf{r}_0)G(\mathbf{r}|\mathbf{r}_0)]dV_0 \\ &= p(\mathbf{r}) - \int_{V_0} f(\mathbf{r}_0)G(\mathbf{r}|\mathbf{r}_0)dV_0. \end{aligned}$$

Finally, using the divergence theorem on the left-hand side above reduces the Laplacian operators to derivatives on the normal. Solving for  $p(\mathbf{r})$  results in the Helmholtz-Kirchhoff integral:

$$p(\mathbf{r}) = \int_{V_0} f(\mathbf{r}_0)G(\mathbf{r}|\mathbf{r}_0)dV_0 + \int_S \left[ G(\mathbf{r}|\mathbf{r}_0)\frac{\partial p(\mathbf{r}_0)}{\partial n_0} - p(\mathbf{r}_0)\frac{\partial G(\mathbf{r}|\mathbf{r}_0)}{\partial n_0} \right]dS$$

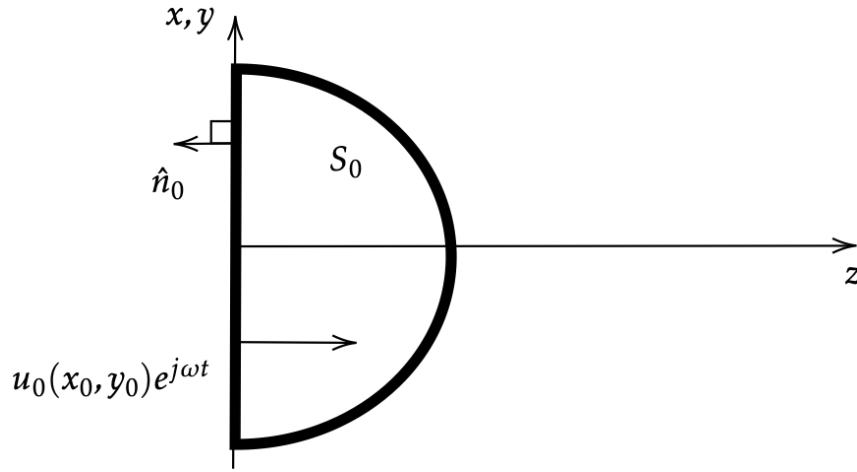
If the medium is reciprocal, the surface integral vanishes in free space because  $p = 0$  at  $r = \infty$ , giving the pressure due to a distribution of sources in free space:

$$p(\mathbf{r}) = \int_{V_0} f(\mathbf{r}_0)G(\mathbf{r}|\mathbf{r}_0)dV_0$$

6. ✨ Use equation (14) to derive the Rayleigh integral of the *first* kind,

$$p(\mathbf{r}) = \frac{j\omega\rho_0}{2\pi} \oint_S \frac{e^{-jkR}}{R} u^{(z)}(\mathbf{r}_0)dS_0 \quad (15)$$

which gives the pressure field due to a *velocity* source. Let  $\mathbf{e}_{n_0}$  be the unit outward normal, and align the surface source in the plane  $z = 0$ , as shown below.



Outline: Let the first integral in equation (14) be zero, i.e., assume no sources distributed in the volume. With the remaining surface integral, write  $\partial p(\mathbf{r}_0)/\partial n_0 = -\partial p(\mathbf{r}_0)/\partial z_0$  in terms of particle velocity using the momentum equation. With the second term, choose  $G$  such that  $\partial G/\partial n_0 = 0$  on the surface. The correct choice (show this) is  $G(\mathbf{r}|\mathbf{r}_0) = g_+(\mathbf{r}|\mathbf{r}_0) + g_-(\mathbf{r}|\mathbf{r}_0)$ , where  $g_{\pm} = e^{-jkR_{\pm}}/4\pi R_{\pm}$ , where  $R_{\pm} = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z \mp z_0)^2}$ . [ANSWER]

Let the first integral in equation (14) be zero, i.e., assume no sources are distributed in the volume:

$$\begin{aligned}
 p(\mathbf{r}) &= \oint_S \left[ G(\mathbf{r}|\mathbf{r}_0) \frac{\partial p(\mathbf{r}_0)}{\partial n_0} - p(\mathbf{r}_0) \frac{\partial G(\mathbf{r}|\mathbf{r}_0)}{\partial n_0} \right] dS_0 \\
 &= \oint_S \left[ -G(\mathbf{r}|\mathbf{r}_0) \frac{\partial p(\mathbf{r}_0)}{\partial z_0} + p(\mathbf{r}_0) \frac{\partial G(\mathbf{r}|\mathbf{r}_0)}{\partial z_0} \right] \Big|_{z_0=0} dS_0 \quad (i)
 \end{aligned}$$

Since an equation that maps particle velocity to pressure is desired, write  $\partial p(\mathbf{r}_0)/\partial n_0$  in terms of particle velocity using the momentum equation.

$$\frac{\partial p(\mathbf{r}_0)}{\partial n_0} = -\frac{\partial p(\mathbf{r}_0)}{\partial z_0} = j\omega \rho_0 u^{(n)}(\mathbf{r}_0)$$

Next, choosing  $G(\mathbf{r}|\mathbf{r}_0) = g_+(\mathbf{r}|\mathbf{r}_0) + g_-(\mathbf{r}|\mathbf{r}_0)$ , where  $g_{\pm} = e^{-jkR_{\pm}}/4\pi R_{\pm}$ , where

$$R_{\pm} = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z \mp z_0)^2},$$

note that

$$\begin{aligned}
 \frac{\partial G}{\partial z_0} \Big|_{z_0=0} &= \frac{\partial g_+}{\partial z_0} \Big|_{z_0=0} + \frac{\partial g_-}{\partial z_0} \Big|_{z_0=0} \\
 &= \frac{\partial}{\partial z_0} \frac{e^{-jkR_+}}{4\pi R_+} \Big|_{z_0=0} + \frac{\partial}{\partial z_0} \frac{e^{-jkR_-}}{4\pi R_-} \Big|_{z_0=0} \\
 &= \left[ \frac{jke^{-jkR_+}(z-z_0)}{4\pi R_+^2} + \frac{(z-z_0)}{4\pi R_+^3} e^{-jkR_+} - \frac{jke^{-jkR_-}(z+z_0)}{4\pi R_-^2} - \frac{(z+z_0)}{4\pi R_-^3} e^{-jkR_-} \right] \Big|_{z_0=0}
 \end{aligned}$$

Note that for  $z_0 = 0$ ,  $R_+ = R_- \equiv R$ , and the above equation becomes

$$\begin{aligned}
 \frac{\partial G}{\partial z_0} \Big|_{z_0=0} &= \frac{jke^{-jkR}(z-z_0)}{4\pi R^2} + \frac{(z-z_0)}{4\pi R^3} e^{-jkR} - \frac{jke^{-jkR}(z+z_0)}{4\pi R^2} - \frac{(z+z_0)}{4\pi R^3} e^{-jkR} \\
 &= 0
 \end{aligned}$$

wooh

Also,  $g_+ = g_- \equiv g$  at  $z_0 = 0$ , so  $G = 2g = e^{-jkR/4\pi R}$ . So equation (i) becomes

$$p(\mathbf{r}) = \oint_S \left( \frac{2e^{-jkR}}{4\pi R} \right) [j\omega \rho_0 u^{(r)}] dS_0$$

Rearranging gives the famous Rayleigh integral of the first kind of acoustics (and the second kind of optics):

$$p(\mathbf{r}) = \frac{j\omega \rho_0}{2\pi} \oint_S \frac{e^{-jkR}}{R} u^{(r)}(\mathbf{r}_0) dS_0$$

In Cartesian coordinates,

$$p(x, y, z) = \frac{j\omega \rho_0}{2\pi} \iint_{-\infty}^{\infty} \frac{e^{-jkR}}{R} u^{(r)}(x_0, y_0, 0) dx_0 dy_0.$$

where  $R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}$ . Note that the contour integral symbol  $\oint$  denoting "integral over a closed surface" is abandoned because there is no sound incident on the surface infinitely far away. (This relates to the *Sommerfeld radiation condition*).

7. Use the Rayleigh integral for a velocity source, Eq. (15) to calculate the on-axis ( $x = y = 0$ ) field due to a uniform circular piston of radius  $a$ . [ANSWER]

Setting  $x = y = 0$  in Cartesian coordinates corresponds to setting  $\sigma = 0$  in cylindrical coordinates in the Rayleigh integral, Eq. (15), and thus  $R = \sqrt{\sigma_0^2 + z^2}$ . Also, since the field is axisymmetric,  $u^{(z)}(\mathbf{r}_0) = u^{(z)}(\sigma_0)$ . Eq. (15) therefore becomes

$$\begin{aligned} p(0, z) &= \frac{j\omega \rho_0}{2\pi} \int_0^{2\pi} \int_0^{\infty} \frac{e^{-jkR}}{R} u^{(z)}(\sigma_0) \sigma_0 d\sigma_0 d\theta \\ &= \frac{jk \rho_0 c_0 u_0}{2\pi} \int_0^{2\pi} \int_0^a \frac{e^{-jk\sqrt{\sigma_0^2 + z^2}}}{\sqrt{\sigma_0^2 + z^2}} \sigma_0 d\sigma_0 d\theta. \end{aligned}$$

The radial integral contains no  $\theta$  dependence. Thus

$$p(0, z) = jk \rho_0 c_0 u_0 \int_0^a \frac{e^{-jk\sqrt{\sigma_0^2 + z^2}}}{\sqrt{\sigma_0^2 + z^2}} \sigma_0 d\sigma_0.$$

Letting  $s = \sqrt{\sigma_0^2 + z^2}$  and thus  $ds = \sigma_0 d\sigma_0 / \sqrt{\sigma_0^2 + z^2}$ , the radial integral is evaluated:

$$\begin{aligned} p(0, z) &= jk \rho_0 c_0 u_0 \int_z^{\sqrt{a^2 + z^2}} e^{-jks} ds \\ &= -\rho_0 c_0 u_0 \exp(-jks) \Big|_z^{\sqrt{a^2 + z^2}} \\ &= \rho_0 c_0 u_0 [\exp(-jkz) - \exp(-jk\sqrt{a^2 + z^2})]. \end{aligned}$$

The above can be factorized as

$$p(0, z) = \rho_0 c_0 u_0 \exp \left[ \frac{jk}{2} (-z - \sqrt{a^2 + z^2}) \right] \left\{ \exp \left[ \frac{jk}{2} (-z + \sqrt{a^2 + z^2}) \right] - \exp \left[ \frac{jk}{2} (z - \sqrt{a^2 + z^2}) \right] \right\}$$

Recalling that  $(e^{jx} - e^{-jx})/2j = \sin x$ , the above becomes

$$p(0, z) = 2j\rho_0 c_0 u_0 \exp[-jk(z + \sqrt{a^2 + z^2})/2] \sin[k(\sqrt{a^2 + z^2} - z)/2].$$

The magnitude of this expression is

$$|p(0, z)| = 2\rho_0 c_0 u_0 \left| \sin[k(\sqrt{a^2 + z^2} - z)/2] \right|.$$

8. ✨ Derive the Rayleigh integral of the *second* kind,

$$p(\mathbf{r}) = \frac{jkz}{2\pi} \oint p(\mathbf{r}_0) \left( 1 + \frac{1}{jkR} \right) \frac{e^{-jkR}}{R^2} dS_0, \quad (16)$$

which gives the pressure field due to a *pressure* source. Start with equation (14), and follow the derivation for the first Rayleigh integral, only now choosing  $G(\mathbf{r}|\mathbf{r}_0) = g_+(\mathbf{r}|\mathbf{r}_0) - g_-(\mathbf{r}|\mathbf{r}_0)$ . What is this integral called in optics? [ANSWER]

As in the derivation of the first Rayleigh integral, let the first integral in equation (14) be zero, i.e., assume no sources are distributed in the volume. Since an equation that maps pressure to pressure is desired, it is desired for  $\partial G(\mathbf{r}|\mathbf{r}_0)/\partial n_0$  to vanish on the boundary. Choosing  $G(\mathbf{r}|\mathbf{r}_0) = g_+(\mathbf{r}|\mathbf{r}_0) - g_-(\mathbf{r}|\mathbf{r}_0)$  does the trick, because at the  $z = 0$ ,  $R_+ = R_- \equiv R$ , and thus  $g_+ = g_- \equiv g$ , so

$$G(\mathbf{r}|\mathbf{r}_0) \Big|_{z_0=0} = g(\mathbf{r}|\mathbf{r}_0) - g(\mathbf{r}|\mathbf{r}_0) = 0.$$

Thus the Helmholtz-Kirchhoff integral becomes

$$\begin{aligned} p(\mathbf{r}) &= \oint_S p(\mathbf{r}_0) \frac{\partial G(\mathbf{r}|\mathbf{r}_0)}{\partial z_0} \Big|_{z_0=0} dS_0 \\ &= \oint p(x_0, y_0, 0) \left[ \frac{\partial}{\partial z_0} \frac{e^{-jkR_+}}{4\pi R_+} - \frac{\partial}{\partial z_0} \frac{e^{-jkR_-}}{4\pi R_-} \right] \Big|_{z_0=0} dx_0 dy_0 \end{aligned} \quad (ii)$$

Note that the term in the square brackets is evaluated separately to be

$$\begin{aligned} \left[ \dots \right] &= \left[ \frac{jke^{-jkR_+}(z - z_0)}{4\pi R_+^2} + \frac{(z - z_0)e^{-jkR_+}}{4\pi R_+^3} + \frac{jke^{-jkR_-}(z + z_0)}{4\pi R_-^2} + \frac{(z + z_0)e^{-jkR_-}}{4\pi R_-^3} \right] \Big|_{z_0=0} \\ &= \frac{jke^{-jkR}z}{4\pi R^2} + \frac{ze^{-jkR}}{4\pi R^3} + \frac{jke^{-jkR}z}{4\pi R^2} + \frac{ze^{-jkR}}{4\pi R^3} \\ &= \frac{jke^{-jkR}z}{2\pi R^2} + \frac{ze^{-jkR}}{2\pi R^3} \\ &= \frac{jkz}{2\pi} \left( 1 + \frac{1}{jkR} \right) \frac{e^{-jkR}}{R^2} \end{aligned}$$

Thus equation (ii) becomes

$$p(\mathbf{r}) = \frac{jkz}{2\pi} \oint p(\mathbf{r}_0) \left( 1 + \frac{1}{jkR} \right) \frac{e^{-jkR}}{R^2} dS_0,$$

or in Cartesian coordinates,

$$p(x, y, z) = \frac{jkz}{2\pi} \iint_{-\infty}^{\infty} p(x_0, y_0, 0) \left(1 + \frac{1}{jkR}\right) \frac{e^{-jkR}}{R^2} dx_0 dy_0$$

where  $R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}$ . Again, the contour integral symbol  $\oint$  denoting "integral over a closed surface" is abandoned because there is no sound incident on the surface infinitely far away.

This integral is called the Rayleigh integral of the first kind in optics *because (need to confirm) usually in optics the source condition is given in terms of the field variable. That is to say,  $E(x_0, y_0, 0)$  is used to compute  $E(x, y, z)$ , while or  $H(x_0, y_0, 0)$  is used to compute  $H(x, y, z)$ .*

9. ✨ Obtain the Fraunhofer approximation of the Rayleigh integral of the first kind (15):

$$p(\mathbf{r}) = \frac{j\omega\rho_0}{2\pi} \frac{e^{-jkr}}{r} \iint_{-\infty}^{\infty} e^{jk_x x_0 + jk_y y_0} u_0(x_0, y_0) dx_0 dy_0 \quad (17)$$

*Outline: Expand the squares in the displacement  $R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}$ , identify  $r^2 = x^2 + y^2 + z^2$ , and take the first-order binomial expansion of  $R$ . Multiply by  $k$  to make the quantity dimensionless, and throw out quadratic terms (terms proportional to  $x_0^2, y_0^2$ ). Finally, interpret  $k_x \equiv kx/r$  and  $k_y \equiv ky/r$ . Do you recognize the resulting 2D integral? At what  $r$  does the Fraunhofer approximation hold? [ANSWER]*

Begin by expanding  $R$ :

$$\begin{aligned} R &= \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2} \\ &= [x^2 + y^2 + z^2 - 2xx_0 - 2yy_0 + x_0^2 + y_0^2]^{1/2} \\ &= [r^2 - 2xx_0 - 2yy_0 + x_0^2 + y_0^2]^{1/2} \\ &= r \left[ 1 - 2\frac{xx_0 + yy_0}{r^2} + \frac{x_0^2 + y_0^2}{r^2} \right]^{1/2} \end{aligned}$$

Now perform the binomial expansion to first order:

$$\begin{aligned} R &= r \left[ 1 - \frac{xx_0 + yy_0}{r^2} + \frac{x_0^2 + y_0^2}{2r^2} \right] \\ &= r - \frac{xx_0 + yy_0}{r} + \frac{x_0^2 + y_0^2}{2r} . \end{aligned}$$

Next, multiply by  $k$  to make the quantity dimensionless (and because this is the combination that appears in the Rayleigh integral), and toss the term that is quadratic:

$$\begin{aligned} kR &= kr - kx_0 \frac{x}{r} - ky_0 \frac{y}{r} \\ &= kr - k_x x_0 - k_y y_0 . \end{aligned}$$

where the interpretation  $k_x \equiv kx/r$  and  $k_y \equiv ky/r$  has been made. Approximate the amplitude in equation (15) to zeroth order and the phase to linear order:

$$p(\mathbf{r}) = \frac{j\omega\rho_0}{2\pi} \frac{e^{-jkr}}{r} \iint_{-\infty}^{\infty} e^{jk_x x_0 + jk_y y_0} u_0(x_0, y_0) dx_0 dy_0$$

(Note that the notation changes here from denoting the velocity source condition from  $u^{(r)}$  to  $u_0$ ). The integral above is simply the 2D spatial Fourier transform of the source condition. This is often denoted  $\hat{u}(k_x, k_y)$ . The directivity is defined as

$$D(\theta, \psi) = \frac{\hat{u}_0(k_x, k_y)}{\hat{u}_0(0, 0)}.$$

The Fraunhofer approximation holds for  $r \gtrsim 3R_0$ , where  $R_0 = ka^2/2$ , the Rayleigh distance.

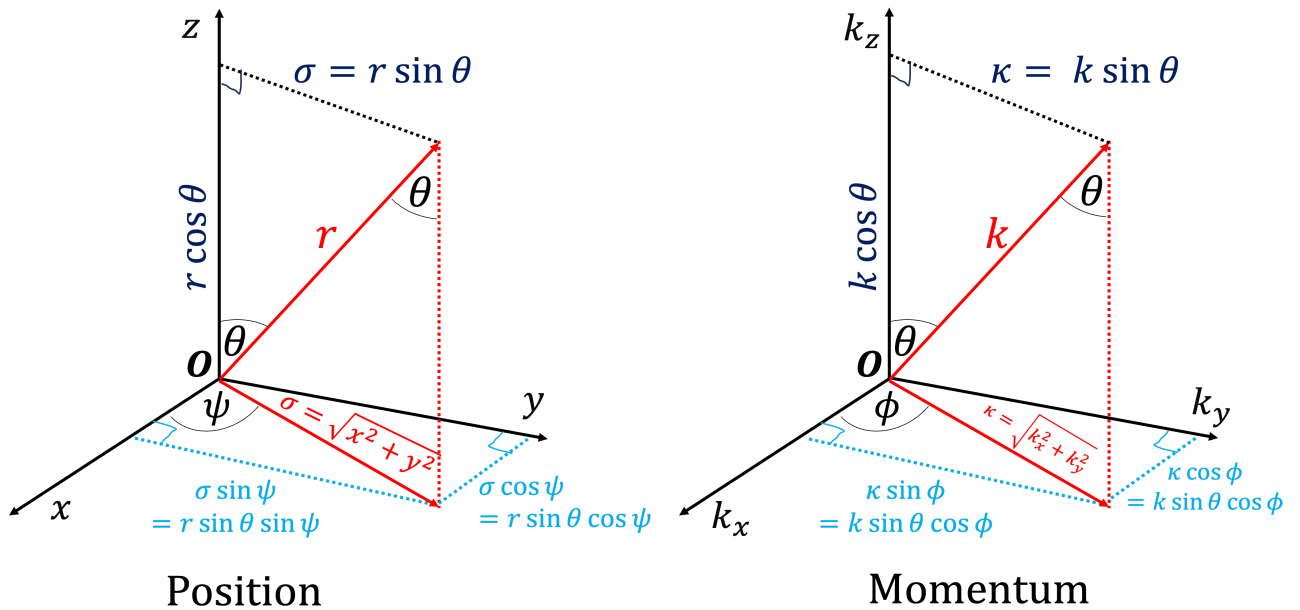
10. What is the  $ka \ll 1$  limit of the Fraunhofer approximation? [ANSWER]

In this limit the exponential in equation (17)  $\rightarrow 1$ . Thus the 2D spatial Fourier transform simply becomes the surface integral over the velocity source condition, which is  $Q_0$ , the volume velocity. Thus the  $ka \ll 1$  limit of the Fraunhofer approximation is an acoustic monopole:

$$p(\mathbf{r}) = \frac{j\omega \rho_0 Q_0}{2\pi} \frac{e^{-jkr}}{r}.$$

11. Convert the Fraunhofer approximation, equation (17), which is given in terms of rectangular coordinates in the source plane, to polar coordinates in the source plane using the change-of-variables in which  $\kappa = k \sin \theta$ , and

$$\begin{aligned} x &= \sigma \cos \psi & y &= \sigma \sin \psi \\ k_x &= \kappa \cos \phi & k_y &= \kappa \sin \phi. \end{aligned}$$



Then simplify the result to the case of an axisymmetric beam (no dependence on  $\psi$ ). [ANSWER]

The argument of the exponential in the integrand of equation (17) becomes

$$\begin{aligned} k_x x_0 + k_y y_0 &= \kappa \cos \phi \sigma_0 \cos \psi_0 + \kappa \sin \phi \sigma_0 \sin \psi_0 \\ &= \kappa \sigma_0 (\cos \phi \cos \psi_0 + \sin \phi \sin \psi_0) \\ &= \kappa \sigma_0 \cos(\phi - \psi_0) \end{aligned}$$

Also,  $dx_0 dy_0 = \sigma_0 d\sigma_0 d\psi_0$ . Thus equation (17) becomes

$$\begin{aligned}
p(\mathbf{r}) &= \frac{j\omega\rho_0}{2\pi} \frac{e^{-jkr}}{r} \int_0^{2\pi} \int_0^\infty e^{j\kappa\sigma_0 \cos(\phi-\psi_0)} u_0(\sigma_0, \psi_0) \sigma_0 d\sigma_0 d\psi_0 \\
&= \frac{j\omega\rho_0}{2\pi} \frac{e^{-jkr}}{r} \int_0^{2\pi} e^{j\kappa\sigma_0 \cos(\phi-\psi_0)} d\psi_0 \int_0^\infty u_0(\sigma_0) \sigma_0 d\sigma_0
\end{aligned}$$

The first line above is the farthest the Fraunhofer approximation can be taken if the source condition has an angular dependence (e.g., a vortex beam). In the second line, it has been assumed that  $u_0 = u_0(\sigma_0)$ . In that case, identify  $2\pi J_0(\kappa\sigma_0) = \int_0^{2\pi} e^{j\kappa\sigma_0 \cos(\phi-\psi_0)} d\psi_0$ , i.e., an integral representation of the Bessel function. Thus

$$p(\mathbf{r}) = \frac{j\omega\rho_0}{2\pi} \frac{e^{-jkr}}{r} \left[ 2\pi \int_0^\infty J_0(\kappa\sigma_0) u_0(\sigma_0) \sigma_0 d\sigma_0 \right]$$

The integral above is the Hankel transform, denoted  $\hat{u}_{0,H}(\kappa)$ , and the quantity in brackets [...] is  $2\pi \hat{u}_{0,H}(\kappa)$ , which is simply the Fourier transform. Also, note that  $J_0(0) = 1$ . Thus the volume velocity is given by  $\hat{u}_{0,H}(0) = Q_0 = 2\pi \int_0^\infty u_0(\sigma_0) \sigma_0 d\sigma_0$ . Multiplying and dividing by the volume velocity gives

$$\begin{aligned}
p(\mathbf{r}) &= \frac{j\omega\rho_0 Q_0}{2\pi} \frac{e^{-jkr}}{r} \frac{2\pi \int_0^\infty J_0(\kappa\sigma_0) u_0(\sigma_0) \sigma_0 d\sigma_0}{2\pi \int_0^\infty u_0(\sigma_0) \sigma_0 d\sigma_0} \\
&= \frac{j\omega\rho_0 Q_0}{2\pi} \frac{e^{-jkr}}{r} \frac{2\pi \int_0^\infty J_0(\kappa\sigma_0) u_0(\sigma_0) \sigma_0 d\sigma_0}{\hat{u}_{0,H}(0)}
\end{aligned}$$

The pressure field is thus written in terms of the directivity  $D(\theta)$ :

$$\begin{aligned}
p(\mathbf{r}) &= \frac{j\omega\rho_0 Q_0}{2\pi} \frac{e^{-jkr}}{r} D(\theta) \\
D(\theta) &= \frac{\hat{u}_{0,H}(\kappa)}{\hat{u}_{0,H}(0)}
\end{aligned}$$

12. Find the far-field pressure field and directivity due to a thin ring of pressure at  $z = 0$  centered about the  $z$  axis, given by the condition

$$u_0(\sigma) = u_0 w \delta(\sigma - a)$$

where  $\sigma$  is the radial coordinate in the source plane. [ANSWER]

Recall from the previous question that the pressure field is given by

$$\begin{aligned}
p(\mathbf{r}) &= \frac{j\omega\rho_0 Q_0}{2\pi} \frac{e^{-jkr}}{r} D(\theta) \\
D(\theta) &= \frac{\hat{u}_{0,H}(\kappa)}{\hat{u}_{0,H}(0)}
\end{aligned}$$

where the Hankel transform is defined as

$$\hat{u}_{0,H}(\kappa) = \int_0^\infty J_0(\kappa\sigma_0) u_0(\sigma_0) \sigma_0 d\sigma_0.$$

Letting  $u_0(\sigma) = u_0 w \delta(\sigma - a)$ , the Hankel transform is calculated using the sifting property of the delta function:

$$\begin{aligned}\hat{u}_{0,H}(\kappa) &= u_0 w \int_0^\infty J_0(\kappa \sigma_0) \delta(\sigma - a) \sigma_0 d\sigma_0 \\ &= u_0 w a J_0(\kappa a)\end{aligned}$$

Thus  $\hat{u}_{0,H}(0) = u_0 w a$ . So the far-field directivity is given by

$$D(\theta) = J_0(\kappa a) = J_0(ka \sin \theta),$$

and the pressure in the far field is given by

$$p(\mathbf{r}) = \frac{j\omega \rho_0 Q_0}{2\pi} \frac{e^{-jkr}}{r} J_0(ka \sin \theta).$$

13. Find the far-field pressure field and directivity due to a uniform circular piston at  $z = 0$  centered about the  $z$  axis, given by the condition

$$u_0(\sigma) = \begin{cases} u_0, & \sigma \in [0, a] \\ 0, & \sigma > a \end{cases},$$

where  $\sigma$  is defined as before. [ANSWER]

Recall from the previous two questions that the pressure field is given by

$$\begin{aligned}p(\mathbf{r}) &= \frac{j\omega \rho_0 Q_0}{2\pi} \frac{e^{-jkr}}{r} D(\theta) \\ D(\theta) &= \frac{\hat{u}_{0,H}(\kappa)}{\hat{u}_{0,H}(0)}\end{aligned}$$

where the Hankel transform is  $\hat{u}_{0,H}(\kappa) = \int_0^\infty J_0(\kappa \sigma_0) u_0(\sigma_0) \sigma_0 d\sigma_0$ . Letting  $u_0(\sigma) = u_0$  for  $\sigma \in [0, a]$  and 0 for  $\sigma > a$ , the Hankel transform is calculated:

$$\begin{aligned}\hat{u}_{0,H}(\kappa) &= u_0 \int_0^a J_0(\kappa \sigma_0) \sigma_0 d\sigma_0 \\ &= u_0 a^2 \frac{J_1(\kappa a)}{\kappa a}.\end{aligned}$$

Since  $\lim_{x \rightarrow 0} J_1(x)/x = 1/2$ ,  $\hat{u}_{0,H}(0) = u_0 a^2/2$ . So the far-field directivity is given by

$$D(\theta) = \frac{2J_1(\kappa a)}{\kappa a} = \frac{2J_1(ka \sin \theta)}{ka \sin \theta},$$

and the pressure in the far field is

$$p(\mathbf{r}) = j\rho_0 c_0 u_0 \frac{ka^2}{2} \frac{e^{j(\omega t - kr)}}{r} \frac{2J_1(ka \sin \theta)}{ka \sin \theta}.$$

14. ✪ Obtain the Fresnel (paraxial) approximation of equation (15). The result is

$$p(x, y, z) = \frac{j\omega \rho_0}{2\pi} \frac{e^{-jkz}}{z} \iint_{-\infty}^{\infty} u_0(x_0, y_0) e^{-jk[(x-x_0)^2 + (y-y_0)^2]/2z} dx_0 dy_0. \quad (18)$$

Outline: Expand  $R$  in powers of  $1/z$  (whereas in the Fraunhofer approximation,  $R$  was expanded in powers of  $1/r$ ). What are the limits on this approximation? [ANSWER]

Start with the first Rayleigh integral, given by equation (15), where

$$R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}.$$

As in the Fraunhofer approximation,  $R$  is the quantity that is approximated. However, this time it is expanded in powers of  $1/z$ :

$$R = z \left[ 1 + \frac{(x - x_0)^2}{z^2} + \frac{(y - y_0)^2}{z^2} \right]^{1/2}$$

The quadratic terms are considered to be small, and the binomial expansion is applied:

$$\begin{aligned} R &= z \left[ 1 + \frac{(x - x_0)^2}{2z^2} + \frac{(y - y_0)^2}{2z^2} \right] \\ &= z + \frac{(x - x_0)^2}{2z} + \frac{(y - y_0)^2}{2z} \end{aligned}$$

The denominator of the integrand of equation (15) is approximated to zeroth order as  $z$ . Meanwhile, the quantity  $kR$  that appears in the exponent becomes

$$kR = kz + \frac{k}{2z} \left[ (x - x_0)^2 + (y - y_0)^2 \right]$$

Note that the next higher order term would be  $\mathcal{O}(z^{-3}) \sim ka^4/8z^3$  which is declared to be  $\ll \pi$ , i.e., the  $\mathcal{O}(z^{-3})$  does not contribute significantly to the phase of  $e^{-jkR}$ , giving the criterion for the paraxial approximation to be  $(z/a)^3 \gtrsim ka/8\pi$ , or simply  $z/a \gtrsim (ka)^{1/3}$ , which corresponds to a region defined by a  $\pm 20^\circ$  angle about the  $z$  axis (the *paraxial* region). Thus the first Rayleigh integral becomes

$$p(x, y, z) = \frac{j\omega\rho_0}{2\pi} \frac{e^{-jkz}}{z} \iint_{-\infty}^{\infty} u_0(x_0, y_0) e^{-jk[(x-x_0)^2 + (y-y_0)^2]/2z} dx_0 dy_0.$$

where  $u^{(r)}(x_0, y_0, 0)$  has been notated  $u_0(x_0, y_0)$  for simplicity. This is known as the "Fresnel diffraction integral." However, it should not be called the "[Fresnel integral](#)," which refers to an integral involved in the diffraction of waves about semi-infinite screens or double-rectangular apertures.

15. Does the paraxial approximation depend on  $ka$ ? Does the paraxial approximation contain evanescent waves? [ANSWER]

While the paraxial approximation is entirely independent of  $ka$ , an implicit assumption underlying the paraxial approximation is that  $ka \gg 1$ . This condition keeps the wavefronts quasi-planar. The "plane wave part" of the Fresnel diffraction integral is the factor  $e^{-ikz}$  in front of the integral.

The paraxial approximation does not contain evanescent waves. This can be seen by noting that the quantity

$$kR = kz + \frac{k}{2z} [(x - x_0)^2 + (y - y_0)^2]$$

is purely real-valued and becomes purely imaginary when multiplied by  $-j$  in the exponential  $e^{-jkR}$ . Thus there are only exponentials of imaginary arguments, which correspond to propagating waves. (In contrast, recall that evanescent waves correspond to exponentials of real arguments).

16. Obtain the axisymmetric form of equation (18), using the rectangular-to-polar mapping

$$\begin{aligned} x &= \sigma \cos \psi & x_0 &= \sigma_0 \cos \psi_0 \\ y &= \sigma \sin \psi & y_0 &= \sigma_0 \sin \psi_0 \end{aligned}$$

and thus  $dx_0 dy_0 = \sigma_0 d\sigma_0 d\psi_0$ . Denote  $\omega$  as  $kc_0$ . [ANSWER]

Begin with the rectangular form of the Fresnel diffraction integral, equation (18), only with  $\omega$  denoted as  $kc_0$ :

$$p(x, y, z) = \frac{jk\rho_0 c_0}{2\pi} \frac{e^{-jkz}}{z} \iint_{-\infty}^{\infty} u_0(x_0, y_0) e^{-jk[(x-x_0)^2 + (y-y_0)^2]/2z} dx_0 dy_0,$$

and note that

$$\begin{aligned} (x - x_0)^2 + (y - y_0)^2 &= \sigma^2 \cos^2 \psi + \sigma_0^2 \cos^2 \psi_0 - 2\sigma\sigma_0 \cos \psi \cos \psi_0 \\ &\quad + \sigma^2 \sin^2 \psi + \sigma_0^2 \sin^2 \psi_0 - 2\sigma\sigma_0 \sin \psi \sin \psi_0 \\ &= \sigma^2 + \sigma_0^2 - 2\sigma\sigma_0(\cos \psi \cos \psi_0 + \sin \psi \sin \psi_0) \\ &= \sigma^2 + \sigma_0^2 - 2\sigma\sigma_0 \cos(\psi - \psi_0) \end{aligned}$$

Thus the Fresnel diffraction integral becomes

$$p(\sigma, \psi, z) = \frac{jk\rho_0 c_0}{2\pi} \frac{e^{-jkz}}{z} \int_0^{2\pi} \int_0^{\infty} u_0(\sigma_0, \psi_0) e^{-jk[\sigma^2 + \sigma_0^2 - 2\sigma\sigma_0 \cos(\psi - \psi_0)]/2z} \sigma_0 d\sigma_0 d\psi_0.$$

This is as far as one can go if  $u_0$  depends on the polar angle  $\psi_0$ . If  $u_0$  does not depend on this angle, the above becomes

$$p(\sigma, \psi, z) = \frac{jk\rho_0 c_0}{2\pi} \frac{e^{-jkz}}{z} e^{-jk\sigma^2/2z} \int_0^{\infty} u_0(\sigma_0) e^{-jk\sigma_0^2/2z} \sigma_0 d\sigma_0 \int_0^{2\pi} e^{j(k\sigma\sigma_0/z) \cos(\psi - \psi_0)} d\psi_0$$

The polar integral is an integral representation of the Bessel function. It equals  $2\pi J_0(k\sigma\sigma_0/z)$ . Thus

$$p(\sigma, \psi, z) = e^{-jkz} \left[ \frac{jk\rho_0 c_0}{z} e^{-jk\sigma^2/2z} \int_0^{\infty} u_0(\sigma_0) J_0(k\sigma\sigma_0/z) e^{-jk\sigma_0^2/2z} \sigma_0 d\sigma_0 \right]. \quad (i)$$

Note that equation (i) is of the form  $p = qe^{-jkz}$ , where  $q$  is the quantity in [...] above. This  $q$  is the quantity that is slowly varying in  $z$ , and  $e^{-jkz}$  is simply the plane wave propagator.

17. How does one include spherical focusing in the Fresnel approximation? Let the focal length be  $d$ . Then a spherical wavefront is proportional to  $e^{jk\sqrt{x^2 + y^2 + (z-d)^2}} / \sqrt{x^2 + y^2 + (z-d)^2}$ . Let the aperture  $a/d$  be small, where  $a$  is the source radius. [ANSWER]

Start with the spherical wave condition at  $z = 0$ , the source:  $e^{jk\sqrt{x^2+y^2+d^2}}/\sqrt{x^2+y^2+d^2}$ . Then phase is

$$jk[x^2 + y^2 + d^2]^{1/2} = jkd \left[ 1 + \frac{x^2 + y^2}{d^2} \right]^{1/2} \\ \simeq jkd + jk \frac{x^2 + y^2}{2d},$$

and the magnitude is simply

$$[x^2 + y^2 + d^2]^{-1/2} = \frac{1}{d} [1 + (x^2 + y^2)/d^2]^{-1/2} \simeq \frac{1}{d}$$

Thus focused wave is proportional to  $\frac{e^{jkd}}{d} e^{jk(x_0^2+y_0^2)/2d}$  in the Fresnel approximation, and accounting for focusing simply amounts to a multiplication of the Fresnel diffraction integral by  $e^{jk(x_0^2+y_0^2)/2d}$ . In polar coordinates, the focusing factor is  $e^{jk\sigma_0^2/2d}$ . That is, the Fresnel diffraction integral becomes

$$p(\sigma, \psi, z) = \frac{jk\rho_0 c_0}{2\pi} \frac{e^{-jkz}}{z} \int_0^{2\pi} \int_0^\infty u_0(\sigma_0, \psi_0) e^{jk\sigma_0^2/2d} e^{-jk[\sigma^2 + \sigma_0^2 - 2\sigma\sigma_0 \cos(\psi - \psi_0)]/2z} \sigma_0 d\sigma_0 d\psi_0, .$$

or, for an axisymmetric beam,

$$p(\sigma, \psi, z) = e^{-jkz} \left[ \frac{jk\rho_0 c_0}{z} e^{-jk\sigma^2/2z} \int_0^\infty u_0(\sigma_0) J_0(k\sigma\sigma_0/z) e^{jk\sigma_0^2/2d} e^{-jk\sigma_0^2/2z} \sigma_0 d\sigma_0 \right].$$

18. In the paraxial approximation, the field in the focal plane of a focused source is given by the \_\_\_\_\_ transform of the source condition in cartesian coordinates and the \_\_\_\_\_ transform in polar coordinates. Why is this? [ANSWER]

(2D spatial) Fourier; Hankel. This is because at  $z = d$ , the factor  $e^{-jk\sigma_0^2/2z}$  equals  $e^{-jk\sigma_0^2/2d}$ , which cancels out with the focusing factor  $e^{jk\sigma_0^2/2z}$ , leaving  $\int_0^\infty u_0(\sigma_0) J_0(k\sigma\sigma_0/d) \sigma_0 d\sigma_0$  [the Hankel transform of  $u_0(\sigma_0)$ ] as the integral in the paraxial approximation. For source conditions that are not axisymmetric, the integral is the 2D spatial Fourier transform of the source condition, though it takes more algebra to show this.

19. Derive the paraxial wave equation,

$$-j2k \frac{\partial q}{\partial z} + \nabla_{\perp}^2 q = 0,$$

where  $\nabla_{\perp}^2$  is the transverse Laplacian,  $\partial/\partial x^2 + \partial/\partial y^2$  in Cartesian coordinates. Start with the Helmholtz equation,  $\nabla^2 p + k^2 p = 0$ , and let  $p(x, y, z) = q(x, y, z) e^{-jkz}$ . Note that  $\partial^2 q/\partial z^2 \ll 1$ .

[ANSWER]

Take the derivatives of  $p$ , and let subscripts indicate partial differentiation with respect to the subscripted variable. By inspection,

$$\frac{\partial^2 p}{\partial x^2} = q_{xx}e^{-jkz} \quad \text{and} \quad \frac{\partial^2 p}{\partial y^2} = q_{yy}e^{-jkz}.$$

Meanwhile,

$$\begin{aligned} \frac{\partial p}{\partial z} &= q_z e^{-jkz} - jkq e^{-jkz} = (q_z - jkq) e^{-jkz} \\ \frac{\partial^2 p}{\partial z^2} &= (q_{zz} - jkq_z) e^{-jkz} - jk(q_z - jkq) e^{-jkz} \\ &= (q_{zz} - j2kq_z - k^2q) e^{-jkz} \end{aligned}$$

The Helmholtz equation in Cartesian coordinates,  $p_{xx} + p_{yy} + p_{zz} + k^2p = 0$ , becomes

$$\begin{aligned} q_{xx}e^{-jkz} + q_{yy}e^{-jkz} + (q_{zz} - j2kq_z - k^2q)e^{-jkz} + k^2qe^{-jkz} &= 0 \\ q_{xx} + q_{yy} + (q_{zz} - j2kq_z - k^2q) + k^2q &= 0 \\ q_{xx} + q_{yy} + q_{zz} - j2kq_z &= 0. \end{aligned}$$

Noting that  $q_{zz}$  is small (Dr. Hamilton showed this rigorously in class), and denoting  $\partial/\partial x^2 + \partial/\partial y^2 \equiv \nabla_{\perp}^2$ , results in

$$-j2k \frac{\partial q}{\partial z} + \nabla_{\perp}^2 q = 0.$$

20. State Babinet's principle. [ANSWER]

Babinet's principle says that "complementary diffracting objects have complementary diffraction patterns" (Blackstock, page 484). For example, the complementary diffracting object of a finite disk of radius  $a$  in free space is a circular aperture of radius  $a$ . Babinet's principle says that the sum of the radiation patterns individually caused by the finite disk and the aperture must be the same as the radiation pattern of the beam in free space.

21. The  $i(kx - \omega t)$  convention is adopted for this problem. Use the 2D Fourier transform pair  $\hat{f}(k_x, k_y) = \iint_{-\infty}^{\infty} f(x, y) e^{-i(k_x x + k_y y)} dx dy$  and  $f(x, y) = (2\pi)^{-2} \iint_{-\infty}^{\infty} \hat{f}(k_x, k_y) e^{i(k_x x + k_y y)} dk_x dk_y$  to solve the Helmholtz equation in terms of the source condition  $p(x, y, z = 0)$ . *Hint: note that the Fourier transform of the  $n^{\text{th}}$  derivative of  $f$  with respect to  $x$  is  $(ik_x)^n \hat{f}(k_x, k_y)$ , and that the  $n^{\text{th}}$  derivative of  $f$  with respect to  $y$  is  $(ik_y)^n \hat{f}(k_x, k_y)$ . Use these relations to reduce the Helmholtz equation to an second-order ordinary differential equation in  $z$ . How does the solution change for a velocity source? [ANSWER]*

See [here](#) for the solution.

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## Chapter 15: Arrays

1. Provide the definition of the directivity function  $D(\theta, \psi)$ , directivity factor  $D_i$ , and directivity index DI. [ANSWER]

The directivity is defined as

$$D(\theta, \psi) = \frac{p(r, \theta, \psi)}{p(r, \theta_m, \psi_m)} = \frac{\hat{u}(k\alpha, k\beta)}{\hat{u}(0, 0)},$$

where  $\theta_m$  and  $\psi_m$  are the polar and azimuthal angles corresponding the direction of the beam's maximum. Note that  $\alpha = \sin \theta \cos \psi$  and  $\beta = \sin \theta \sin \psi$ . The  $z$  axis is aligned such that  $\theta_m = 0$ .

The directivity factor is

$$D_i = \frac{\langle I_{\max} \rangle}{\langle I_{\text{mono}} \rangle},$$

where  $\langle I_{\max} \rangle$  is the maximum intensity in the beam (along the  $z$  axis) and  $\langle I_{\text{mono}} \rangle$  is the intensity at the same distance due to a monopole radiating the same power  $W$ .

The directivity index is simply  $\text{DI} = 10 \log_{10} D_i$  (dB).

2. ✪ Derive an expression for the directivity factor  $D_i$  in terms of the directivity function  $D(\theta, \psi)$ . Note from the definition of directivity in the previous problem that

$$|D(\theta, \psi)|^2 = \frac{\langle I(\theta, \psi) \rangle}{\langle I_{\max} \rangle}.$$

Combine  $\langle I_{\text{mono}} \rangle = W/4\pi r^2$  and  $W = \oint \langle I(\theta, \psi) \rangle dS$  to eliminate  $W$  and invoke  $\langle I_{\max} \rangle |D(\theta, \psi)|^2 = \langle I(\theta, \psi) \rangle$ . [ANSWER]

Eliminate  $W$ :

$$\begin{aligned} \langle I_{\text{mono}} \rangle &= \frac{1}{4\pi r^2} \oint \langle I(\theta, \psi) \rangle dS \\ &= \frac{1}{4\pi r^2} \oint \langle I(\theta, \psi) \rangle r^2 d\Omega \\ &= \frac{1}{4\pi} \oint \langle I(\theta, \psi) \rangle d\Omega \end{aligned}$$

Invoke  $\langle I(\theta, \psi) \rangle = \langle I_{\max} \rangle |D(\theta, \psi)|^2$ :

$$\begin{aligned} \langle I_{\text{mono}} \rangle &= \frac{1}{4\pi} \oint \langle I_{\max} \rangle |D(\theta, \psi)|^2 d\Omega \\ \langle I_{\text{mono}} \rangle &= \frac{\langle I_{\max} \rangle}{4\pi} \oint |D(\theta, \psi)|^2 d\Omega \end{aligned}$$

Therefore, the directivity factor is

$$D_i = \frac{\langle I_{\max} \rangle}{\langle I_{\text{mono}} \rangle} = \frac{4\pi}{\oint |D(\theta, \psi)|^2 d\Omega}$$

3. Calculate the directivity  $D(\theta, \psi)$  due to

$$u_0(x, y) = Q_0[\delta(x - d/2) + \delta(x + d/2)]\delta(y)$$

i.e., two in-phase monopoles separated by distance  $d$ . [ANSWER]

The directivity is simply given by

$$D(\theta, \psi) = \frac{\hat{u}_0(k_x, k_y)}{\hat{u}_0(0, 0)}$$

The numerator is evaluated using the sifting property of the  $\delta$  function:

$$\begin{aligned} \hat{u}_0(k_x, k_y) &= Q_0 \iint_{-\infty}^{\infty} [\delta(x - d/2) + \delta(x + d/2)]\delta(y) e^{j(k_x x + k_y y)} dx dy \\ &= Q_0 \int_{-\infty}^{\infty} [\delta(x - d/2) + \delta(x + d/2)] e^{jk_x x} dx \int_{-\infty}^{\infty} \delta(y) e^{jk_y y} dy \\ &= Q_0 (e^{jk_x d/2} + e^{-jk_x d/2}) e^0 \\ &= 2Q_0 \cos(k_x d/2) \end{aligned}$$

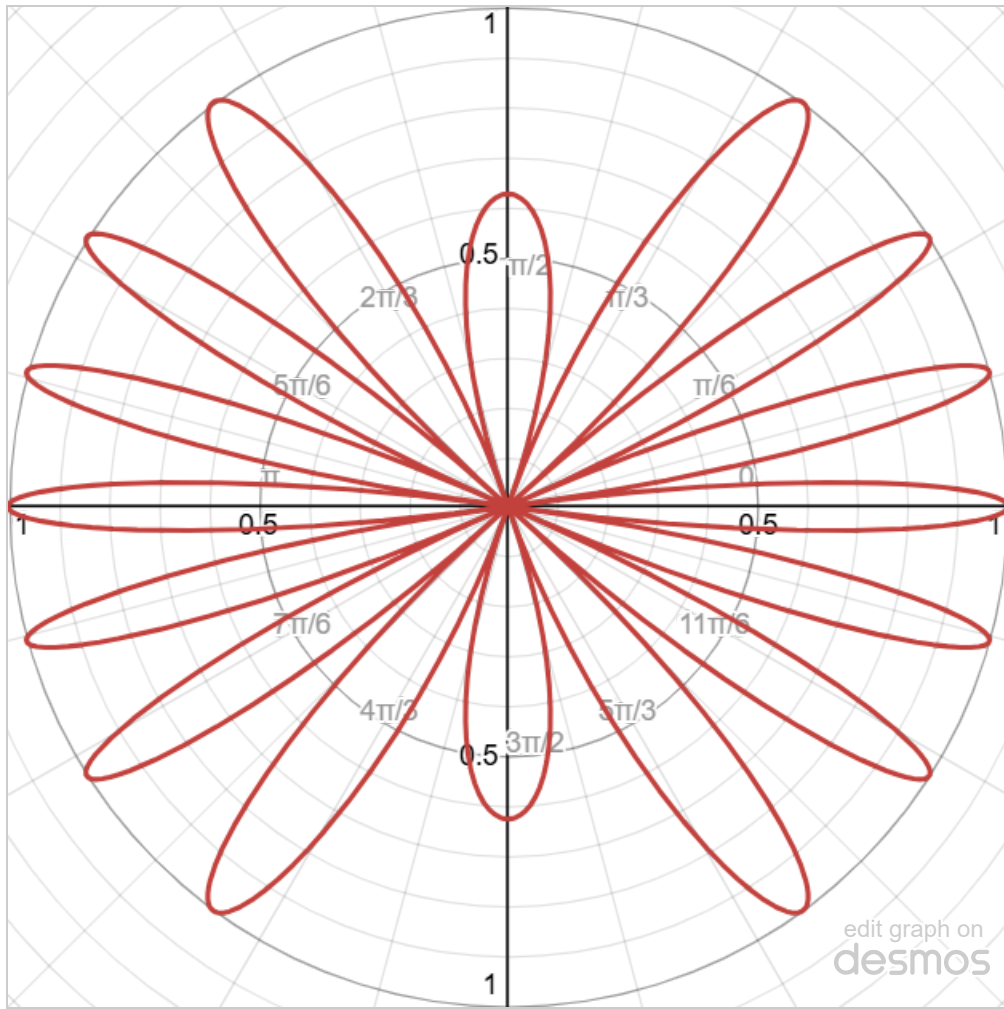
Recalling that  $k_x = k \sin \theta \cos \psi$ , the numerator is

$$\hat{u}_0(k_x, k_y) = 2Q_0 \cos[(kd/2) \sin \theta \cos \psi].$$

Meanwhile, the denominator is simply  $\hat{u}_0(k_x, k_y) = 2Q_0$ . Thus the directivity is

$$D(\theta, \psi) = \cos[(kd/2) \sin \theta \cos \psi].$$

The directivity is plotted below for  $kd \in [0, 35]$ .



4. A line array of  $N$  point sources radiates a pressure field given by

$$p(r, \theta) = A \frac{e^{-jkr}}{r} \sum_{n=0}^{N-1} e^{jnkd \sin \theta},$$

where  $\theta = 0$  points in the direction perpendicular to the line of sources. Given that

$$\sum_{n=0}^{N-1} e^{j2n\phi} = \frac{1 - e^{j2N\phi}}{1 - e^{j2\phi}},$$

show that the directivity of the line array is

$$D(\theta) = \frac{\sin N\phi}{N \sin \phi},$$

where  $\phi = \frac{kd}{2} \sin(\theta)$ . Evaluate the result for  $\theta = 0^\circ$ . What is this type of radiation called? What is the type of radiation for  $\theta = 90^\circ$  called? [ANSWER]

The right-hand side of the given identity is manipulated into the desired form.

$$\begin{aligned}
\sum_{n=0}^{N-1} e^{jnkd \sin \theta} &= \frac{1 - e^{j2N\phi}}{1 - e^{j2\phi}} \\
&= \frac{e^{jN\phi}}{e^{j\phi}} \left( \frac{e^{-jN\phi} - e^{jN\phi}}{e^{-j\phi} - e^{j\phi}} \right) \\
&= \frac{e^{jN\phi}}{e^{j\phi}} \left( \frac{e^{jN\phi} - e^{-jN\phi}}{e^{j\phi} - e^{-j\phi}} \right) \\
&= e^{j(N-1)\phi} \frac{\sin(N\phi)}{\sin \phi} \\
&= e^{j(N-1)kd \sin(\theta)/2} \frac{\sin[Nkd \sin(\theta)/2]}{\sin[kd \sin(\theta)/2]}
\end{aligned}$$

The pressure field is therefore

$$p(r, \theta) = A \frac{e^{-jkr}}{r} e^{j(N-1)kd \sin(\theta)/2} \frac{\sin[Nkd \sin(\theta)/2]}{\sin[kd \sin(\theta)/2]}$$

Meanwhile, plotting reveals that the maximum occurs at  $\theta_m = 0$ , for which the pressure is

$$p(r, \theta_m) = A \frac{e^{-jkr}}{r} N$$

The directivity is therefore

$$D(\theta) = \frac{p(r, \theta)}{p(r, \theta_m)} = \frac{\sin[Nkd \sin(\theta)/2]}{N \sin[kd \sin(\theta)/2]} .$$

For  $\theta = 0$ ,  $D(0) = 1$ . This is referred to as "broadside radiation." For  $\theta = 90^\circ$ , the radiation is called "end-fire."

5. Obtain the directivity  $D(\theta, \psi)$  for a continuous line array whose source condition is  $u_0(x, y) = u_0 w \delta(y)$  from  $-L/2 \leq x \leq L/2$  and 0 otherwise. [ANSWER]

The directivity is calculated using Fourier transforms:

$$D(\theta, \psi) = \frac{\hat{u}_0(k_x, k_y)}{\hat{u}_0(0, 0)} .$$

The numerator is given by

$$\begin{aligned}
\hat{u}_0(k_x, k_y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_0 w \delta(y) e^{jk_x x} e^{jk_y y} dx dy \\
&= u_0 w \int_{-L/2}^{L/2} e^{jk_x x} dx \\
&= \frac{u_0 w}{jk_x} e^{jk_x x} \Big|_{-L/2}^{L/2} \\
&= \frac{u_0 w}{jk_x} [e^{jk_x L/2} - e^{-jk_x L/2}] \\
&= \frac{2u_0 w}{k_x} \sin k_x L/2 \\
&= u_0 L w \frac{\sin k_x L/2}{k_x L/2}
\end{aligned}$$

Meanwhile, the denominator is

$$\hat{u}_0(0, 0) = u_0 L w .$$

Recall that  $k_x = k \sin \theta \cos \psi$ . Thus the directivity is

$$\begin{aligned}
D(\theta, \psi) &= \frac{\hat{u}_0(k_x, k_y)}{\hat{u}_0(0, 0)} \\
&= \frac{\sin [(kL/2) \sin \theta \cos \psi]}{(kL/2) \sin \theta \cos \psi} .
\end{aligned}$$

6. Consider a circular piston, whose directivity is given by the jinc function,  $D(\theta) = 2J_1(\kappa a) / \kappa a$ . The wavenumber is

$$\begin{aligned}
\kappa &= \sqrt{k_x^2 + k_y^2} \\
k_x &= k \sin \theta \cos \psi \\
k_y &= k \sin \theta \sin \psi
\end{aligned}$$

Suppose the beam is now steered with  $k_x$  with  $k(\sin \theta \cos \psi - \sin \theta_0)$ . What will the new directivity be? *[ANSWER]*

$$D(\theta) = \frac{2J_1[ka(\sin \theta - \sin \theta_0)]}{ka(\sin \theta - \sin \theta_0)}$$

7. ✱ A distribution of  $N$  velocity sources, each with a source condition of  $u_1(x - x_n, y - y_n)$ , has a total source condition of

$$u_0(x, y) = \sum_{n=1}^N u_1(x - x_n, y - y_n) .$$

Prove that the directivity of the distribution is the product of the directivity of each source and the directivity of the array, where the array is treated as a collection of point sources. *Hint: write*

the above source condition in terms of  $\delta$ -function,

$$u_0(x, y) = \sum_{n=1}^N \iint_{-\infty}^{\infty} u_1(x - x_n, y - y_n) \delta(x - x_n) \delta(y - y_n) dx dy,$$

and take the 2D Fourier transform. [ANSWER]

The Fourier transform of the velocity source condition

$$u_0(x, y) = \sum_{n=1}^N \iint_{-\infty}^{\infty} u_1(x - x_n, y - y_n) \delta(x - x_n) \delta(y - y_n) dx dy$$

is taken by noting that the Fourier transform of  $\delta(x - x_n)$  is  $e^{jk_x x_n}$ , and similarly the Fourier transform of  $\delta(y - y_n)$  is  $e^{jk_y y_n}$ :

$$\hat{u}_0(k_x, k_y) = \hat{u}_1(k_x, k_y) \sum_n e^{j(k_x x_n + k_y y_n)}.$$

Dividing by  $\hat{u}_0(0, 0) = \hat{u}_1(0, 0) \sum_n e^{j(0x_n + 0y_n)} = \hat{u}_1(0, 0)$  gives the directivity (see problem 1 of this section),

$$D(\theta, \psi) = D_1(\theta, \psi) D_A(\theta, \psi).$$

8. Suppose there are two circular pistons of radius  $a$  separated by a distance  $d$  from each other (center-to-center). What is the directivity of the two pistons? [ANSWER]

The product theorem makes the problem easy. The directivity of each piston is

$$D_1(\theta) = \frac{2J_1(ka \sin \theta)}{ka \sin \theta}$$

while the directivity for the  $N = 2$  line array is

$$D_A(\theta, \psi) = \cos[(kd/2) \sin \theta \cos \psi].$$

The directivity of the two circular pistons of radius  $a$  separated by a distance  $d$  is thus

$$D(\theta, \psi) = \frac{2J_1(ka \sin \theta)}{ka \sin \theta} \cos[(kd/2) \sin \theta \cos \psi].$$

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# Ultrasonics

The questions below cover only the theoretical aspects of the ultrasonics course taught by M. R. Haberman in the spring 2022 semester at UT Austin. Introductory topics covered in the course are not listed below because they can be found elsewhere on this website. For example, index notation is covered in [the vector algebra and vector calculus section](#) of the Math page. Basic wave phenomena are covered in the "[What is a wave?](#)" section of the Physical Acoustics page. The final sections of that page cover [diffraction](#) and [arrays](#).

Much of the content covered in the course can be found in [Introduction to Elastic Wave Propagation by A. Bedford and D. S. Drumheller](#).

1. [Elasticity](#)
2. [Derivation of elastic wave equation](#)
3. [Reflection and transmission](#)
4. [Rayleigh waves](#)
5. [Waves in plates](#)
6. [Signal processing](#)

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## Elasticity

This section is based on the first chapter of Bedford and Drumheller's book linked above.

1. What is the difference between  $\mathbf{X}$  and  $\mathbf{x}$ ? Quantities written in terms of  $\mathbf{X}$  belong in the \_\_\_\_\_ description while quantities written in terms of  $\mathbf{x}$  belong in the \_\_\_\_\_ description. Which description is more conventionally used in acoustics, and which is more conventionally used in elasticity? *[ANSWER]*

$\mathbf{X}$  is the position vector of an object in the reference state, while  $\mathbf{x}$  is the position vector of an object in the current state. Quantities written in terms of  $\mathbf{X}$  belong in the Lagrangian (material) description while quantities written in terms of  $\mathbf{x}$  belong in the Eulerian (spatial) description. The Lagrangian description is more commonly used in elasticity, because deformations of an object are studied with respect to the initial state, where the Eulerian description is more common in acoustics because microphones measure Eulerian quantities.

2. Use Cartesian index notation, i.e.,  $\mathbf{X} \mapsto X_k$ ,  $\mathbf{x} \mapsto x_k$ ,  $k = 1, 2, 3$ . Define the motion and inverse motion. *[ANSWER]*

The motion is defined as

$$x_k = \hat{x}_k(X_m, t),$$

where the "hat" indicates that a function depends on the variables  $X_k$  and  $t$ . The inverse motion is defined as

$$X_k = X_k(x_m, t).$$

3. The displacement is defined as  $\mathbf{u} = \mathbf{x} - \mathbf{X}$ . The velocity is the time derivative of the displacement, and the acceleration is the time derivative of the velocity. Evaluate the displacement, velocity, and acceleration in the Lagrangian and Eulerian descriptions. *[ANSWER]*

See [here](#) for the definitions.

4. Noting that

$$dx_k = \frac{\partial \hat{x}_k}{\partial X_m} dX_m,$$

and defining

$$\begin{aligned} dS^2 &= dX_k dX_k \\ ds^2 &= dx_k dx_k = \frac{\partial \hat{x}_k}{\partial X_m} dX_m \frac{\partial \hat{x}_k}{\partial X_n} dX_n, \end{aligned}$$

show that

$$ds^2 - dS^2 = 2E_{mn}dX_m dX_n,$$

where

$$E_{mn} = \frac{1}{2} \left( \frac{\partial \hat{u}_m}{\partial X_n} + \frac{\partial \hat{u}_n}{\partial X_m} + \frac{\partial \hat{u}_k}{\partial X_m} \frac{\partial \hat{u}_k}{\partial X_n} \right).$$

What is  $E_{mn}$  called? *[ANSWER]*

Using the provided definitions,

$$ds^2 - dS^2 = \frac{\partial \hat{x}_k}{\partial X_m} dX_m \frac{\partial \hat{x}_k}{\partial X_n} dX_n - dX_k dX_k$$

The  $dX_m dX_n$  term can be factored out by using a Kronecker delta  $\delta_{mn}$  in the second term:

$$ds^2 - dS^2 = \left( \frac{\partial \hat{x}_k}{\partial X_m} \frac{\partial \hat{x}_k}{\partial X_n} - \delta_{mn} \right) dX_m dX_n.$$

Now the definition of the Lagrangian displacement,  $\hat{u}_k = \hat{x}_k - X_k$ , or  $\hat{x}_k = \hat{u}_k + X_k$ , is invoked:

$$\begin{aligned}
ds^2 - dS^2 &= \left[ \frac{\partial}{\partial X_m} (\hat{u}_k + X_k) \frac{\partial}{\partial X_n} (\hat{u}_k + X_k) - \delta_{mn} \right] dX_m dX_n \\
&= \left[ \left( \frac{\partial \hat{u}_k}{\partial X_m} + \frac{\partial X_k}{\partial X_m} \right) \left( \frac{\partial \hat{u}_k}{\partial X_n} + \frac{\partial X_k}{\partial X_n} \right) - \delta_{mn} \right] dX_m dX_n \\
&= \left[ \left( \frac{\partial \hat{u}_k}{\partial X_m} + \delta_{km} \right) \left( \frac{\partial \hat{u}_k}{\partial X_n} + \delta_{kn} \right) - \delta_{mn} \right] dX_m dX_n \\
&= \left[ \frac{\partial \hat{u}_k}{\partial X_m} \frac{\partial \hat{u}_k}{\partial X_n} + \frac{\partial \hat{u}_k}{\partial X_n} \delta_{km} + \frac{\partial \hat{u}_k}{\partial X_m} \delta_{kn} \right] dX_m dX_n \\
&= \left[ \frac{\partial \hat{u}_m}{\partial X_n} + \frac{\partial \hat{u}_n}{\partial X_m} + \frac{\partial \hat{u}_k}{\partial X_m} \frac{\partial \hat{u}_k}{\partial X_n} \right] dX_m dX_n
\end{aligned}$$

Defining the *Lagrangian strain tensor*

$$E_{mn} = \frac{1}{2} \left( \frac{\partial \hat{u}_m}{\partial X_n} + \frac{\partial \hat{u}_n}{\partial X_m} + \frac{\partial \hat{u}_k}{\partial X_m} \frac{\partial \hat{u}_k}{\partial X_n} \right),$$

the final equation above reads

$$ds^2 - dS^2 = 2E_{mn}dX_m dX_n.$$

In linear elasticity, the Lagrangian strain tensor becomes

$$E_{mn} = \frac{1}{2} \left( \frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} \right),$$

or in index notation,

$$E_{mn} = \frac{1}{2} (u_{m,n} + u_{n,m}).$$

5. To what does the stress-strain relation  $T_{km} = c_{kmij}E_{ij}$  reduce for isotropic linear elastic materials?

[ANSWER]

$$T_{km} = \lambda \delta_{km} E_{jj} + 2\mu E_{km}.$$

$\lambda$  is the dilation constant, and  $\mu$  is the shear modulus.

## Derivation of elastic wave equation

See section 1.6 of Bedford and Drumheller for a more thorough development of Newton's second law and its linearization.

1. The linearized differential form of Newton's second law in the absence of body forces reads

$$\rho a_m = \frac{\partial T_{mk}}{\partial x_k}.$$

In linear elasticity the left-hand side can be written as  $\rho a_m = \rho_0 \partial^2 u_m / \partial t^2$ , while the stress tensor for a linear isotropic material, upon substitution of the linear strain tensor, becomes

$$T_{mk} = \lambda \delta_{mk} \frac{\partial u_j}{\partial x_j} + \mu \left( \frac{\partial u_m}{\partial x_k} + \frac{\partial u_k}{\partial x_m} \right).$$

Obtain

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}$$

by combining Newton's second law and the linear strain tensor. [\[ANSWER\]](#)

Inserting

$$T_{mk} = \lambda \delta_{mk} \frac{\partial u_j}{\partial x_j} + \mu \left( \frac{\partial u_m}{\partial x_k} + \frac{\partial u_k}{\partial x_m} \right).$$

into the right-hand side of

$$\rho a_m = \frac{\partial T_{mk}}{\partial x_k}$$

gives

$$\rho_0 \frac{\partial^2 u_m}{\partial t^2} = (\lambda + \mu) \frac{\partial^2 u_k}{\partial x_k \partial x_m} + \mu \frac{\partial^2 u_m}{\partial x_k \partial x_k},$$

which in vector form gives the desired result.

2. The vector Laplacian identity  $\nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$  can be used to write the result of question (1) above as

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u})$$

Apply the Helmholtz decomposition  $\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\psi}$  to the above to obtain two wave equations, one corresponding to compressional waves, and the other corresponding to shear waves. Which one travels faster? [\[ANSWER\]](#)

See [here](#) for the decomposition. The compressional wave travels faster, as can be seen from the expression for the sound speed.

## Reflection and transmission

1. What are the boundary conditions at the boundary of two media for normally incident compressional waves? Use knowledge from Acoustics I to immediately write the *stress*-reflection and transmission coefficients. What are  $R$  and  $T$  for the *displacement*-reflection and transmission coefficients? Does the same relation hold for normally incident shear waves?

[ANSWER]

The boundary conditions for normally incident compressional and shear waves are identical. They are:

1. The normal component of the traction is continuous at the boundary.
2. The normal component of the displacement is continuous at the boundary.

Noting that the displacement and particle velocity are off by factor  $j\omega$ , the derivation for stress-reflection and transmission coefficients is identical to that for  $R$  and  $T$  at a fluid-fluid interface:

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$T = \frac{2Z_2}{Z_2 + Z_1}$$

Note that the same relations hold for normally incident shear waves, the only difference being a substitution of  $\lambda + 2\mu \mapsto \mu$  in the impedances.

2. How are the *displacement* reflection and transmission coefficients related to the *stress* reflection and transmission coefficients? [ANSWER]

The *stress* reflection and transmission coefficients are ratios of reflected and transmitted stresses to the incident stress amplitude,

$$R_{\text{stress}} = \frac{A_{\text{r, stress}}}{A_{\text{i, stress}}}, \quad T_{\text{stress}} = \frac{A_{\text{t, stress}}}{A_{\text{i, stress}}},$$

while *displacement* reflection and transmission coefficients are ratios of reflected and transmitted *displacements* to the incident *displacement* amplitude,

$$R_{\text{disp}} = \frac{A_{\text{r, disp}}}{A_{\text{i, disp}}}, \quad T_{\text{disp}} = \frac{A_{\text{t, disp}}}{A_{\text{i, disp}}},$$

To obtain the displacement reflection and transmission coefficients, one should start with the general relationship between stress and strain for linear isotropic media,  $T_{km} = \lambda \delta_{km} E_{jj} + 2\mu E_{km}$ . Since the strain for normal incidence is purely normal to the surface (e.g., purely in the  $x_1$  direction), it suffices to consider  $T_{11} = \lambda \delta_{11} E_{11} + 2\mu E_{11} = (\lambda + 2\mu) u_{1,1} = M u_{1,1}$ , where  $M \equiv \lambda + 2\mu$ . For a harmonic stress wave (i.e.,  $\propto e^{j(kx - \omega t)}$ ), the stress coefficient  $A_{\text{stress}}$  is therefore related to the displacement coefficient  $A_{\text{disp}}$  by the *spatial derivative* of the displacement coefficient, i.e.,

$$A_{\text{stress}} = -jk_n M_n A_{\text{disp}},$$

where  $k$  is the wavenumber, and where  $n$  denotes the medium (1 or 2). However there is an additional complication: stress is defined to be positive when pushing inward. This contributes an additional minus sign to the reflection coefficient:

$$A_{i, \text{ disp}} = -\frac{1}{jk_1 M_1} A_{i, \text{ stress}}, \quad A_{r, \text{ disp}} = \frac{1}{jk_1 M_1} A_{r, \text{ stress}} .$$

Thus the displacement reflection coefficient is off by a sign from the stress reflection coefficient:

$$\begin{aligned} R_{\text{disp}} &= \frac{A_{r, \text{ disp}}}{A_{i, \text{ disp}}} \\ &= \frac{A_{r, \text{ stress}}}{jk_1 M_1} \frac{-jk_1 M_1}{A_{t, \text{ stress}}} \\ &= -\frac{A_{r, \text{ stress}}}{A_{t, \text{ stress}}} \\ &= -R_{\text{stress}} . \end{aligned}$$

Meanwhile, the displacement and stress transmission coefficients are related by

$$A_{t, \text{ disp}} = -\frac{1}{jk_2 M_2} A_{t, \text{ stress}} .$$

Thus the displacement transmission coefficient is off by  $Z_1/Z_2$  from the stress transmission coefficient:

$$\begin{aligned} T_{\text{disp}} &= \frac{A_{t, \text{ disp}}}{A_{i, \text{ disp}}} \\ &= \frac{A_{t, \text{ stress}}}{-jk_2 M_2} \frac{-jk_1 M_1}{A_{i, \text{ stress}}} \\ &= \frac{Z_1}{Z_2} T_{\text{stress}} . \end{aligned}$$

3. What are the boundary conditions for reflection and transmission between two elastic solids? Summarize qualitatively how to find the reflection and transmission coefficients. *[ANSWER]*

The following quantities must be equal on either side of the boundary:

1. the normal component of the displacement:  $u_y^{(1)} = u_y^{(2)}$
2. the transverse components of the displacement:  $u_z^{(1)} = u_z^{(2)}$
3. the normal component of the stress:  $T_{yy}^{(1)} = T_{yy}^{(2)}$
4. the transverse component of the stress:  $T_{yz}^{(1)} = T_{yz}^{(2)}$

The problem is worked out in Cartesian 2-space [here](#), first for an incident longitudinal wave, and then for an incident shear wave (I typed up these notes for Dr. Haberman--quite the ordeal).

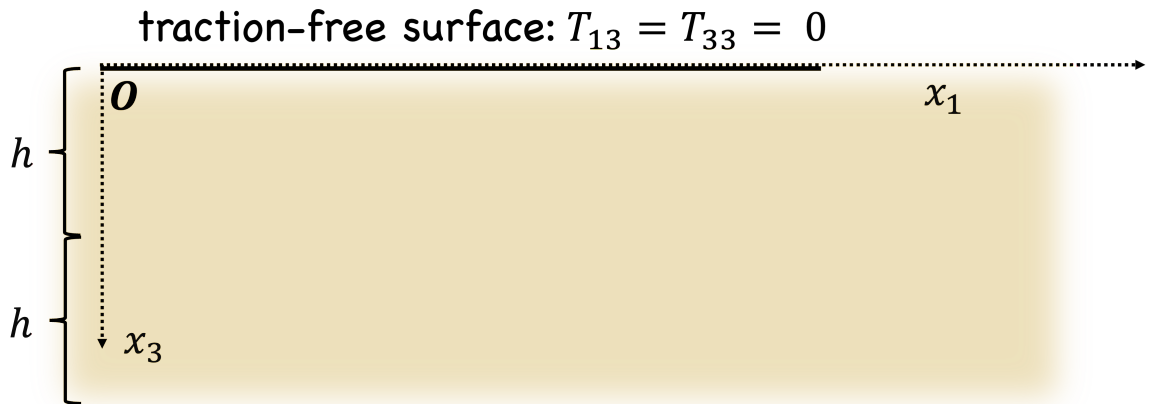
In summary, the displacements are written in terms of their Cartesian components, and two equations are obtained from points (1) and (2) above. Meanwhile, note that the longitudinal strains are  $E_{ii} = u_{i,i}$  and the shear strains are  $E_{mn} = \frac{1}{2}(u_{m,n} + u_{n,m})$ . The normal stresses are thus calculated using the stress-strain relation  $T_{ii} = \lambda \delta_{ii} E_{jj} + 2\mu E_{ii}$ , and the transverse components

of the stress are calculated using  $T_{km} = 2\mu E_{km}$ ,  $k \neq m$ . Two additional equations are thus obtained by invoking (3) and (4) above. These four equations are cast in a matrix form, with the column vector consisting of the two reflection and two transmission coefficients. The  $4 \times 4$  matrix must be inverted (numerically, for practical purposes) to calculate the reflection and transmission coefficients.

*As you can tell, reflection and transmission for oblique incidence between two solids (or between a solid a fluid) is an algebraically messy problem. I doubt this topic will show up on the qualifier, beyond perhaps a conceptual understanding of reflection and transmission problems.*

## Rayleigh waves

1. Consider an elastic half-space with a traction-free surface. Limit wave propagation to two dimensions, where  $x_3$  points downward, and  $x_1$  is oriented along the surface of the half-space:



**no wave motion in direction of  $x_2$  (out of the screen)**

This "plane-strain" condition requires that there is no displacement in the  $x_2$  direction, i.e.,  $u_2 = 0$ . What are the potential functions  $\phi$  and  $\psi$  that describe the displacement field  $\mathbf{u} = \nabla\phi + \nabla \times \psi$ ? [ANSWER]

Note that in index notation,

$$\nabla\phi = \phi_{,1}\hat{x}_1 + \phi_{,2}\hat{x}_2 + \phi_{,3}\hat{x}_3$$

and

$$\nabla \times \psi = (\psi_{3,2} - \psi_{2,3})\hat{x}_1 + (\psi_{1,3} - \psi_{3,1})\hat{x}_2 + (\psi_{2,1} - \psi_{1,2})\hat{x}_3.$$

The component  $u_2$  of  $\mathbf{u} = \nabla\phi + \nabla \times \psi$  must vanish. To achieve this,  $\phi$  cannot depend on  $x_2$ , and  $\psi_1 = \psi_3 = \text{constant} = 0$ . Thus  $\psi = \psi\hat{x}_2$ . This gives the displacement components

$$\begin{aligned} u_1 &= \phi_{,1} - \psi_{,3} \\ u_3 &= \phi_{,3} + \psi_{,1}. \end{aligned}$$

2. Seek a solution to the compressional wave equation  $\nabla^2 \phi - \ddot{\phi}/c_L^2 = 0$  and the shear wave equation  $\nabla^2 \psi - \ddot{\psi}/c_T^2 = 0$  by assuming the form  $\phi = A(x_3)e^{j(\omega t - kx_1)}$  and  $\psi = B(x_3)e^{j(\omega t - kx_1)}$ .

[ANSWER]

Inserting the forms of solution into the corresponding wave equations (see [here](#) for details) gives

$$A(x_3)'' + k^2[(c_R/c_L)^2 - 1]A(x_3) = 0$$

and

$$B(x_3)'' + k^2[(c_R/c_T)^2 - 1]B(x_3) = 0,$$

where  $c_R = \omega/k$  (the Rayleigh wave speed).

The solution to the above ODEs is

$$\begin{aligned} A(x_3) &= Ae^{\pm kqx_3}, & q &= \sqrt{1 - (c_R/c_L)^2} \\ B(x_3) &= Be^{\pm ksx_3}, & s &= \sqrt{1 - (c_R/c_T)^2}. \end{aligned}$$

Substituting these results into the assumed form of solution gives

$$\begin{aligned} \phi(\mathbf{x}, t) &= Ae^{-kqx_3} e^{j(\omega t - kx_1)} \\ \psi(\mathbf{x}, t) &= Be^{-ksx_3} e^{j(\omega t - kx_1)}. \end{aligned}$$

3. The boundary conditions of a traction-free surface in 2D are that the normal and shear components of the stress vanish:

$$\begin{aligned} T_{13} &= 2\mu E_{13} = 0 \\ T_{33} &= (\lambda + 2\mu)E_{33} + \lambda E_{11} = 0. \end{aligned}$$

Use the definition of the linear strain tensor to write these relations in terms of the displacements. Use the displacement potentials calculated previously to obtain a characteristic equation that satisfies these boundary conditions. *Hint: Define  $r \equiv s^2 + 1 = 2 - (c_R/c_T)^2$ .*

*Answer:  $r^2 - 4sq = 0$ .* [ANSWER]

Using the linear strain tensor,  $E_{mn} = \frac{1}{2}(u_{m,n} + u_{n,m})$ , the boundary conditions are written in terms of the displacements.

$$\begin{aligned} T_{13} &= 2\mu E_{13} = 2\mu(u_{1,3} + u_{3,1})/2 = 0 \\ T_{33} &= (\lambda + 2\mu)E_{33} + \lambda E_{11} = \lambda u_{1,1} + (\lambda + 2\mu)u_{3,3} = 0. \end{aligned}$$

Using the displacement potentials

$$\begin{aligned} \phi(\mathbf{x}, t) &= Ae^{-kqx_3} e^{j(\omega t - kx_1)} \\ \psi(\mathbf{x}, t) &= Be^{-ksx_3} e^{j(\omega t - kx_1)}. \end{aligned}$$

the displacements are found to be

$$\begin{aligned}
u_1 &= \phi_{,1} - \psi_{,3} \\
&= (-jkAe^{-kqx_3} + ksBe^{-ksx_3})e^{j(\omega t - kx_1)} \\
u_3 &= \phi_{,3} + \psi_{,1} \\
&= (-kqAe^{-kqx_3} - jkBe^{-ksx_3})e^{j(\omega t - kx_1)}
\end{aligned}$$

The derivatives of the displacement are taken in preparation for satisfying the boundary conditions:

$$\begin{aligned}
u_{1,1} &= (-k^2Ae^{-kqx_3} - jk^2sBe^{-ksx_3})e^{j(\omega t - kx_1)} \\
u_{1,3} &= (jk^2qAe^{-kqx_3} - k^2s^2Be^{-ksx_3})e^{j(\omega t - kx_1)} \\
u_{3,1} &= (jk^2qAe^{-kqx_3} - k^2Be^{-ksx_3})e^{j(\omega t - kx_1)} \\
u_{3,3} &= (k^2q^2Ae^{-kqx_3} + jk^2sBe^{-ksx_3})e^{j(\omega t - kx_1)}
\end{aligned}$$

Inserting the above derivatives in the continuity conditions at the boundary  $x_3 = 0$  gives

$$\begin{aligned}
T_{13} &= 2\mu(u_{1,3} + u_{3,1})/2 = 0 \\
\Rightarrow jk^2qA - k^2s^2B + jk^2qA - k^2B &= 0 \\
2jsA - (1 + s^2)B &= 0
\end{aligned}$$

Recall that  $s^2 = 1 - (c_R/c_T)^2$  and define  $r \equiv s^2 + 1 = 2 - (c_R/c_T)^2$ , giving

$$2jqA - rB = 0. \quad (i)$$

Meanwhile, the other boundary condition gives

$$\begin{aligned}
0 &= T_{33} = \lambda u_{1,1} + (\lambda + 2\mu)u_{3,3} \\
\Rightarrow \lambda(-k^2A - jk^2sB) + (\lambda + 2\mu)(k^2q^2A + jk^2sB) \\
&= \lambda(-A - jsB) + (\lambda + 2\mu)(q^2A + jsB) \\
&= -\lambda A - js\lambda B + (\lambda + 2\mu)q^2A + js(\lambda + 2\mu)B \\
&= [(\lambda + 2\mu)q^2 - \lambda]A + js[-\lambda + (\lambda + 2\mu)]B \\
&= [(\lambda + 2\mu)q^2 - \lambda]A - 2js\mu B
\end{aligned}$$

Dividing the last equation above by  $\mu$  gives

$$[q^2(\lambda + 2\mu)/\mu - \lambda/\mu]A - 2jsB = 0$$

Note that  $(\lambda + 2\mu)/\mu = c_L^2/c_T^2$ , and recall that  $q^2 = 1 - c_R^2/c_L^2$ . Write

$$\lambda/\mu = (\lambda + 2\mu)/\mu - 2$$

in the square brackets, giving

$$\begin{aligned}
0 &= [q^2(\lambda + 2\mu)/\mu - (\lambda + 2\mu)/\mu + 2]A - 2jsB \\
&= [(1 - c_R^2/c_L^2)(c_L/c_T)^2 - (c_L/c_T)^2 + 2]A - 2jsB \\
&= [(c_L/c_T)^2 - c_R^2/c_T^2 - (c_L/c_T)^2 + 2]A - 2jsB \\
&= [2 - c_R^2/c_T^2]A - 2jsB
\end{aligned}$$

Identifying  $r = 2 - (c_R/c_T)^2$  gives

$$rA - 2jsB = 0 \quad (\text{ii})$$

(Only someone who truly loves waves would have the patience for this kind of algebra). Combining equations (i) and (ii) gives the system of linear equations

$$\begin{pmatrix} 2jq & -r \\ r & -2js \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Nontrivial solutions ( $A \neq 0, B \neq 0$ ) exist for the case when the  $2 \times 2$  matrix above is not invertible, i.e.,

$$\det \begin{pmatrix} 2jq & -r \\ r & -2js \end{pmatrix} = 0$$

$$r^2 - 4sq = 0.$$

4. Define  $\eta \equiv c_R/c_T$  and  $\zeta = c_T/c_L$ . This gives

$$r = 2 - (c_R/c_T)^2 = 2 - \eta^2,$$

$$s = \sqrt{1 - (c_R/c_T)^2} = \sqrt{1 - \eta^2}$$

$$q = \sqrt{1 - (c_R/c_L)^2} = \sqrt{1 - \eta^2 \zeta^2}.$$

Write the characteristic equation for Rayleigh waves at a traction-free surface,  $r^2 - 4sq = 0$ , in terms of these parameters. How many roots does this equation have, and how many of them are physical? *Answer:*

$$\eta^6 - 8\eta^4 + 8\eta^2(3 - 2\zeta^2) + 16(\zeta^2 - 1) = 0.$$

*(I am having a hard time showing this. Let me know if you know how it's done.) [ANSWER]*

I can't figure this out :(

The 6th order equation generally has six complex roots, but only the real positive roots are physical.

5. Are Rayleigh waves dispersive? [ANSWER]

Rayleigh waves are not dispersive since  $\eta$  is only a function of the Poisson's ratio.

6. In what case can a Rayleigh wave travel along a curved surface? [ANSWER]

A Rayleigh wave travel along a curved surface if the wavelength is much smaller than the radius of curvature.

## Waves in plates

This section covers waves in plates of two types: horizontally polarized shear waves and Lamb waves. Note that Lamb waves were not discussed in depth in class, and the relevant homework assignments were numerical exercises. Their derivation is similar to that for Rayleigh waves.

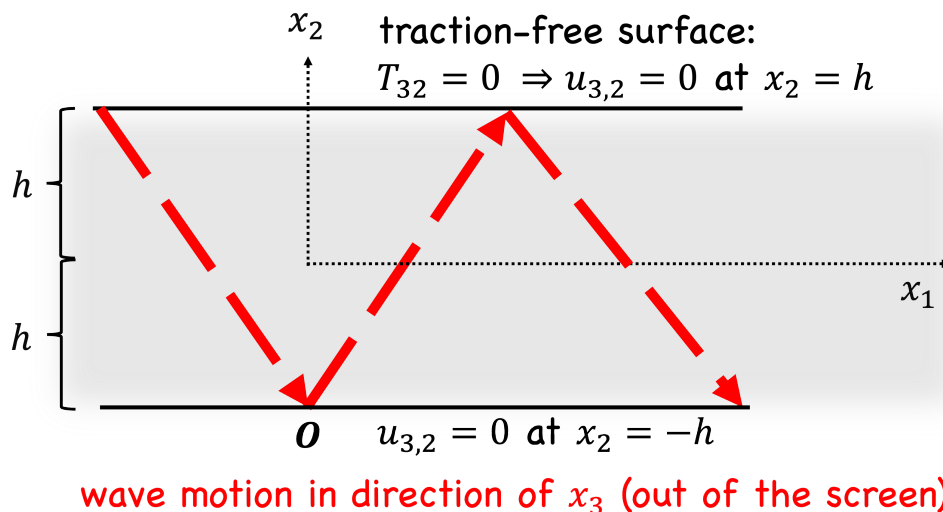
1. What is an SH wave? How are they different from Lamb waves? *[ANSWER]*

An SH wave is a "horizontally polarized shear wave," which are waves with a strain orthogonal to the direction of propagation (studied here in a plate with traction-free surfaces). Often the direction of polarization is chosen to be in the direction of  $\hat{x}_3$ . SH waves have the same form as acoustic waves. Lamb waves are waves in plates with traction-free surfaces, but with plane strain conditions (I am not sure if this true about all Lamb waves, but it is true for the Lamb waves in Dr. Haberman's notes). That is, Lamb waves neglect SH waves.

2. The wave equation for SH waves is very similar to the acoustic wave equation in 2D:

$$\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} - \frac{1}{c_T^2} \frac{\partial^2 u_3}{\partial t^2} = 0.$$

Assume the wave motion is of the form  $u_3(x_1, x_2, t) = f(x_2)e^{i(\omega t - kx_1)}$ , where the coordinates are defined below.



Obtain the function  $f(x_2)$ , where  $q = \sqrt{k_T^2 - k^2}$ . What kinds of modes does the application of the boundary conditions at  $h$  and  $-h$  give rise to? *[ANSWER]*

The function  $f(x_2)$  must be harmonic, as can be seen by inspection from the wave equation (or, formally, by separation of variables):

$$f(x_2) = B_1 \sin(qx_2) + B_2 \cos(qx_2).$$

$q$  is determined by application of the boundary conditions of the traction-free surface, which require that the derivative of  $f$  vanishes at  $x = h$  and  $x = -h$ :

$$\begin{aligned} B_1 \cos(qh) + B_2 \sin(qh) &= 0 \\ B_1 \cos(-qh) + B_2 \sin(-qh) &= 0, \end{aligned}$$

or, using the even and odd properties of the trigonometric functions,

$$B_1 \cos(qh) + B_2 \sin(qh) = 0$$

$$B_1 \cos(qh) - B_2 \sin(qh) = 0.$$

Both equations are satisfied by

$$B_1 = 0, \quad qh = \frac{n\pi}{2}, \quad n = 0, 2, 4 \dots \quad \text{even}$$

$$B_2 = 0, \quad qh = \frac{n\pi}{2}, \quad n = 1, 3, 5 \dots \quad \text{odd}$$

Evidently, the application of the boundary conditions gives rise to "symmetric" and "antisymmetric" modes, named for the symmetry/antisymmetry of the even/odd  $n$  modes about the midplane.

3. Given that  $q_n = n\pi/2h$ , obtain the dispersion relation  $\Omega(K)$ , where  $\Omega = 2h\omega/\pi c_T$  is the dimensionless frequency, and  $K = 2kh/\pi$  is the dimensionless wavenumber. Qualitatively describe the dispersion relation. Obtain the dimensionless group and phase speeds of the SH waves. [ANSWER]

Solve

$$q_n^2 = (n\pi/2h)^2 = k_T^2 - k^2$$

for  $k^2 = \omega/c_T$  and divide by  $(\pi/2h)^2$  to obtain

$$n^2 + (2kh/\pi)^2 = (2h\omega/\pi c_T)^2,$$

which gives

$$n^2 + K^2 = \Omega^2.$$

The dispersion relation consists of an infinite number of continuous branches, each corresponding to an integer value of  $n$ . The lowest ( $n = 0$ ) mode is non-dispersive, and all the higher order modes are hyperbolas with  $\Omega = K$  asymptotes.

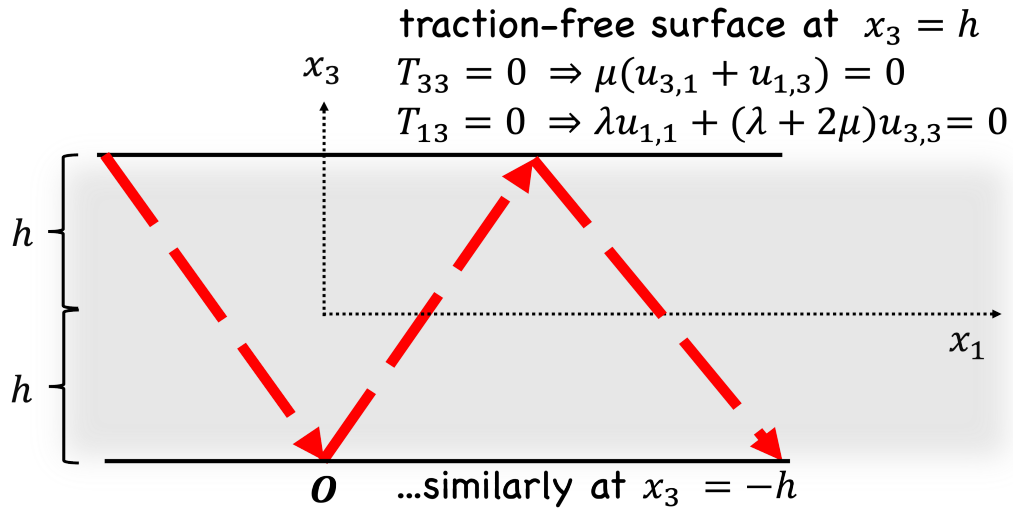
The dimensionless phase speed is

$$\frac{\Omega}{K} = \pm \sqrt{(n/K)^2 + 1},$$

and the dimensionless group speed is

$$\frac{d\Omega}{dK} = \pm \frac{1}{\sqrt{1 + (n/K)^2}}.$$

4. Outline the derivation for Lamb waves between two layers, where the coordinates are as defined below.



**no wave motion in  $x_2$  (out of the screen)**

Limit the derivation to plane strain, i.e.,  $u_2 = 0$ , and do not attempt to satisfy the boundary conditions. [ANSWER]

First note that for plane-strain propagation (see question 1 of the Rayleigh wave section)

$$u_1 = \phi_{,1} - \psi_{,3}$$

$$u_3 = \phi_{,3} - \psi_{,1}$$

Similar to the Rayleigh wave problem, it is assumed that the potentials have the form

$$\phi(\mathbf{x}, t) = f(x_3)e^{j(\omega t - kx_1)}$$

$$\psi(\mathbf{x}, t) = g(x_3)e^{j(\omega t - kx_1)}$$

To determine  $f(x_3)$  and  $g(x_3)$ , the potentials are inserted into their respective wave equations,

$$\phi_{,11} + \phi_{,33} - \frac{1}{c_L^2} \ddot{\phi} = 0$$

$$f''(x_3) + (\omega^2/c_L^2 - k^2)f(x_3) = 0$$

and

$$\psi_{,11} + \psi_{,33} - \frac{1}{c_T^2} \ddot{\psi} = 0$$

$$g''(x_3) + (\omega^2/c_T^2 - k^2)g(x_3) = 0$$

Solving the above second-order ODEs gives

$$f(x_3) = A_1 \sin(px_3) + A_2 \cos(px_3), \quad p = k\sqrt{(k_L/k)^2 - 1},$$

$$g(x_3) = B_1 \sin(qx_3) + B_2 \cos(qx_3), \quad q = k\sqrt{(k_T/k)^2 - 1}.$$

The displacements are calculated by  $u_1 = \phi_{,1} - \psi_{,3}$  and  $u_3 = \phi_{,3} - \psi_{,1}$ . As is the case with the SH waves,  $u_1$  and  $u_3$  can be classified as symmetric and antisymmetric (about the midplane of the plate).

The boundary conditions are that the normal and shear stresses vanish at both  $x_3 = h$  and  $x_3 = -h$ :

$$\begin{aligned}T_{33} = 0 &\Rightarrow \lambda u_{1,1} + (\lambda + 2\mu)u_{3,3} = 0 \\T_{13} = 0 &\Rightarrow \mu(u_{1,3} + u_{3,1}).\end{aligned}$$

Application of the boundary conditions to the even and odd modes results in the so-called Rayleigh-Lamb relations. These relations must be solved numerically for almost all practical cases.

## Signal processing

I just looked over the course notes for this topic.

# Nonlinear acoustics

This page is organized by the first and second halves of Dr. Hamilton's Nonlinear Acoustics class:

1. [Linear lossy, nonlinear exact, nonlinear approximate, and Burgers equations](#)
2. [Rankine-Hugoniot relations, weak shock theory, radiation force, and acoustic streaming](#)

I mainly refer to the two review guides I wrote for Dr. Hamilton's Nonlinear Acoustics course in spring 2023, linked below. These reviews skip over some applications of nonlinear acoustics (the parametric array, HIFU, phase conjugation, etc.), but they cover the major theoretical developments of the course. I have also included some additional (mostly conceptual) questions.

The textbook used in the course is *Nonlinear Acoustics*, 3rd ed., edited by M. F. Hamilton and D. T. Blackstock (Springer, 2024). The textbook is open-access and can be downloaded [here](#).

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## Linear lossy, nonlinear exact, nonlinear approximate, and Burgers equations

The first half of the course is investigated in a Socratic method (similar to the physical acoustics page) [here](#). The [detailed solutions](#) are also available.

Below are a few conceptual questions about the governing equations of nonlinear acoustics. Answering these questions may help acquire a bigger picture of nonlinear acoustics.

1. What sort of wave motion does

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} = -\frac{\delta}{c_0^4} \frac{\partial^3 p}{\partial t^3} \quad (1)$$

describe? Qualitatively describe the solutions of this wave equation. [\[ANSWER\]](#)

This is the linear wave equation with attenuation, where  $\delta$  is the so-called "diffusivity of sound." In retarded time, this wave equation can be transformed into a diffusion-type equation (except with space and time-flipped from the conventional diffusion equation). This means that in retarded time, the acoustic pressure diffuses in  $\tau$ . For example, for a delta function at  $x = 0$ , the solution is a Gaussian in  $\tau$ . *Also think about this evolution in terms of the attenuation coefficient  $\alpha = \delta\omega^2/2c_0^3$ .*

2. How does

$$\frac{\partial p}{\partial x} = \frac{\delta}{2c_0^3} \frac{\partial^2 p}{\partial \tau^2} \quad (2)$$

relate to equation (1)? Which of these two equations is invariant under the exchange  $t \mapsto -t$ ? How can this time-reversal symmetry be consistent with the second law of thermodynamics? Resolve this paradox. [\[ANSWER\]](#)

Equation (2) is derived from equation (1) by making the substitution  $(\eta x, t - x/c_0) \mapsto (x_1, \tau)$ , where  $\eta$  is the acoustic Stokes number [See problem 1(j) of the [first review guide](#)]. The even derivative in  $\tau$  makes the second equation invariant under the exchange  $t \mapsto -t$ , but this is an artificial manifestation of time-reversal symmetry, because the second equation is a progressive wave equation. Progressive wave equations describe disturbances that go in one direction as time increases, so by definition, progressive wave equations do not obey time-reversal symmetry.

3. If the source condition is  $p(0, \tau) = \sin(\omega \tau)$ , qualitatively describe the evolution of  $p(x, \tau)$ , the solution of equation (2). [\[ANSWER\]](#)

$p(x, \tau)$  will simply decrease in amplitude, but the waveform will not distort, since it only contains a single frequency.

4. What sort of wave motion does

$$\frac{\partial u}{\partial t} + (c + u) \frac{\partial u}{\partial x} = 0 \quad (3)$$

describe? Qualitatively describe its solutions. After whom are these solutions named? [\[ANSWER\]](#)

This is the exact lossless nonlinear progressive wave equation. Comparison of this progressive wave equation to the linear lossless progressive wave equation suggests that these solutions, named after Poisson, travel faster if they have a larger amplitude, and slower if they have a smaller amplitude. Thus the Poisson solution generally hosts waveform-steepening. These solutions are valid up to the shock formation distance,  $\bar{x}$ . See [these notes](#) for a derivation of this equation from first principles (a good exercise).

5. What was Earnshaw's contribution to the solution of equation (3)? [\[ANSWER\]](#)

Earnshaw matched the Poisson solution to a boundary condition.

6. What is the relationship between

$$\frac{\partial u}{\partial x} = \frac{\beta}{c_0^2} u \frac{\partial u}{\partial \tau} \quad (4)$$

and equation (3)? [\[ANSWER\]](#)

Equation (4) is the  $\mathcal{O}(\epsilon^2)$  approximation of equation (3).

7. What is the name of the exact explicit solution of equation (4) in terms of a Fourier sine series?

[ANSWER]

This is Fubini solution.

8. If one has a solution  $p(x, \tau)$  or  $u(x, \tau)$  to a nonlinear wave equation, what is the first step in finding the shock-formation distance  $x_{sh}$ ? [ANSWER]

Since a shock is where the slope of the waveform of the solution becomes infinite, the  $\tau$ -derivative of the solution should be set to infinity, i.e.,

$$\frac{\partial p}{\partial \tau} = \infty, \quad \text{or} \quad \frac{\partial u}{\partial \tau} = \infty$$

9. What is the name and order  $\mathcal{O}$  of

$$\frac{\partial p}{\partial x} - \frac{\delta}{2c_0^3} \frac{\partial^2 p}{\partial \tau^2} = \frac{\beta p}{\rho_0 c_0^3} \frac{\partial p}{\partial \tau} \quad ? \quad (5)$$

How is this equation qualitatively related to equations (2)-(4)? How does the Burgers equation of acoustics differ from the more widely known Burgers equation (found on [Wikipedia](#), for example)? What is the significance of the Burgers equation in nonlinear acoustics? [ANSWER]

This is the Burgers equation, and it is  $\mathcal{O}(\epsilon^2)$ . This equation can be thought of as a combination of equation (4) (the quadratic order lossless nonlinear progressive wave equation) and equation (2) (the linear lossy progressive wave equation). The Burgers equation is the simplest model of nonlinear and absorptive wave propagation, which allows for the study of waves beyond the point of shock formation.

10. When nonlinearity dominates absorption, what is the name of the solution of equation (5) in the region  $\sigma = x/\bar{x} \gtrsim 3$ ? [ANSWER]

This is the Fay solution.

11. Nonlinearity is due to finite displacement and nonlinear governing equations, while dispersion is due to thermoviscous effects, boundary layers, and/or relaxation, but are any of these dispersion effects nonlinear? In other words, are nonlinearity and dispersion separate concepts, or are there situations in which dispersion manifests as nonlinearity? [ANSWER]

This was in fact a question I asked Dr. Hamilton. His response:

We do not account for nonlinearity in the absorption and dispersion terms, which is to say that we do not account for terms of  $\mathcal{O}(\epsilon^2 \eta)$ , so yes, we treat the effects as separate concepts. The nonlinear viscous terms, for example, can be taken into account, and there are studies of this, but the associated effects are significant only in special cases.

# Rankine-Hugoniot relations, weak shock theory, radiation force, and acoustic streaming

The second half of the course is investigated [in this review](#), and the solutions are available [here](#).

Below are some additional questions, which are mostly conceptual. The final three questions are more involved and are included mainly for completeness of the important equations of nonlinear acoustics.

1. What is weak shock theory? *[ANSWER]*

Weak shock theory simplifies the study of shock waves to isentropic and reflectionless jumps at  $\mathcal{O}(\epsilon^2)$ . A corollary of these simplifications is that the loss of energy is localized to the shock front.

2. Qualitatively describe the speed at which a weak shock travels. *[ANSWER]*

The weak shock speed is related to the average of the particle velocity ahead and behind the shock front.

3. Qualitatively describe what determines the location of a shock front in retarded time in terms of the geometry of an overturned waveform. *[ANSWER]*

The location of a shock front in retarded time is given by the point  $\tau_{\text{sh}}$  at which the positive and negative areas of the overturned wavefront are equal. This is called "Landau's equal-area rule."

4. Which two solutions are bridged together by Blackstock's bridging function

$$B_n^{(1)} = \frac{2P_{\text{sh}}}{n\pi}$$
$$B_n^{(2)} = \frac{2}{n\pi\sigma} \int_{\Phi_{\text{sh}}}^{\pi} \cos[n(\Phi - \sigma \sin \Phi)] d\Phi ?$$

Recall that  $B_n$  is the Fourier sine series expansion coefficients, i.e.,  $P(\sigma, \theta) = \sum_{n=1}^{\infty} B_n(\sigma) \sin n\theta$ . What is the significance of Blackstock's bridging function? *[ANSWER]*

The Blackstock bridging function bridges the Fubini solution (which solves the exact lossless nonlinear progressive wave equation) to the Fay solution (which solves the Burgers equation for  $\sigma \gtrsim 3$ ). The significance of Blackstock's bridging function is the full evolution it provides of a progressive wave at  $\mathcal{O}(\epsilon^2)$  for  $0 \leq \sigma < \infty$ . It recovers the Fubini solution for  $0 \leq \sigma \leq 1$  and the Fey solution for  $\sigma \gtrsim 3$ .

5. True/False: the lossless exact progressive wave equation, the Burgers equation, and weak shock theory all describe isentropic wave propagation. *[ANSWER]*

True. The lossy linear progressive wave equation, and hence the Burgers equation, describes isentropic wave propagation. Weak shock theory at  $\mathcal{O}(\epsilon^2)$  is also isentropic. This somewhat

rationalizes how Blackstock's bridging function miraculously "bridges" a lossless exact solution (Fubini) to a lossy quadratic-order solution (Fay).

6. Qualitatively discuss the phenomenon of *waveform freezing*. [ANSWER]

Geometric spreading competes with nonlinear shock formation.

7. Coordinates that are fixed in space are called \_\_\_\_\_ coordinates, analogous to sitting on the bank of a river and watching a fallen leaf go by. [ANSWER]

Eulerian

8. Coordinates that are move along with particles are called \_\_\_\_\_ coordinates, analogous to swimming in the river alongside the fallen leaf. [ANSWER]

Lagrangian

9. *Not a conceptual question*: Transform the linear wave equation

$$\nabla^2 p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} = 0 \quad (6)$$

from  $(x, y, z, t)$  to  $(x, y, z, \tau)$  coordinates, where  $\tau = t - z/c_0$ . The correct result is

$$\frac{\partial^2 p}{\partial \tau \partial z} = \frac{c_0}{2} \nabla^2 p, \quad (7)$$

as given by P. V. Yuldashev and V. A. Khokhlova, "Simulation of Three-Dimensional Nonlinear Fields of Ultrasound Therapeutic Arrays," *Acoustical Physics*, **57**, 334-343, 2011, equation (3).

[ANSWER]

To facilitate the coordinate transformation, denote the new spatial coordinates with primes, i.e.,  $(x', y', z', \tau)$ . Start by writing the partial derivatives in the old  $(x, y, z, t)$  coordinates in terms of the new  $(x', y', z', \tau)$  coordinates:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'} \frac{\partial y'}{\partial x} + \frac{\partial}{\partial z'} \frac{\partial z'}{\partial x} + \frac{\partial}{\partial \tau} \frac{\partial \tau}{\partial x} = \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial x'} \frac{\partial x'}{\partial y} + \frac{\partial}{\partial y'} \frac{\partial y'}{\partial y} + \frac{\partial}{\partial z'} \frac{\partial z'}{\partial y} + \frac{\partial}{\partial \tau} \frac{\partial \tau}{\partial y} = \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial x'} \frac{\partial x'}{\partial z} + \frac{\partial}{\partial y'} \frac{\partial y'}{\partial z} + \frac{\partial}{\partial z'} \frac{\partial z'}{\partial z} + \frac{\partial}{\partial \tau} \frac{\partial \tau}{\partial z} = \frac{\partial}{\partial z'} - \frac{1}{c_0} \frac{\partial}{\partial \tau} \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial}{\partial y'} \frac{\partial y'}{\partial t} + \frac{\partial}{\partial z'} \frac{\partial z'}{\partial t} + \frac{\partial}{\partial \tau} \frac{\partial \tau}{\partial t} = \frac{\partial}{\partial \tau} \end{aligned}$$

Using these relations, note that

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial x'^2} \\ \frac{\partial^2}{\partial y^2} &= \frac{\partial^2}{\partial y'^2} \\ \frac{\partial^2}{\partial z^2} &= \frac{\partial^2}{\partial z'^2} - \frac{2}{c_0} \frac{\partial}{\partial z'} \frac{\partial}{\partial \tau} + \frac{1}{c_0^2} \frac{\partial^2}{\partial \tau^2} \\ \frac{\partial^2}{\partial t^2} &= \frac{\partial^2}{\partial \tau^2}\end{aligned}$$

Thus equation (6) becomes

$$\left( \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{2}{c_0} \frac{\partial^2}{\partial z' \partial \tau} + \frac{1}{c_0^2} \frac{\partial^2}{\partial \tau^2} \right) p - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial \tau^2} = 0.$$

Canceling common terms and rearranging gives

$$\nabla^2 p = \frac{2}{c_0} \frac{\partial^2 p}{\partial z' \partial \tau}.$$

Finally, recalling that  $z'$  is but a placeholder, and that in fact  $z' = z$ , gives the desired equation (7),

$$\frac{\partial^2 p}{\partial \tau \partial z} = \frac{c_0}{2} \nabla^2 p.$$

10. Equation (5) for propagation in the  $z$  direction is  $\frac{\partial p}{\partial z} = \frac{\delta}{2c_0^3} \frac{\partial^2 p}{\partial \tau^2} + \frac{\beta p}{\rho_0 c_0^3} \frac{\partial p}{\partial \tau}$ . Take the partial derivative with respect to  $\tau$  and add the result to the right-hand side of equation (7) to obtain

$$\frac{\partial^2 p}{\partial \tau \partial z} = \frac{c_0}{2} \nabla^2 p + \frac{\delta}{2c_0^3} \frac{\partial^3 p}{\partial \tau^3} + \frac{\beta p}{2\rho_0 c_0^3} \frac{\partial^2 p^2}{\partial \tau^2}. \quad (8)$$

What is the name of the resulting wave equation? *[ANSWER]*

Taking the partial derivative with respect to  $\tau$  gives

$$\frac{\partial^2 p}{\partial \tau \partial z} = \frac{\delta}{2c_0^3} \frac{\partial^3 p}{\partial \tau^3} + \frac{\beta p}{\rho_0 c_0^3} \frac{\partial^2 p}{\partial \tau^2}$$

Adding the above to the right-hand side of equation (7) gives the *Westervelt equation*, equation (8)

11. The KZK equation is

$$\frac{\partial^2 p}{\partial \tau \partial z} = \frac{c_0}{2} \nabla_{\perp}^2 p + \frac{\delta}{2c_0^3} \frac{\partial^3 p}{\partial \tau^3} + \frac{\beta}{2\rho_0 c_0^3} \frac{\partial^2 p^2}{\partial \tau^2}. \quad (1)$$

Qualitatively describe the difference between the Westervelt and KZK equations. Why would someone describe a sound beam using the KZK equation instead of using the Westervelt equation? *[ANSWER]*

Again quoting Dr. Hamilton:

The KZK equation simply uses the Fresnel (paraxial) approximation of the Helmholtz equation operator ( $\nabla^2 + k^2$ ) that is on the left-hand side of the traditional form of the Westervelt equation...Analytical and numerical solutions of the KZK equation are much more easily obtained than the corresponding solutions of the Westervelt equation.

# Math

This section covers the list of math topics provided [here](#):

solutions of ordinary and partial differential equations, power series solutions, linear independence, elementary linear algebra, determinants, Taylor series, eigenvalue problems, orthogonality, Fourier series and integrals, integration by parts, vector algebra, complex numbers, special functions (e.g., Bessel functions and Legendre polynomials).

I have organized these topics in a way that progresses from basic mathematics to more difficult concepts:

1. [Review of basics](#)
2. [Taylor series](#)
3. [Fourier series and transforms](#)
4. [Linear algebra](#)
5. [Ordinary differential equations](#)
6. [Orthogonality and special functions](#)
7. [Vector algebra](#)
8. [Vector calculus](#)
9. [Partial differential equations](#)

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## Review of basics

1. State the fundamental theorem of algebra (d'Alambert's theorem). [ANSWER]

Every non-constant single-variable polynomial  $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$  with complex coefficients  $a_n, a_{n-1}, \dots, a_0$  has at least one complex root.

2. State the rational root theorem. [ANSWER]

Each rational root of  $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$  is given by  $x = p/q$ , where  $p$  is an integer factor of  $a_0$ , and where  $q$  is an integer factor of  $a_n$ .

3. Solve  $x^3 - x^2 - 5x - 3 = 0$ . [ANSWER]

$$x = -1, -1, 3$$

4. Solve  $x^3 - x^2 - 8x - 6 = 0$ . [ANSWER]

$$x = -1, 1 + \sqrt{7}, 1 - \sqrt{7}$$

5. Solve  $x^4 - 2x^3 - 3x^2 + 8x - 4 = 0$ . [ANSWER]

$$x = -2, 1, 1, 2$$

6. Decompose  $1/(x^2 + 2x - 3)$  into partial fractions. [ANSWER]

$$\frac{1}{4} \left( \frac{-1}{x+3} + \frac{1}{x-1} \right)$$

7. Decompose  $(x^3 + 16)/(x^3 - 4x^2 + 8x)$  into partial fractions. [ANSWER]

See example 2 [here](#).

8. Decompose  $1/(x^3 - 1)$  into partial fractions. [ANSWER]

See example 5 [here](#) and note that the limit method need not be used.

9. Find the roots of  $\exp(2z) = 2i$ . [ANSWER]

Write  $i = e^{i \arctan \infty} = e^{i\pi/4}$ . Then take the log of both sides, giving  $z = \frac{\ln 2}{2} + i \frac{\pi}{4}$ .

10. Evaluate  $(-i)^{1/3}$ . [ANSWER]

$$\begin{aligned}
(-i)^{1/3} &= (e^{-i\pi/2})^{1/3} \\
&= e^{-i\pi/6} \\
&= (\cos \pi/6 - i \sin \pi/6) \\
&= (\sqrt{3}/2 - i/2)
\end{aligned}$$

11. Find the real part of  $e^{-ix}/(1 + e^{a+ib})$ . [ANSWER]

$$\begin{aligned}
\frac{e^{-ix}}{1 + e^{a+ib}} &= \frac{\cos x - i \sin x}{1 + e^a e^{ib}} \\
&= \frac{\cos x - i \sin x}{1 + e^a(\cos b + i \sin b)} \\
&= \frac{\cos x - i \sin x}{1 + e^a(\cos b + i \sin b)} \frac{1 + e^a(\cos b - i \sin b)}{1 + e^a(\cos b - i \sin b)} \\
&= \frac{\cos x + e^a \cos x(\cos b - i \sin b) - i \sin x - ie^a \sin x(\cos b - i \sin b)}{1 + e^a(\cos b + i \sin b) + e^a(\cos b - i \sin b) + e^{2a}(\cos b + i \sin b)(\cos b - i \sin b)} \\
&= \frac{\cos x + e^a \cos x \cos b - ie^a \cos x \sin b - i \sin x - ie^a \sin x \cos b - e^a \sin x \sin b}{1 + e^a(\cos b) + e^a(\cos b) + e^{2a}(\cos^2 b + \sin^2 b)} \\
&= \frac{\cos x + e^a \cos x \cos b - e^a \sin x \sin b}{1 + e^{2a}(\cos^2 b + \sin^2 b)} - i \frac{e^a \cos x \sin b + \sin x + e^a \sin x \cos b}{1 + 2e^a \cos b + e^{2a}(\cos^2 b + \sin^2 b)}
\end{aligned}$$

The real part is  $\frac{\cos x + e^a \cos x \cos b - e^a \sin x \sin b}{1 + e^{2a}(\cos^2 b + \sin^2 b)}$ .

12. State the fundamental theorem of calculus. [ANSWER]

The derivative and integral are inverses.

13. What is the relationship between differentiability and continuity? [ANSWER]

Differentiability implies continuity, but continuity does not imply differentiability. For an example of a continuous function that is not differentiable, consider  $\sqrt{|x|}$  at  $x = 0$ .

14.  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x^2 + x} =$  [ANSWER]

1 by L'Hopital's rule.

15.  $\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x - \sin x} =$  [ANSWER]

6 by repetitive use of L'Hopital's rule.

16.  $\lim_{x \rightarrow \infty} x^n e^{-x} =$  [ANSWER]

0 by L'Hopital's rule. Repeat L'Hopital's rule until  $x^{n-1} = x^0$ .

17.  $d \sinh x / dx =$  [ANSWER]

cosh  $x$

18.  $d \cosh x/dx =$  [ANSWER]

$\sinh x$

19.  $d \tan x/dx =$  [ANSWER]

$\sec^2 x$

20.  $d \cot x/dx =$  [ANSWER]

$-\csc^2 x$

21.  $d \sec x/dx =$  [ANSWER]

$\sec x \tan x$

22.  $d \csc x/dx =$  [ANSWER]

$-\cot x \csc x$

23.  $\int dx(x^4 + x^3 + x^2 + 1)/(x^2 + x - 2) =$  [ANSWER]

See example 6 [here](#).

24. Show that  $\int udv = uv - \int vdu$ . [ANSWER]

Start with the product rule,  $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ , integrate both sides over  $x$ , and rearrange.

25.  $\int \ln x dx =$  [ANSWER]

$x \ln(x) - x$

26.  $\int e^x \sin x dx =$  [ANSWER]

Integrate by parts twice to get  $\int e^x \sin x dx = e^x[\sin(x) - \cos(x)]/2$ .

27. What are some general guidelines about trigonometric substitution? [ANSWER]

If a quantity  $a^2 - x^2$  is involved, set  $x = a \sin \theta$ . If a quantity  $x^2 + a^2$  is involved, set  $x = a \tan \theta$ .  
If a quantity  $x^2 - a^2$  is involved, set  $x = a \sec \theta$ .

28.  $\int \sqrt{a^2 - x^2} dx =$  [ANSWER]

$$\frac{a^2}{2} \arcsin \frac{x}{a} + \frac{a^2}{4} \frac{x}{a} [1 - (x/a)^2]^{1/2} + C.$$

See [this page](#) for more on trigonometric substitution.

29.  $\int \frac{dx}{\sqrt{a^2 - x^2}} =$  [ANSWER]

$$\arcsin(x/a) + C$$

$$30. \int \frac{dx}{a^2+x^2} = \text{[ANSWER]}$$

$$\frac{1}{a} \arctan(x/a) + C$$

$$31. \otimes \int \sqrt{a^2 + x^2} dx = \text{[ANSWER]}$$

See [this page](#) for the solution. The [integral of secant cubed](#) is needed.

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[→ Advance to next topic](#)

# Taylor series

1. What is the difference between a Taylor series and a Maclaurin series? [\[ANSWER\]](#)

A Taylor series of  $f(x)$  is

$$f(x - a) = f(a) + f'(a)(x - a) + \frac{1}{2!}f''(a)(x - a)^2 + \dots,$$

while the Maclaurin series is the special case for  $a = 0$ .

2. Write the first three nonzero terms of the Maclaurin series representation of  $e^x$ ,  $\sin x$ ,  $\cos x$ , and  $\tan x$ . [\[ANSWER\]](#)

$$\begin{aligned}e^x &\simeq 1 + x + \frac{x^2}{2!} \\ \sin x &\simeq x - \frac{x^3}{3!} + \frac{x^5}{5!} \\ \cos x &\simeq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\ \tan x &\simeq x + \frac{x^3}{3}\end{aligned}$$

Sorry, I will go no higher.

3. Show that  $\cos x = (e^{jx} + e^{-jx})/2$ ,  $\sin x = (e^{jx} - e^{-jx})/2j$ .  
4. Write the first three terms of the Maclaurin series representation of  $\ln(1 + x)$ . [\[ANSWER\]](#)

$$\ln(1 + x) \simeq x - \frac{x^2}{2} + \frac{x^3}{3} \text{ near } x = 0$$

5. Write the first three terms of the Maclaurin series representation of  $\ln(1 - x)$  [\[ANSWER\]](#)

$$\ln(1 - x) \simeq -x - \frac{x^2}{2} - \frac{x^3}{3} \text{ near } x = 0$$

6. Write the first three terms of the Taylor series representation of  $\ln(-x)$  about  $x = -1$ . Verify your answer for by checking that it can be obtained from the previous expansion for  $\ln(1 - x)$  by shifting the latter one unit in the negative  $x$  direction. [\[ANSWER\]](#)

$$\ln(-x) \simeq -(x + 1) - \frac{(x + 1)^2}{2} - \frac{(x + 1)^3}{3} \text{ near } x = -1$$

This matches the previous result when the previous result is shifted one unit in the  $-x$  direction.



# Fourier series and transforms

1. The sine-cosine form of the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)].$$

Why is  $a_0$  not included in the sum? [ANSWER]

$a_0$  is not included in the sum to preserve a symmetric form of the definitions of  $a_n$  and  $b_n$ .

2. Use orthogonality to determine  $a_0$ ,  $a_n$ , and  $b_n$ . [ANSWER]

First multiply both sides of the form above by  $\cos(mx)dx$  on both sides and integrate from  $-\pi$  to  $\pi$ . Do the same for  $\sin(mx)dx$ . Use the relations

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \sin nxdx &= \delta_{nm}\pi \\ \int_{-\pi}^{\pi} \cos mx \cos nxdx &= \delta_{nm}\pi \quad (2\pi \text{ for } m = n = 0) \\ \int_{-\pi}^{\pi} \sin mx \cos nxdx &= 0 \end{aligned}$$

and obtain

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nxdx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nxdx \end{aligned}$$

3. If  $f(x)$  is periodic in  $2\pi$ , has a finite number of maxima, minima, and discontinuities, and if  $\int_{-\pi}^{\pi} |f(x)|dx$  is finite, what is the relationship between  $f(x)$  and its Fourier series? What are these conditions called? [ANSWER]

The Fourier series of  $f(x)$  will converge to  $f(x)$  at all points where  $f(x)$  is continuous. For points at which  $f(x)$  is discontinuous (i.e., has a jump), the Fourier series converges to the midpoint of the jump.

These are the Dirichlet conditions.

4. Obtain the Fourier series expansion coefficients of a periodic square wave, which is 1 for  $-\pi \leq x \leq 0$  and  $-1$  for  $0 < x \leq \pi$ . [ANSWER]

Using the definitions above,  $a_0 = 0$ ,  $a_n = 0$ , and  $b_n = -4/n\pi$  for odd  $n$  and 0 for even  $n$ .

5. Obtain the Fourier series expansion coefficients of a periodic sawtooth wave, which is  $x/\pi$  for  $-\pi \leq x \leq \pi$ . [ANSWER]

Using the definitions above,  $a_0 = 0$ ,  $a_n = 0$ , and  $b_n = -(-1)^n 2/n\pi$ .

6. Obtain the Fourier series expansion coefficients for some of the periodic functions listed [here](#). [ANSWER]

The expansion coefficients are listed on that page. The half-wave rectified sine appears in the Penn State math packet.

7. The complex-exponential form of the Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Use orthogonality to determine  $c_n$ . [ANSWER]

The orthogonality relation is

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx} dx = 2\pi \delta_{nm}.$$

Multiplying both sides by  $e^{-imx}$  and integrating from  $-\pi$  to  $\pi$  gives

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

8. [Attempt some of these problems](#) from Boas's *Methods* book, using the sine-cosine and complex exponential Fourier series. [ANSWER]

The answers are provided in that document.

9. State Parseval's theorem. [ANSWER]

Parseval's theorem says that the average value of  $|f(x)|^2$  over a period is the sum of the magnitude-squared of the expansion coefficients. This is most straightforwardly written in terms of the complex exponential Fourier series (because there's only one expansion coefficient):

$$|f(x)|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Parseval's theorem is sometimes referred to as the "completeness relation," because if any one of the harmonics were left out, then  $|f(x)|^2 > \sum_{n=-\infty}^{\infty} |c_n|^2$ , which is known as Bessel's inequality.

10. In what sense is the Fourier transform a generalization of the Fourier series? What are the conditions for the convergence of the Fourier transform? [ANSWER]

The Fourier transform is the "continuum limit" of the Fourier series, expressing a function as the integral of a continuous spectrum of waves, rather than as just the sum of a discrete spectrum of

waves. The Fourier transform applies to arbitrary functions (not necessarily periodic).

The form of the Fourier transform can readily be derived by considering the complex-exponential form of the Fourier series, given above. In the limit that  $n$  is a continuous index  $k$ , the series becomes an integral,

$$f(x) = \int_{-\infty}^{\infty} c(k)e^{ikx} dk,$$

and the coefficients  $c(k)$  are

$$c(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-ikx} dx.$$

Note that the appearance of  $1/2\pi$  in the expression for  $c(k)$  is merely a convention.

The conditions for the convergence of the Fourier transform are the same as those for the Fourier series (the Dirichlet conditions).

11. Given the function

$$f(x) = \begin{cases} 1, & x \in [-1, 1] \\ 0, & |x| \in (1, \infty) \end{cases}.$$

calculate its Fourier transform  $c(k)$ . Write the integral representation of  $f(k)$  in terms of the calculated  $c(k)$ . [ANSWER]

Calculating the Fourier transform in this case is straightforward:

$$\begin{aligned} c(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-ikx} dx = \frac{1}{2\pi} \int_{-1}^1 e^{-ikx} dx \\ &= \frac{i}{2\pi k} (e^{-ik} - e^{ik}) \\ &= -\frac{i}{2\pi k} (e^{ik} - e^{-ik}) \\ &= -\frac{i}{2\pi k} 2i \sin k \\ &= \frac{\sin k}{\pi k} \end{aligned}$$

So the integral representation of the given function is  $f(x) = \int_{-\infty}^{\infty} \frac{\sin k}{\pi k} e^{ikx} dk$ .

12. Given the function

$$f(x) = \begin{cases} x, & x \in [0, 1] \\ 0, & \text{otherwise} \end{cases}.$$

calculate its Fourier transform  $c(k)$ . Write the integral representation of  $f(k)$  in terms of the calculated  $c(k)$ . [ANSWER]

$$\begin{aligned}
c(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-ikx} dx = \frac{1}{2\pi} \int_0^1 xe^{-ikx} dx \\
&= \frac{ixe^{-ikx}}{2\pi k} + \frac{e^{-ikx}}{2\pi k^2} \Big|_{x=0}^1 \\
&= \frac{ie^{-ik}}{2\pi k} + \frac{e^{-ik} - 1}{2\pi k^2}
\end{aligned}$$

So the integral representation of the given function is

$$f(x) = \int_{-\infty}^{\infty} \left( \frac{ie^{-ik}}{2\pi k} + \frac{e^{-ik} - 1}{2\pi k^2} \right) e^{ikx} dk.$$

13. What does a Gaussian pulse in the time domain i.e.,  $\exp(-t^2/T^2)$  look like in the frequency domain? [\[ANSWER\]](#)

Since the Fourier transform of a Gaussian is a Gaussian, the signal in the frequency domain is a Gaussian.

14. ✳ Now consider a sine wave modulated by a Gaussian envelope in the time domain, i.e.,  $\exp(j\omega_0 t) \exp(-t^2/T^2)$ . What does this signal look like in the frequency domain? [\[ANSWER\]](#)

The Fourier transform of this signal is also a pure Gaussian. I confirmed this analytically and numerically.

At first, it was not intuitively clear to me why this signal is a pure Gaussian in the frequency domain, rather than a combination of a Gaussian and a delta function in the frequency domain. I was expecting a  $\delta$ -function to appear at the frequency  $\omega_0$ . This can be rationalized by making use of the convolution theorem, which states that the Fourier transform of the product of two functions is equal to the convolution of their individual Fourier transforms. We wish to find the Fourier transform of a sinusoid  $\times$  a Gaussian, and we know that the Fourier transform of a sinusoid is a delta function, while that of a Gaussian is a Gaussian. By the convolution theorem, the Fourier transform of the product is the convolution of the delta function and the Gaussian, which is a pure Gaussian.

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# Linear algebra

These questions come from *Introduction to Linear Algebra* by Gilbert Strang and the appendix of *Introduction to Quantum Mechanics* by D. J. Griffiths.

Towards the end of this section there is some overlap with *tensor* algebra.

1. Define linear independence. [ANSWER]

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$ , are linearly independent iff

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \dots + a_n\mathbf{v}_n = \mathbf{0}$$

for  $a_1, a_2, a_3, \dots, a_n = 0$ .

2. The columns of an invertible matrix are \_\_\_\_\_ . [ANSWER]

linearly independent

3. How can one easily check to see that  $n$  vectors, each one  $n \times 1$ , are linearly independent? [ANSWER]

One can assemble a matrix of the  $n$  vectors as  $n \times 1$  columns and see if the determinant is nonzero. If so, the matrix can be inverted, and the columns are linearly independent. If the determinant is 0, the columns are not linearly independent.

4. Determine whether the vectors  $\mathbf{v}_1 = \mathbf{e}_x + \mathbf{e}_y$  and  $\mathbf{v}_2 = -3\mathbf{e}_x + 2\mathbf{e}_y$  are linearly independent, where  $\mathbf{e}_x$  and  $\mathbf{e}_y$  are the Cartesian basis vectors. [ANSWER]

The matrix representation of  $\mathbf{v}_1$ ,  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , and the matrix representation of  $\mathbf{v}_2$  is  $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ . Then, by the definition of linear independence,

$$a_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This can be written as

$$\begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Performing row reduction yields the identity matrix. Therefore the columns are linearly independent.

5. Taking  $\mathbf{v}_1 = \mathbf{e}_x + \mathbf{e}_y$  and  $\mathbf{v}_2 = -3\mathbf{e}_x + 2\mathbf{e}_y$  as basis vectors, what linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$  gives the vector  $\mathbf{v}_3 = 2\mathbf{e}_x + \mathbf{e}_y$ ? [ANSWER]

The question at hand is to explore

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0}.$$

Perform row operations on the matrix representation of this equation,

$$\begin{pmatrix} 1 & -3 & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

gives

$$\begin{pmatrix} 1 & 0 & 7/5 \\ 0 & 1 & -1/5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus  $a_1 = -\frac{7}{5}a_3$  and  $a_2 = \frac{1}{5}a_3$ . To answer the question at hand, set  $a_3 = -1$  to find that  $a_1 = \frac{7}{5}$  and  $a_2 = -\frac{1}{5}$ . That is,  $\frac{7}{5}\mathbf{v}_1 - \frac{1}{5}\mathbf{v}_2 = \mathbf{v}_3$ .

6. Invert

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

using row operations. Check the result by using Cramer's rule for the inverse,  $A^{-1} = \frac{1}{\det A}C^T$ , where  $C$  is the matrix of cofactors, which is  $(-1)^{i+j}$  times the determinant of  $A$  when the  $i$ th row and  $j$ th column are crossed out. [ANSWER]

See [here](#) for the inverse taken using row operations.

7. Invert

$$A = \begin{pmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

using row operations. Check the result by using  $A^{-1} = \frac{1}{\det A}C^T$ . [ANSWER]

See [here](#) for the solution.

8. What is the rank of a matrix? How does one find the rank of a matrix? [ANSWER]

The rank of a matrix is the dimension of the vector space spanned by its columns (or equivalently, rows). To find the rank of a matrix, the number of linearly independent columns of the matrix must be determined.

9. Find the rank of

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{pmatrix}$$

using row operations. [ANSWER]

The rank is 2. See [here](#).

10. Find the rank of

$$\begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{pmatrix}$$

What is the rank of the transpose of this matrix? [ANSWER]

Using row operations leads to

$$\begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

from which it can be seen that the third column is  $-2\mathbf{v}_1 + \mathbf{v}_2$ , where  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are the column vectors of the original matrix. However, the rest of the column vectors are linearly independent. Thus the rank of the matrix is 3. In general the rank of the transpose of a matrix is rank of its transpose, i.e., the rank of the transpose of this matrix is 3.

11. ✪ What are the four fundamental subspaces corresponding to an  $m \times n$  matrix? State the fundamental theorem of linear algebra. [ANSWER]

The four subspaces of an  $m \times n$  matrix  $A$  are listed below:

1. The column space is the vector space spanned by the linearly independent columns of  $A$ . That is to say, the linearly independent columns  $A$  form the basis of the column space.
2. The nullspace is the vector space spanned by all the vectors  $\mathbf{x}$  that satisfy  $A\mathbf{x} = \mathbf{0}$ .
3. The row space is just the column space of  $A^T$ .
4. The left nullspace is just the nullspace of  $A^T$ .

The fundamental theorem of linear algebra provides the following relations between the four subspaces:

- The column space and row space both have dimension  $r$ , which is the rank of the matrix. The nullspace has dimension  $n - r$ , and the left nullspace has dimension  $m - r$ .
- The nullspace is orthogonal to the row space, and the column space is orthogonal to the left nullspace.

The first statement makes sense because if  $n$  is the rank of  $A$  (i.e.,  $r = n$ ), then the matrix is full rank, i.e.,  $A$  is invertible. Thus its nullspace has dimension 0; indeed,  $n - r = n - n = 0$ . Now suppose that the rank of  $A$  is less than  $n$ , i.e.,  $n > r$ . Then, there are  $n - r > 0$  free variables in the solution to  $A\mathbf{x} = \mathbf{0}$ . That is to say,  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n-r}$  form a basis of dimension  $n - r$  in the nullspace of  $A$ .

The second statement says that if  $\mathbf{v}$  is in the nullspace of  $A$ , and if  $\mathbf{w}$  is in the row space of  $A$ , then  $\mathbf{v} \cdot \mathbf{w} = 0$ . This makes sense because each row vector  $\mathbf{w}$  of  $A$ , when multiplied by a vector  $\mathbf{v}$  from the nullspace, gives 0 on the right-hand side of the equation  $A\mathbf{v} = 0$ . (A similar argument can be made for the column space and the left nullspace).

12. Prove the triangle inequality,  $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ . [ANSWER]

Note that

$$\begin{aligned} |\mathbf{u} + \mathbf{v}|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2|\mathbf{u}||\mathbf{v}| \cos \theta. \end{aligned} \quad (\text{i})$$

Meanwhile note that

$$(|\mathbf{u}| + |\mathbf{v}|)^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2|\mathbf{u}||\mathbf{v}|. \quad (\text{ii})$$

Comparing equations (i) and (ii) and noting that  $|\cos \theta| \leq 1$  results in

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 + 2|\mathbf{u}||\mathbf{v}| \cos \theta \leq |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2|\mathbf{u}||\mathbf{v}|$$

which is equivalent to

$$|\mathbf{u} + \mathbf{v}|^2 \leq (|\mathbf{u}| + |\mathbf{v}|)^2 \quad (\text{iii})$$

Taking the square root of equation (iii) gives the triangle inequality,

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|.$$

13.  $a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z$  is a Cartesian 3-vector. Does the subset of all vectors with  $a_z = 0$  constitute a vector space? If so, what is its dimension; if not, why not? [ANSWER]

Yes,  $a_x \mathbf{e}_x + a_y \mathbf{e}_y$  constitutes a vector space in the  $x$ - $y$  plane of dimension 2.

14. Does the subset of all vectors with  $a_z = 1$  constitute a vector space? Explain why or why not. [ANSWER]

No, the subset of vectors with  $a_z = 1$  does not span the space for two reasons. The first reason is that linear combinations of these vectors result in vectors outside the space. For example, adding two vectors from this subset would result in a vector with  $a_z = 2$ , which is not in the subset. The second reason is that this subset does not have a null vector,  $(0, 0, 0)$ .

15. Does the subset of all vectors with  $a_x = a_y = a_z$  constitute a vector space? Explain why or why not. [ANSWER]

Yes, this combination constitutes a vector space with dimension 1.

16. Consider the collection of all polynomials in  $x$  with complex coefficients of degree less than  $N$ . Does this set constitute a "vector" space? If so, suggest a convenient basis and provide its dimensions. [ANSWER]

Yes, this combination constitutes a vector space with dimension  $N - 1$ . A convenient basis is  $1, x, x^2, \dots, x^{N-1}$ .

17. Do polynomials that are even functions constitute a "vector" space? What about polynomials that are odd functions? *[ANSWER]*

Yes, even polynomials constitute a vector space with dimension  $(N)/2$ . A convenient basis is  $1, x^2, x^4, \dots, x^N$ . The dimension is  $N/2$ . Meanwhile, odd polynomials constitute a vector space with dimension  $(N - 1)/2$ . A convenient basis is  $x, x^3, x^5, \dots, x^{N-1}$ .

18. Do polynomials whose leading coefficient is 1 constitute a "vector" space? *[ANSWER]*

No, because the sum of two such polynomials would have a leading coefficient not equal to one, and therefore the sum would not be in the space.

19. Do polynomials whose value of 0 at  $x = 1$  constitute a "vector" space? *[ANSWER]*

Yes;  $1, x - 1, (x - 1)^2, \dots, (x - 1)^{N-1}$ . The dimension is  $N - 1$ .

20. Do polynomials whose value of 1 at  $x = 0$  constitute a "vector" space? *[ANSWER]*

No, because the sum of two such polynomials would not be 1 at  $x = 0$ , and therefore the sum would not be in the space.

21. Provide definitions of the following types of matrices: symmetric, Hermitian, skew-symmetric, skew-Hermitian, orthogonal, unitary. *[ANSWER]*

$$\text{Symmetric: } A = A^T.$$

$$\text{Hermitian: } A = A^\dagger.$$

$$\text{Skew-symmetric: } A = -A^T.$$

$$\text{Skew-Hermitian: } A = -A^\dagger.$$

$$\text{Orthogonal: } A^{-1} = A^T.$$

$$\text{Unitary: } A^{-1} = A^\dagger.$$

22. ✪ Prove that a real symmetric matrix  $S$  has real eigenvalues and orthogonal eigenvectors.

*[ANSWER]*

First the eigenvalues: the eigenvalue equation is  $S\mathbf{x} = \lambda\mathbf{x}$ , which, upon taking the conjugate and noting that  $S$  is real, gives  $S\mathbf{x} = \lambda^*\mathbf{x}^*$ . Now consider the quantity

$$\lambda\mathbf{x}^* \cdot \mathbf{x} = \mathbf{x}^* \cdot \lambda\mathbf{x}$$

Invoking the eigenvalue equation  $S\mathbf{x} = \lambda^*\mathbf{x}^*$  gives

$$\lambda\mathbf{x}^* \cdot \mathbf{x} = \mathbf{x}^* \cdot S\mathbf{x}$$

By the definition of the transpose,

$$\begin{aligned}\lambda \mathbf{x}^* \cdot \mathbf{x} &= S^T \mathbf{x}^* \cdot \mathbf{x} \\ &= S \mathbf{x}^* \cdot \mathbf{x} \\ &= (S^* \mathbf{x}^*) \cdot \mathbf{x}\end{aligned}$$

where the second two equalities are because  $S$  is symmetric and real, respectively. Again invoking the eigenvalue equation gives

$$\begin{aligned}\lambda \mathbf{x}^* \cdot \mathbf{x} &= \lambda^* \mathbf{x}^* \cdot \mathbf{x} \\ \lambda x^2 &= \lambda^* x^2,\end{aligned}$$

or  $\lambda = \lambda^*$  i.e.,  $\lambda$  is real.

Now for the eigenvectors. Consider two distinct eigenvalue-eigenvector pairs of  $S$ :

$$S\mathbf{x} = \lambda\mathbf{x}, \quad S\mathbf{y} = \mu\mathbf{y}$$

Then consider the quantity  $S\mathbf{x} \cdot \mathbf{y}$ . Using the definition of the transpose and the fact that  $S$  is symmetric gives

$$\begin{aligned}S\mathbf{x} \cdot \mathbf{y} &= \mathbf{x} \cdot S^T \mathbf{y} \\ &= \mathbf{x} \cdot S\mathbf{y} \\ \lambda \mathbf{x} \cdot \mathbf{y} &= \mathbf{x} \cdot \mu \mathbf{y},\end{aligned}$$

where the eigenvalue equations have been used in the last equality. Rearranging terms gives

$$\begin{aligned}\lambda \mathbf{x} \cdot \mathbf{y} &= \mu \mathbf{x} \cdot \mathbf{y} \\ (\lambda - \mu) \mathbf{x} \cdot \mathbf{y} &= 0.\end{aligned}$$

By the zero product property, and by noting that  $\lambda \neq \mu$  (i.e., they're distinct),  $\mathbf{x} \cdot \mathbf{y} = 0$ , which means the eigenvectors are orthogonal.

23. ✪ When solving  $A\mathbf{x} = \lambda\mathbf{x}$  where  $\mathbf{x} \neq \mathbf{0}$ , what is the meaning of setting  $\det(A - I\lambda) = 0$ ? [ANSWER]

Note that the equation  $A\mathbf{x} = \lambda\mathbf{x}$  can equivalently be written as  $(A - I\lambda)\mathbf{x} = \mathbf{0}$ . Call  $(A - I\lambda) \equiv B$ . The situation  $B\mathbf{x} = \mathbf{0}$  needs to be considered.

Recall two theorems:

*Thm. 1*.....  $B$  is invertible if and only if  $\text{Null}(B) = \{\mathbf{0}\}$ , i.e., the only solution to  $B\mathbf{x} = \mathbf{0}$  is  $\mathbf{x} = \mathbf{0}$ ,

and

*Thm. 2*.....  $B$  is invertible if and only if  $\det B \neq 0$ .

Combining *Thm. 1* and *2* to eliminate the statement " $B$  is invertible" gives

The only solution to  $B\mathbf{x} = \mathbf{0}$  is  $\mathbf{x} = \mathbf{0}$  if and only if  $\det B \neq 0$ .

The contrapositive of this statement (which is logically equivalent to any conditional statement) is:

$\det B = 0$  if and only if  $\mathbf{x} = \mathbf{0}$  is not the only solution to  $B\mathbf{x} = \mathbf{0}$ .

Since the statement is an iff statement, its converse is also true:

$\mathbf{x} = \mathbf{0}$  is not the only solution to  $B\mathbf{x} = \mathbf{0}$  if and only if  $\det B = 0$ .

We are interested in cases in which  $\mathbf{x} = \mathbf{0}$  is not the only solution to  $(A - I\lambda)\mathbf{x} = \mathbf{0}$ , and the above statement says that  $\mathbf{x}$  is not the only solution if and only if  $\det(A - I\lambda) = 0$ . Thus we evaluate  $\det(A - I\lambda) = 0$  and solve for the eigenvalues and eigenvectors.

24. Suppose  $A\mathbf{x} = \lambda\mathbf{x}$  and  $A$  is not full rank (i.e., it has at least two columns that are linearly dependent). What then must be at least one of its eigenvalues? *[ANSWER]*

At least one of the eigenvalues of must be 0 if  $A$  is not full rank.

25. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

as well as the eigenvalues and eigenvectors of  $A^2$ ,  $A^{-1}$ , and  $A + 4I$  (without actually computing the eigenvalues and eigenvectors of the latter four matrices). *[ANSWER]*

The eigenvalues and eigenvectors of  $A$  are

$$\begin{aligned} \lambda = 1, \quad \mathbf{x}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda = 3, \quad \mathbf{x}_1 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

$A^2$ ,  $A^{-1}$ , and  $A + 4I$ , have the same eigenvectors as  $A$ , but the eigenvalues are respectively squared (1, 9), inverted (1, 1/3), and added by four (5, 7).

26. Find the eigenvalues and eigenvectors of

$$S = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Does the relative orientation of the eigenvectors make sense? *[ANSWER]*

The eigenvalues and eigenvectors of  $S$  are

$$\lambda = 0, \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda = 1, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda = 3, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Since the matrix is symmetric, it makes sense that the eigenvalues are real and the eigenvectors are orthogonal (see the proof above).

27. ✳  $A\mathbf{x} = \lambda\mathbf{x}$  is a vector equation because the left- and right-hand sides are both vectorial. Recast  $A\mathbf{x} = \lambda\mathbf{x}$  into a matrix form (in which the LHS and RHS are matrices), where  $X$  is a matrix whose columns consist of the eigenvectors  $\mathbf{x}$ , and where  $\Lambda = I\lambda$ . Solve the resulting matrix equation for  $A$ . How does this equation simplify if  $A$  is a symmetric matrix? What is the result for the symmetric matrix called? [ANSWER]

Writing the eigenvalue problem  $A\mathbf{x} = \lambda\mathbf{x}$  as a matrix equation gives

$$AX = \Lambda X.$$

The right hand side can be written as  $\Lambda X = I\lambda X = IX\lambda = XI\lambda = X\Lambda$ :

$$AX = X\Lambda.$$

Multiplying both sides by  $X^{-1}$  on the right solves for  $A$ :

$$A = X\Lambda X^{-1}.$$

For a symmetric matrix,  $A = A^T$ , so  $A = (X\Lambda X^{-1})^T = (X^{-1})^T \Lambda^T X^T$ . Since  $X$  consists of orthonormal eigenvectors (remember, the eigenvectors of a symmetric matrix are orthonormal), the matrix itself is orthogonal, i.e.,  $X^T = X^{-1}$ . Thus  $A = X\Lambda X^T$  for a symmetric matrix  $A$ . This is called *spectral decomposition*, and the fact that any symmetric matrix can be written this way is called the *spectral theorem*.

28. ✳ What is the recasting of of a matrix  $A$  as  $\Lambda = X^{-1}AX$  called? For what matrices can this procedure be carried out? What is the significance of the matrix  $X$ , and why would someone want to perform this operation? How does this particular operation relate to general changes of basis? [ANSWER]

This is called diagonalization. It can be carried out if  $X^{-1}$  exists. Since the columns of  $X$  are simply the eigenvectors of  $A$ , the eigenvectors of  $A$  must be linearly independent for  $X^{-1}$  to exist. In other words, the determinant of  $A$  cannot be 0 for  $A$  to be diagonalized.

The significance of the matrix  $X$  is that it represents the linear transformation from one basis into a better basis. In the first basis, the matrix  $A$  may not be diagonal, but in the better basis, the

matrix is diagonalized into  $\Lambda$ . This linear transformation, into a basis in which the matrix is diagonal, is a special type of *change of basis*, which can be thought of as a "change to the best basis." The more general change-of-base operation (to basis that does not necessarily diagonalize a matrix) is considered in the final three problems of this section. The form of the general change-of-basis is still "transformation  $\times$  matrix in old basis  $\times$  inverse transformation."

At this juncture, it is helpful to keep in mind the distinction between the absolute and the relative.  $A$  is an absolute quantity (for example, a tensor or operator), and it is represented by a relative quantity (a matrix), that changes in different bases. In one basis, it is represented by  $A$ , and in the better basis, it is represented by the diagonal matrix  $\Lambda$ , where the eigenvalues of  $A$  lie along the diagonal of  $\Lambda$ . To get from the first basis to the better basis, one multiplies vectors by the linear transformation  $X$ , and one diagonalizes matrices calculating  $\Lambda = X^{-1}AX$ .

29. ✪ Find the eigenvalues and eigenvectors of

$$M = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}.$$

Is it possible to diagonalize the matrix, i.e., find the decomposition  $M = X\Lambda X^{-1}$ ? Suppose there is a vector represented by  $\mathbf{v} = (1, 0, 0)$  in the first basis, calculate the representation of this vector in the new (eigen)basis. [ANSWER]

After a considerable amount of manipulation, it is found that

$$\begin{aligned} \lambda_1 = 0, \quad \mathbf{x}_1 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ \lambda_2 = 1, \quad \mathbf{x}_2 &= \begin{pmatrix} 2 \\ 1 - i \\ 1 \end{pmatrix} \\ \lambda_3 = i, \quad \mathbf{x}_3 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

The eigenvectors span the space, because a matrix whose columns consist of these vectors has a nonzero determinant. Therefore it is possible to diagonalize the  $M$ . To find the decomposition  $M = X\Lambda X^{-1}$ , note that  $X$  consists of columns that are the eigenvectors of  $M$ ,

$$X = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 - i & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

the inverse of which is  $C^T/|X|$ , where  $C$  is the cofactor matrix:

$$X^{-1} = \frac{1}{|X|} \begin{pmatrix} -1 & 1 & i-1 \\ 0 & 0 & 1 \\ 2 & -1 & 1-i \end{pmatrix}^T = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ i-1 & 1 & 1-i \end{pmatrix}$$

The decomposition is thus

$$M = X\Lambda X^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1-i & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & i \end{pmatrix} \begin{pmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ i-1 & 1 & 1-i \end{pmatrix}$$

which, upon multiplying, does indeed recover

$$M = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}.$$

To rotate  $\mathbf{v} = (1, 0, 0)$  into the new basis, multiply  $X^{-1}\mathbf{v} = (-1, 1, i-1)$ . Why is the multiplication by  $X^{-1}$  and not  $X$ ? See question 36.

30. Factor

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

into  $A = X\Lambda X^{-1}$ . Without actually performing the diagonalization again, find the factorization for  $A^3$  and  $A^{-1}$ . [ANSWER]

The eigenvalues and eigenvectors are found to be

$$\lambda_1 = 1, \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 3, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The eigendecomposition for  $A$  is thus

$$A = X\Lambda X^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

The eigendecomposition for  $A^3$  is obtained by simply cubing the eigenvalues,

$$A^3 = X\Lambda^3 X^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 27 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

and the eigendecomposition for  $A^{-1}$  is obtained by simply inverting the eigenvalues,

$$A^{-1} = X\Lambda^{-1} X^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

I verified these calculations in MATLAB.

31. ✳ What is the difference between a vector  $\mathbf{v}$  and its representation as a list of numbers  $\mathbf{v}$ ? Similarly, what is the difference between a tensor  $\mathbf{A}$  (sans-serif) and a matrix  $A$  (italicized)?

[ANSWER]

$A$  is a representation of  $A$  in a chosen basis; similarly,  $\mathbf{v}$  is the vector  $\mathbf{v}$  expressed in a particular basis.

The distinction between the abstraction and its representation is helpful when the following questions.

32. Let  $A$  and  $B$  be two dyadic tensors (dyads). Show that  $(AB)^T = B^T A^T$ . Hint: introduce two auxiliary vectors  $\mathbf{u}$  and  $\mathbf{v}$  and use the definition of transpose,  $(A^T \mathbf{u}) \cdot \mathbf{v} = (A \mathbf{v}) \cdot \mathbf{u}$  twice. [ANSWER]

Multiply  $\mathbf{u}$  on the right-hand side by the identity matrix  $I$  twice:

$$\begin{aligned} [(AB)^T \mathbf{u}] \cdot \mathbf{v} &= [(AB) \mathbf{v}] \cdot \mathbf{u} \\ &= A[B \mathbf{v}] \cdot \mathbf{u} \\ &= (A^T \mathbf{u}) \cdot [B \mathbf{v}] \\ &= B \mathbf{v} \cdot A^T \mathbf{u} \\ &= B^T (A^T \mathbf{u}) \cdot \mathbf{v} \\ &= B^T A^T \mathbf{u} \cdot \mathbf{v} \end{aligned}$$

Since  $\mathbf{u}$  and  $\mathbf{v}$  are arbitrary, the proof is complete.

33. Let  $A$  and  $B$  be two invertible dyads. Show that  $(AB)^{-1} = B^{-1} A^{-1}$ . [ANSWER]

Multiply  $\mathbf{u}$  on the right-hand side by the identity tensor  $I$  twice:

$$\begin{aligned} (AB)^{-1} \mathbf{u} &= (AB)^{-1} I I \mathbf{u} \\ &= (AB)^{-1} I A A^{-1} \mathbf{u} \\ &= (AB)^{-1} A I A^{-1} \mathbf{u} \\ &= (AB)^{-1} A B B^{-1} A^{-1} \mathbf{u} \\ &= B^{-1} A^{-1} \mathbf{u} \end{aligned}$$

Since  $\mathbf{u}$  is arbitrary, the proof is complete.

34. Prove that any tensor  $T$  can be written as the sum of a symmetric tensor  $S$  and an antisymmetric tensor  $A$ . Similarly prove that any tensor can be written as the sum of a real tensor  $R$  and an imaginary tensor  $M$ . Finally prove that any tensor can be written as the sum of a symmetric tensor  $H$  and an antisymmetric tensor  $K$ . [ANSWER]

Note that a symmetric matrix is constructed by adding  $T$  and its transpose, while an asymmetric matrix is constructed by subtracting the transpose from  $T$ .

$$\begin{aligned} S &= \frac{1}{2} (T + T^T) \\ A &= \frac{1}{2} (T - T^T) \end{aligned}$$

Adding the above equations shows that  $T = S + A$ .

Next, note that a real matrix is constructed by adding  $T$  and its conjugate, while an imaginary matrix is constructed by subtracting the conjugate from  $T$ .

$$R = \frac{1}{2}(T + T^*)$$

$$M = \frac{1}{2}(T - T^*)$$

Adding the above equations shows that  $T = R + M$ .

Finally, note that a Hermitian matrix is constructed by adding  $T$  and its Hermitian conjugate, while a skew-Hermitian matrix is constructed by subtracting the Hermitian conjugate from  $T$ .

$$H = \frac{1}{2}(T + T^\dagger)$$

$$K = \frac{1}{2}(T - T^\dagger)$$

Adding the above equations shows that  $T = H + K$ .

35. How does one test to see if  $A$  and  $B$  share the same independent eigenvectors? Provide examples in physics of two linear operators that share eigenvectors, and two linear operators that do not.

[ANSWER]

They share the same independent eigenvectors iff  $AB = BA$ , i.e., they commute. An example of linear operators that commute and therefore share eigenvectors are  $L^2$  and  $L_z$ ; an example of matrices that do not commute and therefore do not share eigenvectors are  $L_x$  and  $L_z$ , where  $L$  is the orbital angular momentum operator of quantum mechanics, and  $L_n$  is its projection on the  $n$  Cartesian axis..

36. ✪ For some tensorial linear equation  $Ax = b$ , suppose there is one representation of this equation in the primed basis  $'$ ,

$$A'x' = b'$$

and another representation in the unprimed basis,

$$Ax = b.$$

Given the unprimed basis is a linear transformation of the primed basis, i.e.,  $Sv' = v$ , find the matrix representation of  $A$  in the unprimed basis in terms of  $A'$  and  $S$ . What is the relationship between the eigenvalues of  $A$ ,  $A$  and  $A'$ ? What are  $A$  and  $A'$  called with respect to one another? How does this change of basis relate to *diagonalization*? [ANSWER]

See [here](#) for the manipulations that lead to  $A = SA'S^{-1}$ . The relationship between the eigenvalues of  $A$ ,  $A$  and  $A'$ , is that they are all equal. The eigenvalues are independent of the basis, and the eigenvalues of  $A$  can be calculated using the so-called principle invariants (see Stern's nonlinear continuum mechanics notes, ch. 1). Matrices  $A = SA'S^{-1}$  and  $A'$  are called *similar*.

Compare the change-of-basis,  $A = SA'S^{-1}$ , to diagonalization,  $\Lambda = X^{-1}A'X$ . Usually the diagonal matrix  $\Lambda$  is the representation in the *new* basis. Thus  $S$  is analogous to  $X^{-1}$ , i.e., when the columns of  $S^{-1}$  are composed of the eigenvectors of  $A'$ , the change of basis recovers the eigendecomposition. Since vectors transform from the old basis to the new basis as  $S\mathbf{v}' = \mathbf{v}$ , the transformation from the old (primed) basis  $'$  to the new (unprimed) basis in which the representation of  $A$  is diagonal is  $X^{-1}\mathbf{v}' = \mathbf{v}$ .

37. ✪ Show that the eigenvalues  $\lambda$  satisfying  $A\mathbf{x} = \lambda\mathbf{x}$  (unprimed basis) are invariant under the transformation to the primed  $'$  basis, where the linear transformation between unprimed and primed bases is  $\mathbf{v} = S\mathbf{v}'$ . In other words, show that  $A'\mathbf{x}' = \lambda\mathbf{x}'$  [ANSWER]

Start with the eigenvalue problem in the unprimed basis. Invoke the linear transformation between unprimed and primed bases is  $\mathbf{x} = S\mathbf{x}'$ . Multiply both sides by the inverse of the linear transformation, and identify  $S^{-1}AS = A'$  (see previous problem). Note that the equation is an eigenvalue problem now in the primed basis, with the same eigenvalue  $\lambda$ .

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ AS\mathbf{x}' &= \lambda S\mathbf{x}' \\ S^{-1}AS\mathbf{x}' &= \lambda\mathbf{x}' \\ A'\mathbf{x}' &= \lambda\mathbf{x}' \end{aligned}$$

Thus the eigenvalues are invariant under linear transformations of matrices.

38. Do the trace and determinant of a tensor depend on the basis? Provide expressions for the trace and determinant of a tensor  $A$  (with an  $n \times n$  matrix representation) terms of the eigenvalues of  $A$ . [ANSWER]

Like the eigenvalues themselves, the trace and the determinant are independent of basis.  $\text{Tr } A = \sum_n \lambda_n$  and  $\det A = \prod_n \lambda_n$ .

39. ✪ Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Comment on the orientation of the eigenvectors. Are they orthogonal? Why or why not? If not, does this contradict the earlier claim that the eigenvectors of a real symmetric matrix (problem 22) are orthogonal? And if the eigenvectors are not orthogonal, can one make them orthogonal? If so, do this. [ANSWER]

See [here](#) for the solution. In summary, two of the eigenvalues  $\lambda_1$  and  $\lambda_2$  are *degenerate*, and their corresponding eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are found to not be orthogonal. Meanwhile, both  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are orthogonal to  $\mathbf{v}_3$ . This is consistent with the claim made in problem 22, since that statement was restricted to non-degenerate eigenvalues. To orthogonalize the eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , the so-called Gram-Schmidt procedure is used. Griffiths notes that this procedure rarely needs to be carried out in practice, but that it can be done in principle.

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# Ordinary differential equations

Each numbered equation in this section represents a unique type of differential equation. For a thorough review of this section, be sure to know how to solve each type of differential equation.

1. Define a homogeneous function. Provide an example of a homogeneous functions. [ANSWER]

Simply put,

$$f(sx_1, \dots, sx_n) = s^k f(x_1, \dots, x_n)$$

is a homogeneous function, where  $k$  is the "degree of homogeneity." For example,  $f(x, y, z) = x^5 y^2 z^3$  is homogeneous of degree 10 because  $f(ax, ay, az) = (ax)^5 (ay)^2 (az)^3 = a^{10} f(x, y, z)$ .

2. Define a homogeneous polynomial and provide an example. [ANSWER]

Similar to a homogeneous function, a homogeneous polynomial is defined by

$$P(sx_1, \dots, sx_n) = s^k P(x_1, \dots, x_n)$$

where  $P$  is some multivariate polynomial. Simply put, a polynomial is homogeneous if its terms all have the same degree. An example of a homogeneous polynomial is  $x^2 + 2xy + y^2$ .

3. What defines the homogeneity of a rational function? [ANSWER]

A rational function is homogeneous if it consists of a ratio of two homogeneous polynomials.

4. Define the linearity of a function  $f(x)$ . Linear functions obey the \_\_\_\_\_ principle. [ANSWER]

The properties  $f(x + y) = f(x) + f(y)$  and  $f(ax) = af(x)$  for all  $a$  together define linearity. Linear functions obey the superposition principle.

5. Is a homogeneous function always linear? Is a linear function always homogeneous? [ANSWER]

A homogeneous function is *not* always linear. For example,  $f(x, y, z) = x^5 y^2 z^3$  is homogeneous but not linear. However, a linear function is always homogeneous, because homogeneity is one of the two properties of linear functions.

6. What defines a homogeneous first-order ordinary differential equation? Provide an example of a first-order ODE that is homogeneous, as well as an example of one that is inhomogeneous.

[ANSWER]

A homogeneous ordinary differential equation is of the form  $\frac{dy}{dx} = f(x, y) = f\left(\frac{y}{x}\right)$ . An example of a first-order ODE that is homogeneous is  $\frac{dy}{dx} = \frac{\sin(y/x)}{y/x} + (y/x)^2$ , while an example of a first-

order ODE that is inhomogeneous is  $\frac{dy}{dx} = y + x$ .

7. Classify the differential equation

$$\frac{dy}{dx} = a(x)y. \quad (1)$$

Find the general solution to equation (1). [ANSWER]

This is a linear, homogeneous, separable ordinary differential equation. The general solution is

$$y = \pm Ce^{\int a(x)dx}.$$

8. Classify and solve  $\frac{dy}{dx} = [x + \cos(x)]y$  for  $y$ . [ANSWER]

This is a linear, homogeneous, separable ordinary differential equation. The general solution is

$$y = \pm Ce^{\int a(x)dx}.$$

9. Classify and solve  $\frac{dy}{dx} = (y - x)^2$ . The solution can remain in an implicit form. *Hint: is there a transformation that can make this ODE separable?* [ANSWER]

As written, this is a nonlinear, homogeneous, inseparable ordinary differential equation. However, it can be made separable by making the substitution  $z = y - x$ . Then, the ODE becomes

$$\frac{dz}{dx} = z^2 - 1,$$

and the general solution is

$$\frac{\ln(z - 1)}{\ln(z + 1)} = 2x + C.$$

10. Classify and solve  $\frac{dy}{dt} = \cos(y - t)$ . Follow the hint above. [ANSWER]

As written, this is a nonlinear, homogeneous, inseparable ordinary differential equation. However, it can be made separable by making the substitution  $z = y - t$ . Then, the ODE becomes

$$\frac{dz}{dx} = \cos(z) - 1,$$

and the general solution is found by performing the integral

$$\int \frac{dz}{\cos z - 1} = t.$$

I used Wolfram to calculate the integral (It is doable, but involves a trigonometric substitution). The solution is

$$\cot \frac{z}{2} = t,$$

or in an explicit form,

$$z = 2\operatorname{arccot}(t),$$

11. Classify and solve  $y + \sqrt{xy} = x \frac{dy}{dx}$ , where  $x > 0$ . *Hint: is there a transformation that can make this ODE separable? This time it a multiplicative transformation, unlike that additive transformations in the previous two examples.* [ANSWER]

First the ODE is rearranged into the form,

$$\begin{aligned} x \frac{dy}{dx} &= y + \sqrt{xy} \\ \frac{dy}{dx} &= \frac{y}{x} + \frac{\sqrt{xy}}{x} \\ \frac{dy}{dx} &= \frac{y}{x} + \sqrt{\frac{y}{x}} \end{aligned}$$

From the last equation it can be seen that the ODE is homogeneous (because it is of the form  $dy/dx = f(y/x)$ ). Making the substitution  $y/x = u$  results in

$$\begin{aligned} x \frac{du}{dx} + u &= u + \sqrt{u} \\ x \frac{du}{dx} &= \sqrt{u} \\ \int u^{-1/2} du &= \int \frac{dx}{x} \\ 2u^{1/2} &= \ln x + C \\ y^{1/2}/x^{1/2} &= \frac{1}{2}(\ln x + C) \\ y &= x\left(\frac{1}{2}\ln x + C\right)^2 \end{aligned}$$

12. Classify the differential equation

$$\frac{dy}{dx} = a(x)y + b(x). \quad (2)$$

Find the general solution to equation (2) by assuming the form of solution to equation (1), but with  $C = C(x)$ . What is the name of this solution? [ANSWER]

This is a linear, inhomogeneous first-order ordinary differential equation.

The form of solution is assumed to be  $y = C(x)e^{\int a(x)dx}$ . Inserting this solution into equation (2) and applying the product rule gives

$$e^{\int a(x)dx} \frac{d}{dx} C(x) + C(x) \frac{d}{dx} e^{\int a(x)dx} = a(x)y + b(x)$$

Note that  $\frac{d}{dx} e^{\int a(x)dx} = e^{\int a(x)dx} \frac{d}{dx} \int a(x)dx = a(x)e^{\int a(x)dx}$ . Therefore,

$$\begin{aligned} e^{\int a(x)dx} \frac{d}{dx} C(x) + C(x)a(x)e^{\int a(x)dx} &= a(x)y + b(x) \\ &= a(x)C(x)e^{\int a(x)dx} + b(x) \end{aligned}$$

Canceling the common term above gives

$$e^{\int a(x)dx} \frac{dC(x)}{dx} = b(x)$$

Solving for  $C(x)$  gives

$$C(x) = \int b(x)e^{-\int a(x)dx} dx$$

Substituting this equation for  $C(x)$  into the assumed form of solution gives

$$y = e^{\int a(x)dx} \left[ \int b(x)e^{-\int a(x)dx} dx \right]$$

This is called the *Cauchy equation*.

13. Classify and solve  $(1 + x^2)y' + 2xy = \cos x$ . [ANSWER]

This is a linear, inhomogeneous first-order ordinary differential equation. Rewriting it as  $y' = \frac{\cos x}{1+x^2} - \frac{2xy}{1+x^2}$  renders it in the form of  $y' = a(x)y + b(x)$ . Thus the Cauchy formula

$$y = e^{\int a(x)dx} \left[ \int b(x)e^{-\int a(x)dx} dx \right]$$

can be applied with

$$\begin{aligned} a(x) &= -\frac{2x}{1+x^2} \\ b(x) &= \frac{\cos x}{1+x^2} \end{aligned}$$

Noting that

$$\begin{aligned} \int a(x)dx &= -\int \frac{2x}{1+x^2} dx \\ &= -\ln(1+x^2) + C \quad \text{and} \\ \int b(x)e^{-\int a(x)dx} dx &= \int \frac{\cos(x)}{1+x^2} e^{\ln(1+x^2)} dx \\ &= \int \cos x dx = \sin x + C \end{aligned}$$

Thus Cauchy equation is

$$\begin{aligned} y &= e^{\ln(1+x^2)^{-1}} (\sin x + C) \\ &= \frac{\sin x}{1+x^2} + C \end{aligned}$$

where  $C$  above varies from line to line (I know it is not technically correct, but one can absorb any arbitrary constant in  $C$ ).

14. Classify the differential equation

$$\frac{dy}{dx} = a(x)y + b(x)y^n \quad (3)$$

where  $n \neq 0, 1$  is real. What is the name of this equation? What substitution does one make to facilitate its solution? [ANSWER]

This is a nonlinear, inhomogeneous first-order ordinary differential equation. It is called the Bernoulli equation.

To solve this equation, one should seek a substitution that makes the equation linear. Let  $y = z^\alpha$ , and therefore  $dy/dx = \alpha z^{\alpha-1} dz/dx$ . Then equation (3) becomes

$$\alpha \frac{dz}{dx} = a(x)z + b(x)z^{\alpha n - \alpha + 1}$$

In order for the above equation to be reduced to a linear inhomogeneous equation [see equation (2)], one should set the exponent of  $z$  multiplying  $b(x)$  equal to 0, which results in

$$\alpha = \frac{1}{1-n}$$

15. Classify and solve  $xydy = (y^2 + x)dx$ . [ANSWER]

Rearrange the equation into its standard form:  $\frac{dy}{dx} = \frac{y^2+x}{xy} = \frac{y}{x} + \frac{1}{y}$ . This is a nonlinear inhomogeneous ordinary differential equation,  $\frac{dy}{dx} = a(x)y + b(x)y^n$  with  $a(x) = \frac{1}{x}$ ,  $b(x) = 1$ , and  $n = -1$ . To transform the ODE into a linear one, set  $y = z^\alpha$ , where  $\alpha = \frac{1}{1-n} = \frac{1}{2}$ , resulting in

$$\begin{aligned} \alpha \frac{dz}{dx} &= a(x)z + b(x) \\ \frac{1}{2} \frac{dz}{dx} &= \frac{1}{x}z + 1 \\ \frac{dz}{dx} &= \frac{2}{x}z + 2 \end{aligned}$$

$\frac{dz}{dx} = \frac{2}{x}z + 2$  is now a linear inhomogeneous ordinary differential equation whose solution is given by the Cauchy equation:

$$\begin{aligned} z &= e^{\int \frac{2}{x} dx} \left( 2 \int e^{-\int \frac{2}{x} dx} dx \right) \\ &= -2x + C \end{aligned}$$

The solution to for  $y$  is thus  $y = \pm \sqrt{C - 2x}$ .

*In my ODE notes from UT Dallas, I had  $y = \pm \sqrt{Cx^2 - 2x}$ , but I disagree with the way the constants of integration were defined in the Cauchy equation that lead to this answer.*

16. Classify the differential equation

$$M(x, y)dx + N(x, y)dy = 0, \quad \text{where } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \quad (4)$$

What is the general solution of equation (4) in integral form? [ANSWER]

This is an exact equation, and the solution in integral form is  $F = \int \frac{\partial F}{\partial x} dx = \int M dx$ . Equivalently, the solution is  $F = \int \frac{\partial F}{\partial y} dy = \int N dy$ . One should take the integral over whichever variable is easier to integrate over. After the taking the integral over one variable, the unknown constant is determined by taking the derivative of  $F$  with respect to the other variable, and setting this equal to  $M$  if the other variable is  $x$ , or  $N$  if the other variable is  $y$ .

17. Classify and solve  $(4x^3 + 3y)dx + (3x + 4y^3)dy = 0$ . [ANSWER]

This is an exact differential equation because

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3$$

Therefore the solution is

$$\begin{aligned} F(x, y) &= \int \frac{\partial F}{\partial x} dx = \int (4x^3 + 3y) dx \\ &= x^4 + 3xy + C(y) \end{aligned}$$

To determine  $C(y)$ , take the partial derivative with respect to  $y$  and set this equal to  $N = 3x + 4y^3$ :

$$\begin{aligned} 3x + C'(y) &= 3x + 4y^3 \\ C'(y) &= 4y^3 \\ C(y) &= y^4 \end{aligned}$$

Thus the solution is

$$F(x, y) = y^4 + x^4 + 3xy.$$

18. How does one solve

$$M(x, y)dx + N(x, y)dy = 0, \quad \text{but now where } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} ? \quad (5)$$

What is the name of this method? [ANSWER]

One can solve equation (5) by multiplying through by  $\mu(x)$  or  $\mu(y)$ , and then forcing that the mixed partials of  $F$  be equal. Note that multiplying through by  $\mu(x, y)$  leads to a partial differential equation, which is not desired in this context.

In this derivation, multiply through by  $\mu(x)$ ,

$$\mu(x)M(x, y)dx + \mu(x)N(x, y)dy = 0,$$

and then set

$$\begin{aligned}\frac{\partial}{\partial y} [\mu(x)M(x, y)] &= \frac{\partial}{\partial x} [\mu(x)N(x, y)] \\ \mu \frac{\partial M}{\partial y} &= \frac{d\mu}{dx} N + \mu \frac{\partial N}{\partial x} \\ \frac{d\mu}{dx} N &= \mu \left[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] \\ \frac{1}{\mu} \frac{d\mu}{dx} &= \frac{\partial M / \partial y - \partial N / \partial x}{N}\end{aligned}\tag{i}$$

Equation (i) can be integrated and solved for  $\mu$ . Thus  $\mu$  is called the integrating factor, and this method is called the "integrating factor method."

Note that the right-hand side of equation (i) must be only a function of  $x$ , in the case above. If it is not (i.e., if it is a function of  $y$  as well), then try multiplying the original equation by  $\mu_y$ , for which

$$\frac{1}{\mu} \frac{d\mu}{dy} = \frac{\partial N / \partial x - \partial M / \partial y}{M}$$

19. Classify and solve  $(x + 2) \sin y dx + x \cos y dy = 0$ . [ANSWER]

This looks like it might be in the form of an exact equation, but since  $\partial M / \partial y = (x + 2) \cos y \neq \partial N / \partial x = \cos y$ , it requires the integrating factor method to turn it into an exact equation. An integrating factor  $\mu(x)$  is attempted.

$$\begin{aligned}\frac{1}{\mu} \frac{d\mu}{dx} &= \frac{(x + 2) \cos y - \cos y}{x \cos y} = 1 + \frac{1}{x} \\ \int \frac{d\mu}{\mu} &= \int \left( 1 + \frac{1}{x} \right) dx \\ \ln \mu &= x + \ln x \\ \mu &= e^x e^{\ln x} = x e^x\end{aligned}$$

Now the original ODE becomes

$$x e^x (x + 2) \sin y dx + x^2 e^x \cos y dy = 0$$

The solution is

$$F(x, y) = \int \frac{\partial F}{\partial x} dx = \dots$$

However, before proceeding with this option, note that in this case that it is much easier to find

$$F(x, y) = \int \frac{\partial F}{\partial y} dy = \int x^2 e^x \cos y dy = x^2 e^x \sin y + C(x).$$

The constant  $C(x)$  is found by differentiating  $F$  with respect to  $x$  and setting this equal to  $xe^x(x + 2) \sin y$ :

$$\frac{\partial F}{\partial x} = (2xe^x + x^2e^x) \sin y + C'(x) = xe^x(x + 2) \sin y$$

Thus

$$C'(x) = x^2e^x \sin y + 2xe^x \sin y - (2xe^x + x^2e^x) \sin y = 0,$$

and thus  $C(x) = \text{const}$ . The solution is therefore  $F(x, y) = x^2e^x \sin y + \text{const}$ .

20. Classify the differential equation

$$ay'' + b' + cy = 0, \tag{6}$$

where  $a$ ,  $b$ , and  $c$  are constants, and where the prime ' will now be used for notational ease to signify derivative. How does one solve this equation? There are the three possible cases that emerge. What are they? *[ANSWER]*

This is a linear, second-order ordinary differential equation with constant coefficients. It is solved by making the substitution  $e^{rx}$ , which leads to a quadratic equation in  $r$ . Note that this generalizes to higher derivatives, leading to polynomials of higher degree to solve.

Three cases emerge.

1. The solutions  $r$  are real (discriminant  $b^2 - 4ac > 0$ ). In this case, the solution to the 2nd order ODE is exponential decay and/or growth.
2. The solutions  $r$  are complex (discriminant  $b^2 - 4ac < 0$ ). In this case, the solution to the 2nd order ODE is waves, which can either be written as complex exponentials, or as sines and cosines.
3. The solutions  $r$  are equal, i.e., a double root (discriminant  $b^2 - 4ac = 0$ ). In this case, the second solution gets multiplied by  $x$  the first. This will be proved later.

21. Classify and solve  $y''' + 4y'' + 9y' + 10y = 0$ . *[ANSWER]*

This is a linear, third-order ordinary differential equation. Follow the procedure above and obtain the characteristic cubic equation in  $r$ ,  $r^3 + 4r^2 + 9r + 10 = 0$ . The three solutions of this equation are  $r = -2$ ,  $r = -1 + 2i$ , and  $r = -1 - 2i$ . Thus the solution is  $y = C_1e^{-2x} + C_2e^{-x} \cos(2x) + C_3e^{-x} \sin(2x)$ .

22. Classify and solve  $y''' + 6y'' + 12y' + 8y = 0$ . *[ANSWER]*

This is also a linear, third-order ordinary differential equation. Following the same procedure as above, obtain the characteristic cubic equation in  $r$ ,  $r^3 + 6r^2 + 12r + 8 = 0$ . The three solutions of this equation are the triple root  $r = -2$ . Thus the solution is therefore  $y = C_1e^{-2x} + C_2xe^{-2x} + C_3x^2e^{-2x}$ .

23. What is the definition of the Wronskian? What information does it provide? *[ANSWER]*

The Wronskian  $W$  is defined as

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1 y_2' - y_2 y_1',$$

where  $y_1$  and  $y_2$  are two solutions to a second-order linear ordinary differential equation, and where  $'$  signifies derivative. Two solutions of the same equation are called a fundamental pair if  $W \neq 0$ .

There is another way to calculate  $W$ : Abel's formula for the Wronskian:  $W = \int e^{-\int p(x)dx}$ , where the ODE is written as  $y'' + p(x)y' + q(x)y = 0$ . However, I don't think it's worth reviewing.

24. Determine whether  $y_1 = x$  and  $y_2 = \ln x$  are a fundamental pair, and if so, on what interval. What about  $y_1 = \arccos \frac{x}{\pi}$  and  $y_2 = \arcsin \frac{x}{\pi}$ ? [ANSWER]

The Wronskian for the first case is  $W = \ln x$ , which is never 0. Thus  $y_1 = x$  and  $y_2 = \ln x$  are a fundamental pair for all  $x > 0$  (since  $\ln x$  is defined on that interval).

The Wronskian for the second case is  $W = \frac{\arccos(x/\pi)}{\pi\sqrt{1-x^2}} + \frac{\arcsin(x/\pi)}{\pi\sqrt{1-x^2}}$ , which (I don't think) has a solution. Thus the solutions are a fundamental pair for  $x \in (-\pi, \pi)$ .

25. ✪ Classify the differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0. \quad (7)$$

Given one solution  $y_1$  to equation (7), how can the other solution  $y_2$  be found? What is this method called? *Hint: Let  $y_2 = uy_1$ . Insert this into equation (7). Introduce  $z = u'$  and then let  $z = u'$  (This is where the name of the method comes from). Solve for  $z$ , integrate, and find  $u$ .  $y_2$  is then found because  $y_2 = uy_1$ . I doubt this would be on the exam as it is too involved.* [ANSWER]

Assume that  $y_2 = uy_1$ . Then the derivatives of  $y_2$  are

$$\begin{aligned} y_2' &= u'y_1 + uy_1' \\ y_2'' &= u''y_1 + u'y_1' + u'y_1' + uy_1'' \end{aligned}$$

Inserting these relations into equation (7) results in

$$a(x)(u''y_1 + u'y_1' + u'y_1' + uy_1'') + b(x)(u'y_1 + uy_1') + c(x)uy_1 = 0.$$

This equation is regrouped:

$$u[a(x)y_1'' + b(x)y_1' + c(x)] + a(x)u''y_1 + 2a(x)u'y_1' + b(x)u'y_1 = 0.$$

The first term in [...] is 0 by equation (7). Therefore,

$$a(x)u''y_1 + u'[2a(x)y_1' + b(x)y_1] = 0.$$

Now, introduce the parameter  $z = u'$ , which kicks all the derivatives above down one:

$$a(x)y_1 \frac{dz}{dx} + z[2a(x)y_1' + b(x)y_1] = 0.$$

$$\frac{dz}{z} = \left[ -\frac{2y_1'}{y_1} - \frac{b(x)}{a(x)} \right] dx$$

Integrating results in

$$\begin{aligned} \ln z &= -2 \int \frac{y_1'}{y_1} dx - \int \frac{b(x)}{a(x)} dx \\ &= -2 \int \frac{dy_1}{dx} \frac{1}{y_1} dx - \int \frac{b(x)}{a(x)} dx \\ &= -2 \int \frac{dy_1}{y_1} - \int \frac{b(x)}{a(x)} dx \\ &= \ln(y_1^{-2}) - \int \frac{b(x)}{a(x)} dx. \end{aligned}$$

Exponentiating gives

$$\begin{aligned} z &= y_1^{-2} e^{\int \frac{b(x)}{a(x)} dx}. \\ u &= \int z dx = \int y_1^{-2} e^{\int \frac{b(x)}{a(x)} dx} dx. \\ y_2 &= y_1 u = y_1 \int y_1^{-2} e^{\int \frac{b(x)}{a(x)} dx} dx. \end{aligned}$$

This method is called reduction of order.

26. Given that  $y_1 = x$  is a solution to  $x^2 y'' - x(x+2)y' + (x+2)y = 0$ , classify this equation and find the general solution. [ANSWER]

This is a linear homogeneous second-order ordinary differential equation with nonconstant coefficients. Reduction of order should be used:

$$\begin{aligned} y_2 &= y_1 \int y_1^{-2} e^{\int \frac{b(x)}{a(x)} dx} dx \\ &= x \int x^{-2} e^{-\int \frac{x(x+2)}{x^2} dx} dx \\ &= x \int x^{-2} e^{-\int (1+2/x) dx} dx \\ &= x \int x^{-2} e^{-x + \ln(x^{-2})} dx \\ &= x \int e^{-x} dx = x e^{-x} \end{aligned}$$

27. Classify the differential equation

$$ay'' + by' + cy = g(x). \quad (8)$$

List the two ways to solve this equation. [ANSWER]

This is linear inhomogeneous second-order ordinary differential equation with constant coefficients. The two methods to solve this are (1) the method of undetermined coefficients and (2) the variation of parameters:

First let us discuss the method of undetermined coefficients, which works when the right-hand side  $g(x)$  is exponential, sinusoidal, polynomial, or a product or sum of exponentials, sinusoids, and polynomials.

1. If  $g(x) = e^{kx}p(x)$ , where  $p$  is a polynomial of degree  $n$ , then use as the form of particular solution  $y_p(x) = e^{kx}q(x)$ , where  $q$  is the  $n$ th degree polynomial. The coefficients of  $q$  are then determined by substituting it into equation (8).
2. If  $g(x) = e^{kx}p(x) \cos(mx)$ , or if  $g(x) = e^{kx}p(x) \sin(mx)$  where  $p$  is a polynomial of degree  $n$ , then use as the form of particular solution  $y_p(x) = e^{kx}q(x) \cos(mx) + e^{kx}s(x) \sin(mx)$ , where  $q$  and  $s$  are  $n$ th degree polynomials with different coefficients.

Importantly, in either case above, if any term of the particular solution is also a solution of the homogeneous equation, then the form of solution should be multiplied by  $x$  (if the particular solution coincides with the homogeneous solution once), or  $x^2$  (if the particular solution coincides with the homogeneous solution twice). This is called resonance.

28. Classify and solve  $y'' + 4y = e^{3x}$ . [ANSWER]

$$y = y_h + y_p = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{13} e^{3x}.$$

29. Classify and solve  $y'' + y = \sin x$ . [ANSWER]

See [here](#) for a video solution. There a resonance here of multiplicity 1. The solution is  $y = C_1 \cos x + C_2 \sin x - \frac{x}{2} \cos x$ .

30. Classify and solve  $y'' - 4y' + 3y = 2xe^x$ . [ANSWER]

The method of undetermined coefficients is used. However, one needs to be careful in this case, because the particular solution resonates with the homogeneous solution. Thus the particular solution be chosen to be  $y_p = xe^{-x}(Ax + B)$ . See [here](#) for the complete solution.

31. Classify and solve  $y'' + 2y' + y = e^{-x}$ . [ANSWER]

The method of undetermined coefficients is used. Again, one needs to be careful because the particular solution resonates with the homogeneous solution, this time with multiplicity 2. Thus the particular solution be chosen to be  $y_p = Ax^2e^{-x}$ . See [here](#) for the complete solution.

32. Determine the form of trial solution for  $y'' - 4y' + 13y = e^{2x} \cos 3x$ . [ANSWER]

Note that the homogeneous solution has characteristic roots  $r = 2 + 3i$  and  $r = 2 - 3i$ . The first of these roots coincides with the driving function,  $e^{2x} \cos 3x$ . Thus the form of solution should be the non-resonant guess multiplied by  $x$ , i.e.,  $xe^{2x}(A \cos 3x + B \sin 3x)$ .

33. What is the variation of parameters? Do not provide the full derivation, but provide the big picture (like, why is it called "variation of parameters"?). In what situations should the variation of parameters be used? [ANSWER]

Variation of parameters provides a more general way of solving  $y'' + p(x)y' + q(x)y = 0$ , a linear 2nd order inhomogeneous ordinary differential equation with non-constant coefficients. It can of course also be used to solve a linear 2nd order inhomogeneous ordinary differential equation with *constant* coefficients, and in fact this is the method that must be used when the right-hand side  $g$  is more complicated than a sum or product of polynomials, trigonometric functions, or exponentials.

To derive the variation of parameters, first one solves the homogeneous equation to obtain solutions  $y_1$  and  $y_2$ . Then, one looks for the form of solution  $y_p = C_1(x)y_1(x) + C_2(x)y_2(x)$ , which gives the method its name  $C_1$  and  $C_2$  are allowed to vary as functions, not constants. An additional constraint

$$C_1'y_1 + C_2'y_2 = 0 \quad (\text{i})$$

is imposed. Taking the derivatives of  $y$ , substituting the result into the ODE, and simplifying results in

$$y_p = C_1'y_1' + C_2'y_2' \quad (\text{ii})$$

Combining equations (i) and (ii) results (by Cramer's rule) in a solution for  $y$ :

$$y_p = -y_1 \int \frac{y_2 g}{W} dx + y_2 \int \frac{y_1 g}{W} dx \quad (\text{iii})$$

Just memorize equation (iii).

34. Solve  $y'' + 4y = \frac{3}{\sin x}$ . [ANSWER]

See [here](#).

35. Solve  $y'' - 2y' + y = \frac{e^t}{t^2+1}$ . [ANSWER]

Using the variation of parameters, the particular solution is found by taking the integrals

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2 g}{W} dx + y_2 \int \frac{y_1 g}{W} dx \\ &\dots \\ &= C_1 e^t + C_2 t e^t - \frac{e^t}{2} \ln(t^2 + 1) + t e^t \arctan t. \end{aligned}$$

36. ✪ What is

$$ax^2 y'' + bxy' + cy = 0 \quad (9)$$

called? (*Why is it a bad name*)? What clever substitution does one make to go about solving it?

[ANSWER]

Equation (9) is known as Euler's equation. A better name is the "Cauchy-Euler equation," because Euler already has so many equations named after him (like  $\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0$  from the

calculus of variations and  $e^{i\pi} + 1 = 0$ , the identity containing the most important numbers of mathematics)!

To solve equation (9), one lets  $x$  be an exponential function of a new independent variable  $t$ . That is,  $x = e^t$ . Then the derivatives of  $y$  with respect to  $t$  are

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{dx} e^t = \frac{dy}{dx} x \\ \frac{d^2y}{dt^2} &= \frac{d}{dt} \frac{dy}{dt} = \frac{d}{dt} \left( \frac{dy}{dx} x \right) = x \frac{d^2y}{dt dx} + \frac{dx}{dt} \frac{dy}{dx}\end{aligned}$$

Note that  $\frac{d^2y}{dt dx} = \frac{d^2y}{dx dt} = \frac{d}{dx} \left( \frac{dy}{dx} \frac{dx}{dt} \right) = \frac{d}{dx} \left( \frac{dy}{dx} \right) \frac{dx}{dt} = \frac{d}{dx} \left( \frac{dy}{dx} \right) x$ . Therefore,

$$\frac{d^2y}{dt^2} = x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx}.$$

Solving the above relations for the derivatives with respect to  $x$  gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x} \frac{dy}{dt} \\ \frac{d^2y}{dx^2} &= \frac{1}{x^2} \left( \frac{d^2y}{dt^2} - x \frac{dy}{dx} \right)\end{aligned}$$

Thus equation (9) is

$$\begin{aligned}ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy &= 0 \\ a \left( \frac{d^2y}{dt^2} - x \frac{dy}{dx} \right) + b \frac{dy}{dt} + cy &= 0 \\ a \left( \frac{d^2y}{dt^2} - \frac{dy}{dt} \right) + b \frac{dy}{dt} + cy &= 0 \\ a \frac{d^2y}{dt^2} + (b - a) \frac{dy}{dt} + cy &= 0\end{aligned}$$

Therefore, the introduction of  $x = e^t$  successfully converts equation (9) into a constant-coefficient differential equation, where the independent variable is  $t$  instead of  $x$ . Thus, the solution to  $a \frac{d^2y}{dt^2} + (b - a) \frac{dy}{dt} + cy = 0$  is  $y(t)$  (and it is solved the same way as equation (6), only in  $t$ , i.e., letting  $y = e^{rt}$ ), which must be converted back to  $y(x)$  by letting  $t = \ln x$ .

37. What are the three cases that arise when solving equation (9)? *[ANSWER]*

It was shown in the previous problem that Euler's equation is a constant-coefficient differential equation where  $r$  is the independent variable:

$$a^2 y'' + (b - a) y' + cy = 0.$$

Three cases arise:

1.  $a^2 y'' + (b - a) y' + cy = 0$  has real roots  $r_1$  and  $r_2$ , and the solution to Euler's differential equation is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$\Rightarrow y(x) = C_1 x^{r_1} + C_2 x^{r_2}.$$

2.  $a^2 y'' + (b - a)y' + cy = 0$  has a double roots  $r_1 = r_2 \equiv r$ , and the solution to Euler's differential equation is

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$

$$\Rightarrow y(x) = C_1 x^r + C_2 \ln x x^r.$$

3.  $a^2 y'' + (b - a)y' + cy = 0$  has a complex roots  $r_1 = \alpha + i\beta$  and  $r_2 = \alpha - i\beta$ , and the solution to Euler's differential equation is

$$y(t) = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

$$\Rightarrow y(x) = C_1 \alpha t \cos(\beta \ln x) + C_2 \alpha t \sin(\beta \ln t)$$

*Thus, to remember how to solve equation (9), one must simply remember that the substitution  $x = e^t$  converts equation (9) into  $a^2 y'' + (b - a)y' + cy = 0$ .*

38. Classify and solve  $2x^2 y'' + 3xy' - y = 0$ . [ANSWER]

This is a Cauchy-Euler differential equation. Recalling that the substitution  $x = e^t$  converts equation (9) into  $a^2 y'' + (b - a)y' + cy = 0$ , the coefficients  $a = 2$ ,  $b = 3$  and  $c = -1$  are identified, and the constant-coefficient equation is

$$2y'' + y' - y = 0,$$

the solution of which is

$$y(t) = C_1 e^{t/2} + C_2 e^{-t}$$

Making the substitution  $x = e^t$  gives the solution of the Cauchy-Euler differential equation:

$$y(x) = C_1 x^{1/2} + C_2 x^{-1}.$$

39. Let  $a_n$  be the expansion coefficients in a series solution  $y = \sum_n a_n x^n$ . Define the radius of convergence of this sum. [ANSWER]

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

40. Calculate the radius of convergence for  $e^x$ . [ANSWER]

Noting that  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ , the radius of convergence is

$$R = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{n!} \right|$$

$$= \lim_{n \rightarrow \infty} (n+1) = \infty.$$

That is, the series converges everywhere.

41. Calculate the radius of convergence for  $\frac{1}{1-x}$ . [ANSWER]

Noting that  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ , the radius of convergence is

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{n!} \right| \\ &= \lim_{n \rightarrow \infty} 1 = 1. \end{aligned}$$

That is, this series converges on the interval  $x = (-1, 1)$ .

42. Solve  $y'' + y = 0$  by series. [ANSWER]

Let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

where it has been noted that the first term in the sum for  $y'$  is 0, and the first two terms in the sum for  $y''$  are 0. Substituting these sums into the ODE gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Now the indices of the first summation term shifted from  $n+2 \mapsto n$ , which means that 2 must be added to each  $n$  in the summand to compensate:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

The two summations may now be combined under the same sum:

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + a_n] x^n = 0$$

Since this relation holds for every  $n$ , one can identify a recurrence relation:

$$\begin{aligned} (n+2)(n+1) a_{n+2} + a_n &= 0 \\ a_{n+2} &= -\frac{a_n}{(n+2)(n+1)} \end{aligned}$$

Now for the fun part, let  $a_0 = 1$  and  $a_1 = 0$ . Then all  $a_n$  for  $n$  odd vanishes, and

$$\begin{aligned}
n = 0 : \quad a_2 &= -\frac{a_0}{(0+2)(0+1)} = -\frac{1}{1 \cdot 2} \\
n = 2 : \quad a_4 &= -\frac{a_2}{(2+2)(2+1)} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \\
n = 4 : \quad a_6 &= -\frac{a_4}{(4+2)(4+1)} = -\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}
\end{aligned}$$

Defining  $k = 2n$ , the pattern is explicitly given by  $a_{2k} = (-1)^k/(2k)!$  and  $a_{2k+1} = 0$ . The solution can be written as

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

Meanwhile, letting  $a_0 = 0$  and  $a_1 = 1$  all  $a_n$  for  $n$  even vanishes, and

$$\begin{aligned}
n = 1 : \quad a_3 &= -\frac{a_1}{(1+2)(1+1)} = -\frac{1}{2 \cdot 3} \\
n = 3 : \quad a_5 &= -\frac{a_3}{(3+2)(3+1)} = \frac{1}{1 \cdot 2 \cdot 4 \cdot 5} \\
n = 5 : \quad a_7 &= -\frac{a_5}{(5+2)(5+1)} = -\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}
\end{aligned}$$

Thus the pattern is explicitly given by  $a_{2k+1} = (-1)^k/(2k+1)!$  and  $a_{2k} = 0$ . The solution can be written as

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

The general solution is therefore

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

which we recognize as linear combinations of sines and cosines.

43. What is the name of the ordinary differential equation  $y'' - xy = 0$ ? What are some of its applications? Solve it by series. *[ANSWER]*

This is the Airy equation, which was developed by Airy to describe caustics in optics, like in a rainbow. It is also used in quantum mechanics, at the turning point in the WKB approximation.

As before, let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

where again it has been noted that the first term in the sum for  $y'$  is 0, and the first two terms in the sum for  $y''$  are 0. Substituting these sums into the Airy equation gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Now the indices must be shifted so the powers of  $x$  match in both sums. This is done by shifting the index of the sum on the right from  $n = 0 \mapsto 1$ , and shifting the index of the sum on the left from  $n = 0 \mapsto 1$ :

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

In order to combine the sums, the lower index must also match. This is done by explicitly removing the  $n = 0$  terms of the sum on the left:

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

The two summations may now be combined under the same sum:

$$2a_2 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - a_{n-1}] x^n = 0$$

Since this relation holds for every  $n$ ,  $a_2 = 0$ , and

$$(n+2)(n+1) a_{n+2} - a_{n-1} = 0$$

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$$

First, let  $a_0 = 1$  and  $a_1 = 0$ . Then all  $a_n$  for  $n$  odd vanishes, and

$$n = 1 : a_3 = \frac{a_0}{(1+2)(1+1)} = \frac{1}{2 \cdot 3} \quad k = 1$$

$$n = 2 : a_4 = \frac{a_1}{(2+2)(2+1)} = 0$$

$$n = 3 : a_5 = \frac{a_2}{(3+2)(3+1)} = 0$$

$$n = 4 : a_6 = \frac{a_3}{(4+2)(4+1)} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} \quad k = 2$$

The pattern is explicitly given by  $a_k = [2 \cdot 3 \cdot 5 \cdot 6 \cdot \dots \cdot (3k + 1)3k]^{-1}$ . The solution can be written as

$$y_1 = 1 + \sum_{k=1}^{\infty} \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \dots \cdot (3k + 1)3k} x^{3k},$$

where the term 1 appears because  $a_0 = 1$  is not included in the summation.

Meanwhile, letting  $a_0 = 0$  and  $a_1 = 1$  gives

$$\begin{aligned} n = 1 : & \quad a_3 = 0 \\ n = 2 : & \quad a_4 = \frac{1}{3 \cdot 4} & \quad k = 1 \\ n = 3 : & \quad a_5 = 0 \\ n = 4 : & \quad a_5 = 0 \\ n = 5 : & \quad a_4 = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} & \quad k = 2 \end{aligned}$$

The pattern is explicitly given by  $a_{3k-1} = [3 \cdot 4 \cdot 6 \cdot 7 \cdot \dots \cdot 3k(3k + 1)]^{-1}$ . The solution can be written as

$$y_2 = 1 + \sum_{k=1}^{\infty} \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot \dots} x^{3k+1},$$

where again the term 1 appears because  $a_1 = 1$  is not included in the summation.

The sum  $y_1 + y_2$  is the general solution to Airy's equation.

44. Solve  $y'' + xy' + y = 0$  by series. [ANSWER]

See [here](#) for the solution.

45. When solving equation (7) (the linear, homogeneous second-order ordinary differential equation with non-constant coefficients  $a(x)y'' + b(x)y' + c(x)y = 0$ ), what must one be weary of if there is an  $x_0$  such that  $a(x_0) = 0$ ? What is the condition on  $x_0$  for equation (7) to be solved by series?

[ANSWER]

$x_0$  is called a singular point. Specifically, if the following limits are finite,  $x_0$  is called a "regular singular point," and the *method of Frobenius* can be used to solve equation (7).

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{b(x)}{a(x)} (x - x_0) \\ \lim_{x \rightarrow x_0} \frac{c(x)}{a(x)} (x - x_0)^2 \end{aligned}$$

If either of the limits above are not finite,  $x_0$  is called a "singular singular point" (very creative!).

46. Identify the differential equation  $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$ , and identify the singular point(s). Classify them as "regular singular" or "singular singular." [ANSWER]

The singular points are  $x_0 = 1$  and  $x_0 = -1$ . For  $x_0 = 1$ ,

$$\lim_{x \rightarrow 1} \frac{-2x}{1-x^2}(x-1) = \lim_{x \rightarrow 1} \frac{2x}{x+1} = 1$$
$$\lim_{x \rightarrow 1} \frac{\alpha(\alpha+1)}{1-x^2}(x-1)^2 = \alpha(\alpha+1) \lim_{x \rightarrow 1} \frac{(x-1)(x-1)}{(1-x)(1+x)} = -\alpha(\alpha+1) \lim_{x \rightarrow 1} \frac{(x-1)}{(1+x)} = 0$$

Therefore the point  $x = 1$  is a regular singular point. Meanwhile, for  $x_0 = -1$ ,

$$\lim_{x \rightarrow -1} \frac{-2x}{1-x^2}(x+1) = \lim_{x \rightarrow -1} \frac{2x}{x-1} = \frac{-2}{-2} = 1$$
$$\lim_{x \rightarrow -1} \frac{\alpha(\alpha+1)}{1-x^2}(x+1)^2 = \alpha(\alpha+1) \lim_{x \rightarrow -1} \frac{(x+1)(x+1)}{(1-x)(1+x)} = -\alpha(\alpha+1) \lim_{x \rightarrow -1} \frac{(x+1)}{(1-x)} = 0$$

Therefore the point  $x = -1$  is also a regular singular point.

47. Solve  $2x^2y'' + xy' - (1+x)y = 0$ . [ANSWER]

See [here](#) for the solution.

48. Identify and solve  $x^2y'' + xy' + (x^2 - \nu^2)y = 0$  for  $\nu = 0$ . What is the name of the solution? Where does it appear in acoustics? [ANSWER]

This is Bessel's equation, and the solutions for  $\nu = 0$  are the Bessel functions  $J_0$  and  $N_0$ . This appears in acoustics as the radial eigenfunction of the Helmholtz equation in cylindrical coordinates for axisymmetric radiation. See [here](#) for the solution for  $J_0$ .

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# Orthogonality and special functions

This section naturally picks up where the previous section on ODEs left off. Also included are problems involving Dirac delta functions.

1. Derive the recursion relation for the power expansion coefficients that solve Bessel's equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

for *arbitrary*  $\nu$ . What choice gives the power expansion coefficients for  $J_\nu$ ? What choice gives the power expansion coefficients for  $N_\nu$ ? [ANSWER]

One can easily show that  $x = 0$  is a regular singular point. Thus the method of Frobenius is taken up:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} \end{aligned}$$

Insertion into Bessel's equation gives

$$\begin{aligned} x^2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + x \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + (x^2 - \nu^2) \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} &= 0. \end{aligned}$$

It is desired for all the summations to be combined into one, and thus for all powers of  $x$  to match. For this, final summation above is rewritten starting from  $n = 2$ :

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0.$$

Now, the  $n = 0$  and  $n = 1$  terms of the first three summations are written explicitly, and the remaining  $\sum_2^\infty$  is written as a single summation:

$$\begin{aligned} [r(r-1) + r - \nu^2] a_0 x^r + [r(r+1) + r + 1 - \nu^2] a_1 x^{r+1} \\ + \sum_{n=2}^{\infty} [a_n (n+r)(n+r-1) + a_n (n+r) - \nu^2 a_n + a_{n-2}] x^{n+r} = 0. \end{aligned}$$

Each term above must vanish, because the right-hand side is 0. Specifically, the coefficient of  $x^r$  gives the indicial equation and the values of  $r$ :

$$r = \pm \nu.$$

Meanwhile, the setting the summand above to 0 gives the recurrence relation:

$$a_n = -\frac{a_{n-2}}{(n+r)^2 - \nu^2}$$

It turns out the choice  $r = \nu$  gives Bessel functions  $J_\nu(x)$ , while the choice  $r = -\nu$  gives Neumann functions:

$$\begin{aligned} a_n &= -\frac{a_{n-2}}{n^2 + 2n\nu} && \text{Bessel} \\ a_n &= -\frac{a_{n-2}}{n^2 - 2n\nu} && \text{Neumann} \end{aligned}$$

## 2. The singular points $x = \pm 1$ of Legendre's equation

$$(1 - x^2)y'' - 2xy' + \lambda y = 0$$

were already found to be to regular singular. Thus derive the recursion relation for the power expansion coefficients that solve Legendre's equation for arbitrary  $\lambda$ . Why is there never much discussion about the second solution of Legendre's equation  $Q_n$ ? *[ANSWER]*

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} \end{aligned}$$

Insertion into Legendre's equation gives

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} - \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} \\ &- 2 \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \lambda \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \end{aligned}$$

The crucial step that was at first throwing me off is to write the first two terms of the first summation explicitly. (This is different from the other Frobenius method problems I have done, where it is the last summation that has terms pulled out so as to match the index of the rest of the summations; this time, it is the first summation, because of the  $1 - x^2$  coefficient of  $y''$  in Legendre's equation).

$$\begin{aligned} &a_0 r(r-1) x^{r-2} + a_1 r(r+1) x^{r-1} + \sum_{n=2}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} - \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} \\ &- 2 \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \lambda \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \end{aligned}$$

In order to write all the summations as a single sum, the first summation is now shifted down by two:

$$a_0 r(r-1)x^{r-2} + a_1 r(r+1)x^{r-1} + \sum_{n=0}^{\infty} [a_{n+2}(n+r+2)(n+r+1) - a_n(n+r)(n+r-1) - 2a_n(n+r) + \lambda a_n]x^{n+r} = 0$$

The indicial equation is found by setting the coefficient of the lowest power of  $x$  to zero, giving

$$r = 0, \quad r = 1.$$

Meanwhile, setting the summand above equal to 0 gives the recursion relation:

$$\begin{aligned} a_{n+2} &= \frac{(n+r)(n+r-1) + 2(n+r) - \lambda}{(n+r+2)(n+r+1)} a_n \\ &= \frac{(n+r)^2 + n+r - \lambda}{(n+r+2)(n+r+1)} a_n \end{aligned}$$

For  $r = 0$ , the above recursion relation gives the Legendre polynomials ( $P_l$  once we pick  $\lambda = l(l+1)$ ), but for  $r = 1$ , the recursion relation gives a dependent solution of the Legendre polynomials. This is a reflection of Fuchs's theorem (See section 21 of Boas's *Methods*). To find the second independent solution of Legendre's equation, one can use reduction of order (see derivation of reduction of order in the ODE section; also See Boas's ch. 12 section 2 problem 4 for the outline of how to find  $Q$ ). It turns out that  $Q$  diverges at the poles,  $x = \pm 1$ , or  $\theta = 0^\circ$  and  $\theta = 180^\circ$  if the argument is  $x = \cos \theta$ . Thus this second solution is not included in problems that include the poles (for the same reason that the Neumann functions are not included in spherical wave problems that include the origin, because the Neumann functions diverge at the origin).

An example that requires  $Q$  is modeling a toroidal bubble in spherical coordinates. The domain of the sound inside such a bubble does not include the poles. Whales and dolphins create these bubbles when they exhale.

3. Given the recurrence relation for Legendre polynomials,

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x),$$

and the integral result

$$\int_{-1}^1 P_n(x)P_m(x)dx = \frac{2}{2n+1} \delta_{nm},$$

show that

$$\int_{-1}^1 x^2 P_{n+1}(x)P_{n-1}(x)dx = \frac{2n(n+1)}{(4n^2-1)(2n+3)}.$$

[ANSWER]

Solve the first equation for  $(2n+1)xP_n(x)$ :

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \quad (i)$$

Shift the indices on the LHS of equation (i) to  $n + 1$  and to  $n - 1$ :

$$(2n + 3)xP_{n+1}(x) = (n + 2)P_{n+2}(x) + (n + 1)P_n(x) \quad (\text{ii})$$

$$(2n - 1)xP_n(x) = nP_n(x) + (n - 1)P_{n-2}(x) \quad (\text{iii})$$

Multiply equations (ii) and (iii):

$$(2n - 1)(2n + 3)x^2P_{n-1}(x)P_{n+1}(x) = n(n + 2)P_n(x)P_{n+2}(x) + n(n + 1)P_n(x)P_n(x) \\ + (n - 1)(n + 2)P_{n+2}(x)P_{n-2}(x) + (n - 1)(n + 1)P_{n-2}(x)P_n(x)$$

Integrate the above over  $x$  from 0 to 1, and employ the orthogonality relation

$$\int_{-1}^1 P_n(x)P_m(x)dx = \frac{2}{2n + 1} \delta_{nm}.$$

This results in

$$(2n - 1)(2n + 3) \int_{-1}^1 P_{n-1}(x)P_{n+1}(x)x^2 dx = \frac{2n(n + 1)}{2n + 1}$$

Dividing by  $(2n - 1)(2n + 3)$  on both sides and noting that  $(2n - 1)(2n + 1) = 4n^2 - 1$  gives the desired result,

$$\int_{-1}^1 P_{n-1}(x)P_{n+1}(x)x^2 dx = \frac{2n(n + 1)}{(4n^2 - 1)(2n + 3)}$$

4. Given  $J_{p-1}(x) - J_{p+1} = 2J'_p$ , the integral relation  $J_0(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin \theta) d\theta$ , show that (part a)

$$J_1 = \frac{2}{\pi} \int_0^{\pi/2} \sin(x \sin \theta) \sin \theta d\theta.$$

Then (part b) obtain

$$x^{-1}J_1(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin \theta) \cos^2 \theta d\theta$$

by integrating the right-hand side of the first result by parts. [ANSWER]

Setting  $p = 0$  and invoking the first identity gives  $J_{-1}(x) - J_1(x) = 2J'_0(x)$ . Noting that  $J_{-1}(x) = -J_1(x)$  gives  $-2J_1(x) = 2J'_0(x)$ , or

$$-J_1(x) = J'_0(x). \quad (\text{i})$$

Attention is now turned to the integral relationship,

$$J_0(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin \theta) d\theta. \quad (\text{ii})$$

The derivative with respect to  $x$  of equation (ii) is taken, giving

$$J'_0 = \frac{d}{dx} J_0(x) = \frac{d}{dx} \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin \theta) d\theta$$

Invoking equation (i) gives

$$\begin{aligned} -J_1(x) &= \frac{2}{\pi} \int_0^{\pi/2} \frac{d}{dx} \cos(x \sin \theta) d\theta \\ -J_1(x) &= -\frac{2}{\pi} \int_0^{\pi/2} \sin(x \sin \theta) \sin \theta d\theta, \end{aligned}$$

or

$$J_1(x) = \frac{2}{\pi} \int_0^{\pi/2} \sin(x \sin \theta) \sin \theta d\theta.$$

Part (b): To integrate

$$J_1(x) = \frac{2}{\pi} \int_0^{\pi/2} \sin(x \sin \theta) \sin \theta d\theta, \quad (\text{iii})$$

by parts, set

$$\begin{aligned} u &= \sin(x \sin \theta), & du &= \cos(x \sin \theta) x \cos \theta d\theta \\ v &= -\cos \theta, & dv &= \sin \theta d\theta. \end{aligned}$$

Equation (iii) integral becomes

$$\begin{aligned} J_1(x) &= -\sin(x \sin \theta) \cos \theta \Big|_0^{\pi/2} + \frac{2}{\pi} \int_0^{\pi/2} x \cos \theta \cos(x \sin \theta) \cos \theta d\theta \\ &= \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin \theta) x \cos^2 \theta d\theta \end{aligned}$$

$x$  is not an integration variable and can thus be divided through on both sides of the equation, giving the desired result,

$$x^{-1} J_1(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin \theta) \cos^2 \theta d\theta.$$

Sometimes when integrating by parts it is helpful to recall the trick "ILATE," (Inverse trig, Logarithmic, Algebraic, Trig, Exponential) which gives the priority of which function to set equal to  $u$  when integrating by parts. For this problem, I just guessed, first trying  $u = \sin \theta$  and then trying  $u = \sin(x \sin \theta)$ . The second choice worked out.

5. Prove that

$$\delta(kx) = \frac{1}{|k|} \delta(x)$$

where  $k$  is any nonzero constant. *Hint: let  $y = kx$ , and integrate a test function  $f(x) = f(y/k)$  times the Dirac delta function of  $y$  from  $-\infty$  to  $\infty$ .* [ANSWER]

$$\begin{aligned}\int_{-\infty}^{\infty} f(x) \delta(kx) dx &= \frac{1}{|k|} \int_{-\infty}^{\infty} f(y/k) \delta(y) dy \\ &= \frac{1}{|k|} f(0) \\ &= \int_{-\infty}^{\infty} f(x) \frac{1}{|k|} \delta(x) dx\end{aligned}$$

where the absolute value has been included to account for the fact that the limits of integration are reversed if  $k < 0$ . Comparing the first and last lines above gives the desired equality,

$$\delta(kx) = \frac{1}{|k|} \delta(x).$$

6.  $\int_2^6 (3x^2 - 2x - 1) \delta(x - 3) dx = \text{[ANSWER]}$

$$3(3)^2 - 2(3) - 1 = 20$$

7.  $\int_0^5 \cos x \delta(x - \pi) dx = \text{[ANSWER]}$

$$\cos \pi = -1$$

8.  $\int_0^3 x^3 \delta(x + 1) dx = \text{[ANSWER]}$

0 (because the limits do not include  $-1$ ).

9.  $\int_{-\infty}^{\infty} \ln(x + 3) \delta(x + 2) dx = \text{[ANSWER]}$

$$\ln(-2 + 3) = 0$$

10.  $\int_{-2}^2 (2x + 3) \delta(3x) dx = \text{[ANSWER]}$

Here it should be noted that  $\delta(kx) = \delta(x)/|k|$ . Therefore the integral evaluates to  $\frac{1}{3} [2(0) + 3] = 1$

11.  $\int_0^2 (x^3 + 3x + 2) \delta(1 - x) dx = \text{[ANSWER]}$

Since  $\delta(kx) = \delta(x)/|k|$ , it follows that  $\delta(-x) = \delta(x)$ . Thus  $\delta(1 - x) = \delta[-(x - 1)] = \delta(x - 1)$ . Thus the integral evaluates to  $1 + 3 + 2 = 6$ .

12.  $\int_{-1}^1 9x^2 \delta(3x + 1) dx = \text{[ANSWER]}$

The delta function can be written as  $\delta[3(x + 1/3)] = \frac{1}{3} \delta(x + 1/3)$ . Thus the integral becomes  $\int_{-1}^1 3x^2 \delta(x + 1/3) dx = \frac{1}{3}$ .

13.  $\int_{-\infty}^a \delta(x - b) dx = \text{[ANSWER]}$

1 for  $a \geq b$  and 0 for  $a$  less than  $b$  (I can't type the less than symbol in HTML!).

14. ✪ Prove that

$$x \frac{d}{dx} [\delta(x)] = -\delta(x).$$

Hint: Integrate  $\int_{-\infty}^{\infty} f(x)x \frac{d}{dx} [\delta(x)] dx$  by parts. [ANSWER]

Following the hint, the following definitions are made,

$$\begin{aligned} u &= f(x)x & v &= \delta(x) \\ du &= \frac{d}{dx} [f(x)x] dx & dv &= \frac{d}{dx} [\delta(x)] dx, \end{aligned}$$

and the integral is taken by parts.

$$\int_{-\infty}^{\infty} f(x)x \frac{d}{dx} [\delta(x)] dx = f(x)x \delta(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x) \frac{d}{dx} [f(x)x] dx \quad (\text{i})$$

The first term on the right-hand side above is zero because  $\delta(\infty) = 0 = \delta(-\infty)$ , so equation (i) becomes

$$\int_{-\infty}^{\infty} f(x)x \frac{d}{dx} [\delta(x)] dx = - \int_{-\infty}^{\infty} \delta(x) \frac{d}{dx} [f(x)x] dx \quad (\text{ii})$$

Applying the product rule to the integral on the right-hand side of equation (ii) gives

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)x \frac{d}{dx} [\delta(x)] dx &= - \int_{-\infty}^{\infty} \delta(x) [f'(x)x + f(x)] dx \\ &= -[-0f'(0) + f(0)] \\ &= - \int_{-\infty}^{\infty} \delta(x) f(x) dx \end{aligned} \quad (\text{iii})$$

where the definition of the delta function has been used in the second and third equalities above. Comparing the integrands of the left- and right-hand sides of equation (iii) gives the desired equality,

$$x \frac{d}{dx} [\delta(x)] = -\delta(x).$$

15. ✪ Prove that

$$\frac{d\theta}{dx} = \delta(x),$$

where  $\theta$  is the Heaviside step function. Hint: Integrate  $\int_{-\infty}^{\infty} f(x) \frac{d\theta}{dx} dx$  by parts. [ANSWER]

Following the hint, the following definitions are made,

$$\begin{aligned} u &= f & v &= \theta \\ du &= \frac{df}{dx} dx & dv &= \frac{d\theta}{dx} dx, \end{aligned}$$

and the integral is taken by parts.

$$\begin{aligned}
\int_{-\infty}^{\infty} f \frac{d\theta}{dx} dx &= f\theta \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \theta \frac{df}{dx} dx && \text{(i)} \\
&= f(\infty)\theta(\infty) - f(-\infty)\theta(-\infty) - \int_{-\infty}^0 0 \frac{df}{dx} dx - \int_0^{\infty} 1 \frac{df}{dx} dx \\
&= f(\infty) - \int_0^{\infty} \frac{df}{dx} dx \\
&= f(\infty) - f(\infty) + f(0) && \text{(1)} \\
&= \int_{-\infty}^{\infty} \delta(x)f(x)dx && \text{(ii)}
\end{aligned}$$

where the definition of the delta function has been used in the last line to write  $f(0)$ . Lines (i) and (ii) give

$$\int_{-\infty}^{\infty} f(x) \frac{d\theta}{dx} dx = \int_{-\infty}^{\infty} \delta(x)f(x)dx .$$

The integrands must be equal, which gives the desired result:

$$\frac{d\theta}{dx} = \delta(x).$$

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# Vector algebra

In the section on [linear algebra](#), different symbols were used for vectors and their representation in a particular bases. In this section, we deal only with the vectors themselves. When vectors need to be expressed in a basis,  $3 \times 1$  matrices are used. It is therefore not necessary to distinguish between  $\mathbf{v}$  and  $v$  (and between  $A$  and  $A$ ).

Some of the problems below come from chapter 1 of *Introduction to Electrodynamics* by D. J. Griffiths.

1. Suppose we have a barrel of fruit that contains  $a_x$  bananas,  $a_y$  pears, and  $a_z$  apples. Denoting  $\mathbf{e}_n$  as the unit vector in the  $n$  direction in space, is  $\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$ , a vector? Explain.

[ANSWER]

No, because  $\mathbf{a}$  does not obey coordinate transformations. For example, choosing a different set of axes does not turn a pear into a banana. By definition, "a vector is any set of three components that transforms in the same manner as a displacement when you change coordinates." (from Griffiths *Introduction to Electrodynamics*, section 1.1.5).

2. How do the components  $a_x$ ,  $a_y$ , and  $a_z$  of a vector  $\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$  transform under the translation of coordinates?

$$\begin{aligned}x' &= x \\y' &= y - a \\z' &= z\end{aligned}$$

In other words, what happens to  $a_x$ ,  $a_y$ , and  $a_z$  when  $\mathbf{a}$  is written as  $\mathbf{a} = a_x\mathbf{e}'_x + a_y\mathbf{e}'_y + a_z\mathbf{e}'_z$ ?

[ANSWER]

The components of a vector are invariant under this transformation.

3. How do the components of a vector transform under the inversion of coordinates?

$$\begin{aligned}x' &= -x \\y' &= -y \\z' &= -z\end{aligned}$$

In other words, what happens to  $a_x$ ,  $a_y$ , and  $a_z$  when  $\mathbf{a}$  is written as  $\mathbf{a} = a_x\mathbf{e}'_x + a_y\mathbf{e}'_y + a_z\mathbf{e}'_z$ ?

[ANSWER]

The components are also inverted.  $a_x \mapsto -a_x$ ,  $a_y \mapsto -a_y$ , and  $a_z \mapsto -a_z$ .

4. How does the cross product of two vectors  $\mathbf{u}$  and  $\mathbf{v}$  transform under the inversion of coordinates? Is the cross product of two vectors really a vector? [ANSWER]

The cross product  $\mathbf{w} = \mathbf{u} \times \mathbf{v}$  is invariant under the inversion, because  $\mathbf{w} = -\mathbf{u} \times -\mathbf{v}$ . Thus  $\mathbf{w}$  is a different kind of quantity than vectors  $\mathbf{u}$  and  $\mathbf{v}$ . It is called a *psuedovector*.

5. How does the scalar triple product of  $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$  transform under the inversion of coordinates? Is the scalar triple product really a scalar? (Griffiths problem 1.10d) *[ANSWER]*

The scalar triple product transforms as  $-\mathbf{w} \cdot (-\mathbf{u} \times -\mathbf{v}) = -\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$ , i.e., the product changes signs when the coordinates are inverted. This is in contrast with the fact that scalars are invariant under coordinate inversions. Thus the scalar triple product is a different kind of quantity than an ordinary scalar. It is called a *psuedoscalar*.

6. Show that  $|\mathbf{u} \times \mathbf{v}|^2 + (\mathbf{u} \cdot \mathbf{v})^2 = |\mathbf{u}|^2|\mathbf{v}|^2$ . *[ANSWER]*

$$\begin{aligned}
 |\mathbf{u} \times \mathbf{v}|^2 + (\mathbf{u} \cdot \mathbf{v})^2 &= \epsilon_{ijk}u_jv_k\epsilon_{ilm}u_lv_m + u_iv_iu_jv_j \\
 &= \epsilon_{ijk}\epsilon_{ilm}u_jv_ku_lv_m + u_iv_iu_jv_j \\
 &= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})u_jv_ku_lv_m + u_iv_iu_jv_j \\
 &= \delta_{jl}\delta_{km}u_jv_ku_lv_m - \delta_{jm}\delta_{kl}u_jv_ku_lv_m + u_iv_iu_jv_j \\
 &= u_lu_lv_kv_k - u_mv_mv_ku_k + u_iv_iu_jv_j \\
 &= u_lu_lv_kv_k - u_iv_iu_jv_j + u_iv_iu_jv_j \\
 &= |\mathbf{u}|^2|\mathbf{v}|^2
 \end{aligned}$$

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# Vector calculus

Vector calculus is not listed as a topic on the math section, but it is worth reviewing. Some of the problems below come from chapter 1 of *Introduction to Electrodynamics* by D. J. Griffiths.

1. In what direction does the gradient of a function point? [ANSWER]

The gradient of a function points in the direction of the steepest ascent.

2. ✪ Prove that  $\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$ . [ANSWER]

It is much easier to go from the right-hand side to the left-hand side.

$$\begin{aligned} & [\mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}]_i \\ &= \epsilon_{ijk} a_j (\epsilon_{klm} \partial_l b_m) + \epsilon_{ijk} b_j (\epsilon_{klm} \partial_l a_m) + (a_j \partial_j) b_i + (b_j \partial_j) a_i \end{aligned}$$

Expanding the above expression and rearranging terms yields

$$\epsilon_{kij} \epsilon_{klm} a_j \partial_l b_m + \epsilon_{kij} \epsilon_{klm} b_j \partial_l a_m + (a_j \partial_j) b_i + (b_j \partial_j) a_i$$

Invoking the product of Kronecker deltas property yields

$$\begin{aligned} & (\delta_{il} \delta_{jm} - \delta_{im} \delta_{lj}) a_j \partial_l b_m + (\delta_{il} \delta_{jm} - \delta_{im} \delta_{lj}) b_j \partial_l a_m + (a_j \partial_j) b_i + (b_j \partial_j) a_i \\ &= a_m \partial_i b_m - a_l \partial_l b_i + b_j \partial_i a_j - b_l \partial_l a_i + (a_j \partial_j) b_i + (b_j \partial_j) a_i \end{aligned}$$

Canceling terms yields

$$a_j \partial_i b_j + b_j \partial_i a_j = \partial_i (a_j b_j),$$

which is the  $i$ th component of  $\nabla(\mathbf{a} \cdot \mathbf{b})$ . Since this holds for all three components, the proof is complete.

3. Prove that  $\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a})$ . [ANSWER]

$$\begin{aligned} \nabla \times (\mathbf{a} \times \mathbf{b}) &= \epsilon_{ijk} \partial_j (\epsilon_{klm} a_l b_m) \\ &= \epsilon_{ijk} \epsilon_{klm} \partial_j a_l b_m \\ &= \epsilon_{kij} \epsilon_{klm} \partial_j a_l b_m \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j a_l b_m \\ &= \delta_{il} \delta_{jm} \partial_j a_l b_m - \delta_{il} \delta_{jm} \partial_j a_l b_m \\ &= \partial_j (a_i b_j) - \partial_j (a_i b_j) \\ &= b_j \partial_j a_i + a_i \partial_j b_j - a_i \partial_j b_j - b_j \partial_j a_i \\ &= \partial_j (a_i b_j) - \partial_j (a_i b_j) \\ &= b_j \partial_j a_i - a_i \partial_j b_j + a_i \partial_j b_j - b_j \partial_j a_i \\ &= (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) \end{aligned}$$

4. Prove that the divergence of the curl is 0. [ANSWER]

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{u}) &= \partial_i (\epsilon_{ijk} \partial_j u_k) \\
&= -\partial_i (\epsilon_{jik} \partial_j u_k) && \text{(permute)} \\
&= -\partial_j (\epsilon_{ijk} \partial_i u_k) && \text{(relabel)} \\
&= -\partial_i (\epsilon_{ijk} \partial_j u_k) && \text{(equality of mixed pt'ls)}
\end{aligned}$$

$-\partial_i (\epsilon_{ijk} \partial_j u_k) = \partial_i (\epsilon_{ijk} \partial_j u_k)$  is of the form  $a = -a$ , which means that  $a = 0$ , i.e.,  $-\partial_i (\epsilon_{ijk} \partial_j u_k) = 0$ , which completes the proof:  $\nabla \cdot (\nabla \times \mathbf{u}) = 0$ .

5. Prove that the curl of the gradient is 0. [ANSWER]

$$\begin{aligned}
\nabla \times (\nabla f) &= \epsilon_{ijk} \partial_j (\nabla f)_k \\
&= \epsilon_{ijk} \partial_j \partial_k f \\
&= -\epsilon_{ikj} \partial_j \partial_k f && \text{(permute)} \\
&= -\epsilon_{ijk} \partial_k \partial_j f && \text{(relabel)} \\
&= -\epsilon_{ijk} \partial_j \partial_k f && \text{(equality of mixed pt'ls)}
\end{aligned}$$

Since  $-\epsilon_{ijk} \partial_j \partial_k f = \epsilon_{ijk} \partial_j \partial_k f$ ,  $\epsilon_{ijk} \partial_j \partial_k f = 0$ , which completes the proof:  $\nabla \times (\nabla f) = 0$ .

6. Show that  $\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$ . [ANSWER]

$$\begin{aligned}
[\nabla \times (\nabla \times \mathbf{a})]_i &= \epsilon_{ijk} \partial_j (\nabla \times \mathbf{a})_k \\
&= \epsilon_{ijk} \partial_j \epsilon_{klm} \partial_l a_m \\
&= \epsilon_{ijk} \epsilon_{klm} \partial_j \partial_l a_m \\
&= \epsilon_{kij} \epsilon_{klm} \partial_j \partial_l a_m \\
&= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j \partial_l a_m \\
&= \delta_{il} \delta_{jm} \partial_j \partial_l a_m - \delta_{im} \delta_{jl} \partial_j \partial_l a_m \\
&= \partial_i \partial_j a_j - \partial_l \partial_l a_i
\end{aligned}$$

The last line equals  $\partial_i (\nabla \cdot \mathbf{a}) - [\nabla^2 \mathbf{a}]_i$ , which is the  $i$ th component of  $\nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$ .

7. In Cartesian coordinates,  $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$  and thus  $r = \sqrt{x^2 + y^2 + z^2}$ . Find  $\nabla r$ . [ANSWER]

$$\begin{aligned}
\nabla r &= \nabla (x^2 + y^2 + z^2)^{1/2} \\
&= \frac{1}{r} (x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) \\
&= \frac{\mathbf{r}}{r}
\end{aligned}$$

8. Let  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$  be the position vector. Thus in Cartesian coordinates,  $R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$ . Find  $\nabla R^2$ . [ANSWER]

$$\nabla R^2 = 2\mathbf{R}$$

9. Find  $\nabla R^{-1}$ . [ANSWER]

$$\nabla 1/R = -\mathbf{R}/R^3 = -\mathbf{e}_r/R^2$$

10. What is the coordinate-free definition of the divergence of a vector field  $\mathbf{F}$ ? [ANSWER]

$$\nabla \cdot \mathbf{F} = \lim_{V \rightarrow 0} \frac{1}{V} \oint_S \mathbf{F} \cdot d\mathbf{a}.$$

11. ✪ Find the divergence of  $\mathbf{e}_r/r^2 = \mathbf{r}/r^3$  in both Cartesian and spherical coordinates. Note that the  $r$  component of the divergence of  $\mathbf{v}$  in spherical coordinates is  $\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_r)$ . Explain the result.

[ANSWER]

In cartesian coordinates,

$$\begin{aligned} \nabla \cdot \frac{\mathbf{e}_r}{r^2} &= \nabla \cdot \frac{\mathbf{r}}{r^3} \\ &= \nabla \cdot \frac{x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z}{(x^2 + y^2 + z^2)^{3/2}} \\ &= \nabla \cdot (x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z)(x^2 + y^2 + z^2)^{-3/2} \\ &= \frac{\partial}{\partial x} [x(x^2 + y^2 + z^2)^{-3/2}] + \frac{\partial}{\partial y} [y(x^2 + y^2 + z^2)^{-3/2}] + \frac{\partial}{\partial z} [z(x^2 + y^2 + z^2)^{-3/2}] \\ &= -\frac{3}{2}(x^2 + y^2 + z^2)^{-5/2} 2x^2 + (x^2 + y^2 + z^2)^{-3/2} + \dots \\ &= \frac{3}{r^3} - \frac{3 \cdot 2(x^2 + y^2 + z^2)}{2r^5} \\ &= \frac{3}{r^3} - \frac{3r^2}{r^5} = \frac{3}{r^3} - \frac{3}{r^3} = 0. \end{aligned}$$

In spherical coordinates,

$$\begin{aligned} \nabla \cdot \frac{\mathbf{e}_r}{r^2} &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{1}{r^2} \right) \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} (1) = 0. \end{aligned}$$

It is a paradoxical result, because  $\mathbf{e}_r/r^2$  is a function that points away from the origin radially in all directions and quadratically falling off in amplitude. That is a pretty divergent function. The resolution to the paradox is that  $\mathbf{e}_r/r^2 \rightarrow \infty$  at the origin, at which the divergence is *not* 0.

12. What is the coordinate-free definition of the curl of a vector field  $\mathbf{F}$ ? [ANSWER]

$$\nabla \times \mathbf{F} = \lim_{S \rightarrow 0} \frac{1}{S} \oint_C \mathbf{F} \cdot d\mathbf{s}.$$

13. Construct a non-constant vector function that has zero divergence and zero curl everywhere.

[ANSWER]

The vector-valued function  $\mathbf{f} = x\mathbf{e}_x - y\mathbf{e}_y$  has zero divergence and zero curl everywhere. Another example is  $\mathbf{f} = y\mathbf{e}_x + x\mathbf{e}_y$ .

14. Calculate the line integral of the function  $\mathbf{v} = y^2\mathbf{e}_x + 2x(y + 1)\mathbf{e}_y$  from  $\mathbf{a} = (1, 1, 0)$  to  $\mathbf{b} = (2, 2, 0)$ , following the path from  $(x, y, z) = (1, 1, 0)$  to  $(2, 1, 0)$  to  $(2, 2, 0)$ . Then calculate the

line integral following the path from  $\mathbf{a} = (1, 1, 0)$  to  $\mathbf{b} = (2, 2, 0)$  directly. Finally calculate  $\oint \mathbf{v} \cdot d\mathbf{s}$  for the loop going from  $(1, 1, 0)$  to  $(2, 1, 0)$  to  $(2, 2, 0)$  to  $(1, 1, 0)$ . [ANSWER]

The integral for first path  $C_1$  from  $(x, y, z) = (1, 1, 0)$  to  $(2, 1, 0)$  (for which  $y = 1$ ) and then to  $(2, 2, 0)$  (for which  $x = 2$ ) is

$$\begin{aligned} \int_{C_1} \mathbf{v} \cdot d\mathbf{s} &= \int_{C_1} [y^2 \mathbf{e}_x + 2x(y+1) \mathbf{e}_y] \cdot [dx \mathbf{e}_x + dy \mathbf{e}_y] \\ &= \int_{C_1} [y^2 dx + 2x(y+1) dy] \\ &= \int_1^2 1^2 dx + \int_1^2 2 \cdot 2(y+1) dy \\ &= \int_1^2 dx + 4 \int_1^2 (y+1) dy \\ &= 1 + 10 = 11 \end{aligned}$$

The integral for the second path  $C_2$  from  $(x, y, z) = (1, 1, 0)$  to  $(2, 2, 0)$  (for which  $y = x$ ) is

$$\begin{aligned} \int_{C_2} \mathbf{v} \cdot d\mathbf{s} &= \int_{C_2} [y^2 \mathbf{e}_x + 2x(y+1) \mathbf{e}_y] \cdot [dx \mathbf{e}_x + dy \mathbf{e}_y] \\ &= \int_{C_2} [x^2 dx + 2y(y+1) dy] \\ &= \int_1^2 x^2 dx + \int_1^2 2y(y+1) dy \\ &= 10 \end{aligned}$$

The integral over the closed loop is easy to calculate, now that the individual paths forming the loop have been calculated. The loop going from  $(1, 1, 0)$  to  $(2, 1, 0)$  to  $(2, 2, 0)$  and finally back to  $(1, 1, 0)$  can be written as

$$\int_{C_1} \mathbf{v} \cdot d\mathbf{s} - \int_{C_2} \mathbf{v} \cdot d\mathbf{s} = 11 - 10 = 1$$

15. Calculate the surface integral of  $\mathbf{v} = 2xz \mathbf{e}_x + (x+2) \mathbf{e}_y + y(z^2 - 3) \mathbf{e}_z$  over five sides (excluding the bottom) of a cubical box whose bottom edge extends from  $x = 0$  to  $x = 2$ . [ANSWER]

The surface integral  $\int_S \mathbf{v} \cdot d\mathbf{a}$  is to be calculated. It will be given by the sum of five integrals: at  $x = 2$  and  $x = 0$  (for which  $d\mathbf{a} = dy dz \mathbf{e}_x$ ), at  $z = 2$  and  $z = 0$  (for which  $d\mathbf{a} = dx dy \mathbf{e}_z$ ), and at  $y = 0$  (for which  $d\mathbf{a} = dx dz \mathbf{e}_y$ ).

$$\begin{aligned} \int_S \mathbf{v} \cdot d\mathbf{a} &= \int_S [2xz \mathbf{e}_x + (x+2) \mathbf{e}_y + y(z^2 - 3) \mathbf{e}_z] \cdot d\mathbf{a} \\ &= \int_0^2 \int_0^2 4z dy dz + \int_0^2 \int_0^2 0 dy dz \\ &\quad + \int_0^2 \int_0^2 y(4-3) dx dy + \int_0^2 \int_0^2 -3y dx dy + \int_0^2 \int_0^2 (x+2) dx dz \\ &= 16 + 0 + 4 - 12 + 12 = 20 \end{aligned}$$

16. Calculate the volume integral of the scalar-valued field  $T = xyz^2$  over a right triangular prism of height  $z = 3$ , whose lower base is the triangle with vertices at the origin,  $(x, y, z) = (1, 0, 0)$ , and  $(x, y, z) = (0, 1, 0)$ . [ANSWER]

The volume integral  $\int_{V'} T dV$  is to be calculated:

$$\begin{aligned}
 \int_{V'} xyz^2 dV &= \int_{z=0}^{z=3} \int_{y=0}^{1-x} \int_{x=0}^{x=1} xyz^2 dx dy dz \\
 &= \int_{z=0}^{z=3} z^2 \int_{x=0}^{x=1} x dx \int_{y=0}^{1-x} y dy dz \\
 &= \int_{z=0}^{z=3} z^2 dz \int_{x=0}^{x=1} x dx \frac{(1-x)^2}{2} \\
 &= \frac{9}{2} \int_{x=0}^{x=1} (x - 2x^2 + x^3) dx \\
 &= \frac{9}{2} \left( \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) \\
 &= \frac{9}{3} \cdot \frac{1}{12} = \frac{3}{8}
 \end{aligned}$$

17. State the gradient theorem. What is the meaning of a conservative field? Provide examples of conservative fields and nonconservative fields from physics. [ANSWER]

The gradient theorem states that

$$\int_a^b \nabla V \cdot ds = V(b) - V(a)$$

A conservative field is one that is given by a scalar potential function. For example,  $\nabla V$  above is a conservative field. The gradient theorem says that the line integral over a conservative field is independent of the path. Examples of conservative fields are the gravitational and electric fields, given by minus the gradient of the gravitational and electrical potential respectively. Meanwhile, the magnetic field is given by the curl of a vector potential, and is therefore not a conservative field.

18. Calculate the line integral  $\int_a^b \nabla T \cdot ds$  for  $T = x^2 + 4xy + 2yz^3$  where  $\mathbf{a} = (0, 0, 0)$  and  $\mathbf{b} = (1, 1, 1)$  for the path  $(0, 0, 0) \rightarrow (1, 0, 0) \rightarrow (1, 1, 0) \rightarrow (1, 1, 1)$ . Then calculate the integral for the path  $(0, 0, 0) \rightarrow (0, 0, 1) \rightarrow (0, 1, 1) \rightarrow (1, 1, 1)$ . Next, calculate the integral for the parabolic path  $z = x^2, y = x$ . Finally, check the result with the gradient theorem. [ANSWER]

For the first path from  $(0, 0, 0) \rightarrow (1, 0, 0) \rightarrow (1, 1, 0) \rightarrow (1, 1, 1)$ ,

$$\begin{aligned}
 \int_a^b \nabla T \cdot ds &= \int_a^b \nabla(x^2 + 4xy + 2yz^3) \cdot (dx\mathbf{e}_x + dy\mathbf{e}_y + dz\mathbf{e}_z) \\
 &= \int_a^b [(2x + 4y)\mathbf{e}_x + (4x + 2z^3)\mathbf{e}_y + (6yz^2)\mathbf{e}_z] \cdot (dx\mathbf{e}_x + dy\mathbf{e}_y + dz\mathbf{e}_z) \\
 &= \int_a^b [(2x + 4y)dx + (4x + 2z^3)dy + (6yz^2)dz]
 \end{aligned}$$

The integral splits into three integrals. For  $(0, 0, 0) \rightarrow (1, 0, 0)$ ,  $y = z = 0$ . For  $(1, 0, 0) \rightarrow (1, 1, 0)$ ,  $x = 1$  and  $z = 0$ . For  $(1, 1, 0) \rightarrow (1, 1, 1)$ ,  $x = y = 1$ :

$$\begin{aligned}\int_{\mathbf{a}}^{\mathbf{b}} [(2x + 4y)dx + (4x + 2z^3)dy + (6yz^2)dz] &= \int_0^1 2x dx + \int_0^1 4dy + \int_0^1 (6z^2)dz \\ &= \int_0^1 2x dx + \int_0^1 4dy + \int_0^1 6z^2 dz \\ &= 1 + 4 + 2 = 7.\end{aligned}$$

For the path  $(0, 0, 0) \rightarrow (0, 0, 1) \rightarrow (0, 1, 1) \rightarrow (1, 1, 1)$ , the integral is again split into three parts. For  $(0, 0, 0) \rightarrow (0, 0, 1)$ ,  $x = y = 0$ . For  $(0, 0, 1) \rightarrow (0, 1, 1)$ ,  $x = 0$  and  $z = 1$ . For  $(0, 1, 1) \rightarrow (1, 1, 1)$ ,  $y = z = 1$ .

$$\begin{aligned}\int_{\mathbf{a}}^{\mathbf{b}} [(2x + 4y)dx + (4x + 2z^3)dy + (6yz^2)dz] &= \int_0^1 0dz + \int_0^1 2dy + \int_0^1 (2x + 4)dx \\ &= 0 + 2 + 5 = 7\end{aligned}$$

Next, for the parabolic path  $z = x^2$ ,  $y = x$ , the integration is reduced to that over a single variable, which is here chosen to be  $x$ . The differentials then  $dz = 2dx$  and  $dy = dx$ , and the integral becomes

$$\begin{aligned}\int_{x=0}^{x=1} [(2x + 4x) + (4x + 2x^6) + (6x^5)2x]dx &= \int_{x=0}^{x=1} (6x + 4x + 2x^6 + 12x^6)dx \\ &= \int_{x=0}^{x=1} (10x + 2x^6 + 12x^6)dx \\ &= 5x^2 + \frac{2}{7}x^7 + \frac{12}{7}x^7 \Big|_{x=0}^{x=1} \\ &= 5 + \frac{2}{7} + \frac{12}{7} = 5 + 14/7 = 7.\end{aligned}$$

Finally, the integral is simply given by

$$\begin{aligned}T(\mathbf{b}) - T(\mathbf{a}) &= x^2 + 4xy + 2yz^3 \Big|_{(0,0,0)}^{(1,1,1)} \\ &= 1 + 4 + 2 = 7.\end{aligned} \quad (\text{much easier!})$$

19. ✪ State and prove the divergence theorem and Stokes's theorem. [ANSWER]

The divergence theorem is

$$\oint \mathbf{F} \cdot d\mathbf{a} = \int \nabla \cdot \mathbf{F} dV,$$

and Stokes's theorem is

$$\oint \mathbf{F} \cdot d\mathbf{s} = \int (\nabla \times \mathbf{F}) \cdot d\mathbf{a}.$$

They are immediate from the definitions of divergence and curl above, but to see what motivates these definitions, see [here](#).

20. List everything that can be concluded about a vector field  $\mathbf{F}$  that has a vanishing curl, i.e.,  $\nabla \times \mathbf{F} = 0$ . [ANSWER]

- The line integral  $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{s}$  is independent of the path.

- The line integral over any closed loop is zero, i.e.,  $\oint_a \mathbf{F} \cdot d\mathbf{s} = 0$ .
- $\mathbf{F}$  is given by the gradient of a scalar potential, i.e.,  $\mathbf{F} = -\nabla V$ .

21. List everything that can be concluded about a vector field  $\mathbf{F}$  that has a vanishing divergence, i.e.,  $\nabla \cdot \mathbf{F} = 0$ . [ANSWER]

- The surface integral  $\int_S \mathbf{F} \cdot d\mathbf{a}$  is independent of the surface (but the line integral  $\int_a^b \mathbf{F} \cdot d\mathbf{s}$  depends on the path.)
- The integral over any closed surface is zero, i.e.,  $\oint_S \mathbf{F} \cdot d\mathbf{a} = 0$ .
- $\mathbf{F}$  is given by the curl of a vector potential, i.e.,  $\mathbf{F} = \nabla \times \mathbf{A}$ .

22. Calculate the line integral  $\oint \mathbf{F} \cdot d\mathbf{s}$  for the vector field  $\mathbf{F} = x^2y\mathbf{e}_x + xy^3\mathbf{e}_y$ , where the closed path of integration is a square whose corners are given by the ordered pairs  $(x, y) = (1, 1)$ ,  $(-1, 1)$ ,  $(-1, -1)$ , and  $(1, -1)$ . Then, use Stokes's theorem to show that the same result is obtained by  $\int (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$ . [ANSWER]

The line integral consists of four sub-paths:

1. From  $(x, y) = (1, -1) \rightarrow (1, 1)$ , the integral is  $\int_{-1}^1 y^3 dy = 0$ .
2. From  $(x, y) = (1, 1) \rightarrow (-1, 1)$ , the integral is  $\int_{-1}^1 x^2 dx = -2/3$ .
3. From  $(x, y) = (-1, 1) \rightarrow (-1, -1)$ , the integral is  $-\int_1^{-1} y^3 dy = 0$ .
4. From  $(x, y) = (-1, -1) \rightarrow (1, -1)$ , the integral is  $-\int_{-1}^1 x^2 dx = -2/3$ .

Apparently, only sub-paths (2) and (4) contribute to  $\oint \mathbf{F} \cdot d\mathbf{s}$ ; their sum is

$$\oint \mathbf{F} \cdot d\mathbf{s} = -\frac{2}{3} - \frac{2}{3} = -\frac{4}{3}. \quad (1)$$

Meanwhile, according to Stokes's theorem,  $\int (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$ , where

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & xy^3 & 0 \end{vmatrix} = (y^3 - x^2)\mathbf{e}_z. \quad (2)$$

Since  $d\mathbf{A} = \mathbf{e}_z dA$  (which is consistent, according to the right-hand rule, with the counter-clockwise manner in which the line integral was evaluated), the surface integral evaluates to

$$\begin{aligned} \int_{-1}^1 \int_{-1}^1 (y^3 - x^2) dx dy &= \int_{-1}^1 [xy^3 - \frac{1}{3}x^3] \Big|_{x=-1}^{x=1} dy \\ &= \int_{-1}^1 (y^3 - \frac{1}{3} - y^3 - \frac{1}{3}) dy \\ &= -\frac{2}{3}y \Big|_{y=-1}^{y=1} = -\frac{4}{3}, \end{aligned}$$

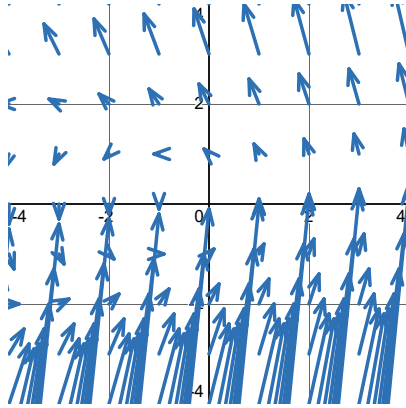
which is consistent with the evaluation of the line integral.

The remaining problems of this section build on one another and should be attempted in order:

23. Calculate the line integral  $\oint \mathbf{G} \cdot d\mathbf{s}$  for the vector field  $\mathbf{G} = -y\mathbf{e}_x + (x + y^2)\mathbf{e}_y$ , where the closed path of integration is a square whose corners are given by the ordered pairs  $(x, y) = (1, 1)$ ,

$(-1, 1)$ ,  $(-1, -1)$ , and  $(1, -1)$ . Then, use Stokes's theorem to show that the same result is obtained by  $\int (\nabla \times \mathbf{G}) \cdot d\mathbf{A}$ . [ANSWER]

The vector field is shown below.



Like in the previous problem, the line integral consists of four sub-paths. This time, the integrals are combined before they are evaluated:

1. From  $(x, y) = (1, -1) \rightarrow (1, 1)$ , the integral is  $\int_{-1}^1 (1 + y^2) dy$ .
2. From  $(x, y) = (1, 1) \rightarrow (-1, 1)$ , the integral is  $\int_{-1}^1 dx$ .
3. From  $(x, y) = (-1, 1) \rightarrow (-1, -1)$ , the integral is  $\int_{-1}^1 (1 - y^2) dy$ , where it has been noted that  $\int_1^{-1} = -\int_{-1}^1$ .
4. From  $(x, y) = (-1, -1) \rightarrow (1, -1)$ , the integral is  $\int_{-1}^1 dx$ .

The integrals add to

$$\oint \mathbf{G} \cdot d\mathbf{s} = 2 \int_{-1}^1 dy + 2 \int_{-1}^1 dx = 8. \quad (3)$$

Meanwhile, according to Stokes's theorem,  $\int (\nabla \times \mathbf{G}) \cdot d\mathbf{A}$ , where

$$\nabla \times \mathbf{G} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x + y^2 & 0 \end{vmatrix} = (1 + 1)\mathbf{e}_z = 2\mathbf{e}_z. \quad (4)$$

Since  $d\mathbf{A} = \mathbf{e}_z dA$ , the surface integral evaluates to

$$2 \int_{-1}^1 \int_{-1}^1 dx dy = 2x|_{-1}^1 y|_{-1}^1 = 2^3 = 8,$$

which is consistent with the evaluation of the line integral.

24. Show in Cartesian coordinates  $(x, y)$  that

$$\int [\nabla \times (G_x \mathbf{e}_x + G_y \mathbf{e}_y)] \cdot \mathbf{e}_z dA = \int [\nabla \cdot (F_x \mathbf{e}_x + F_y \mathbf{e}_y)] dA, \quad (5)$$

where  $F_x = G_y$  and  $F_y = -G_x$ . Then, verify the result by using  $\mathbf{G}$  from the previous problem, i.e.,  $G_x = -y$  and  $G_y = x + y^2$  for the same path of integration. [ANSWER]

The desired identity is easily shown by noting that

$$\nabla \times (G_x \mathbf{e}_x + G_y \mathbf{e}_y) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ G_x & G_y & 0 \end{vmatrix} = \left( \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} \right) \mathbf{e}_z. \quad (6)$$

Thus

$$[\nabla \times (G_x \mathbf{e}_x + G_y \mathbf{e}_y)] \cdot \mathbf{e}_z = \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}.$$

Replacing  $G_y$  with  $F_x$  and  $G_x$  with  $-F_y$  shows that

$$[\nabla \times (G_x \mathbf{e}_x + G_y \mathbf{e}_y)] \cdot \mathbf{e}_z = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \nabla \cdot (F_x \mathbf{e}_x + F_y \mathbf{e}_y), \quad (7)$$

completing the proof.

The result is now verified for  $G_x = -y$  and  $G_y = x + y^2$ , for which  $F_x = x + y^2$  and  $F_y = y$ . The divergence of  $\mathbf{F}$  is

$$\nabla \cdot \mathbf{F} = 1 + 1 = 2. \quad (8)$$

Thus

$$\int \nabla \cdot \mathbf{F} dA = 2 \int_{-1}^1 \int_{-1}^1 dx dy = 8, \quad (9)$$

recovering the result of the previous problem.

25. The mapping  $F_x = G_y$  and  $F_y = -G_x$  and can be written as a *rotation* of the vector field  $\mathbf{F}$ . The rotation in Cartesian coordinates reads

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} G_x \\ G_y \end{pmatrix}. \quad (10)$$

In words, the equation above says that  $\mathbf{F}$  equals a [clockwise rotation](#) by  $90^\circ$  of  $\mathbf{G}$ . Given that  $\oint \mathbf{G} \cdot d\mathbf{s} = \int (\nabla \times \mathbf{G}) \cdot d\mathbf{A}$  (Stokes's theorem) and the identity proved in the previous problem [written symbolically as  $\int (\nabla \times \mathbf{G}) \cdot d\mathbf{A} = \int \nabla \cdot \mathbf{F} dA$ ], show that

$$\int \nabla \cdot \mathbf{F} dA = \oint \mathbf{F} \cdot \mathbf{e}_n ds,$$

where the *unit normal vector* to the boundary is  $\mathbf{e}_n$ , which equals  $R \cdot ds$ , where  $R$  is the rotation matrix for the clockwise rotation. Note that the rotation matrix is orthogonal, i.e.,  $R^{-1} = R^T$ .

[ANSWER]

In terms of  $R$ , the equation of rotation is

$$\mathbf{F} = R \cdot \mathbf{G}. \quad (11)$$

Invoking the fact that  $R$  is an orthogonal matrix allows the equation of rotation to be written as  $\mathbf{G} = R^T \cdot \mathbf{F}$ , insertion of which into Stokes's theorem yields

$$\oint (R^T \cdot \mathbf{F}) \cdot d\mathbf{s} = \int (\nabla \times \mathbf{G}) \cdot d\mathbf{A} = \int \nabla \cdot \mathbf{F} dA, \quad (12)$$

where the second equality holds from the previous problem. By the definition of the transpose operation,  $(R^T \cdot \mathbf{F}) \cdot d\mathbf{s} = \mathbf{F} \cdot (R \cdot d\mathbf{s})$ . Noting that  $R \cdot d\mathbf{s} = \mathbf{e}_n ds$  yields the desired equality,

$$\oint \mathbf{F} \cdot \mathbf{e}_n ds = \int \nabla \cdot \mathbf{F} dA. \quad (13)$$

The above result is surprisingly hard to find in books, but it can be found on [this](#) Wikipedia page (where it is discussed slightly differently).

The above exercise shows that for two-dimensional vector fields, the divergence theorem for the vector field  $\mathbf{F}$  is equivalent to Stokes's theorem for the vector field  $\mathbf{G}$ , where  $\mathbf{F} = R \cdot \mathbf{G}$ . See [this discussion](#) for more.

26. Calculate  $\oint \mathbf{F} \cdot \mathbf{e}_n ds$ , where  $\mathbf{F} = (x + y^2)\mathbf{e}_x + y\mathbf{e}_y$ , and where the domain of integration is the same as in problems 23 and 24. Check to see that the result is equivalent to that of two problem prior, i.e.,  $\oint \mathbf{F} \cdot \mathbf{e}_n ds = \int \nabla \cdot \mathbf{F} dA = 8$ . *[ANSWER]*

The integral  $\oint \mathbf{F} \cdot \mathbf{e}_n ds$  consists of four contributions. For  $F_x = x + y^2$  and  $F_y = y$ , the contributions are

1.  $\int_{-1}^1 F_x|_{x=1} dy = \int_{-1}^1 (1 + y^2) dy = 2 + \frac{2}{3}$
2.  $\int_{-1}^1 F_y|_{y=1} dx = \int_{-1}^1 dx = 2$
3.  $-\int_{-1}^1 F_x|_{x=-1} dy = -\int_{-1}^1 (-1 + y^2) dy = 2 - \frac{2}{3}$
4.  $-\int_{-1}^1 F_y|_{y=-1} dx = \int_{-1}^1 dx = 2$

The minus sign arise for contribution (3) because  $\mathbf{e}_x \cdot \mathbf{e}_n = -1$ . Similarly, for contribution (4),  $\mathbf{e}_y \cdot \mathbf{e}_n = -1$ .

Adding up the contributions yields  $\oint \mathbf{F} \cdot \mathbf{e}_n ds = 2 + 2 + 2 + 2 = 8$ , matching the results of problems 23 and 24.

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# Partial differential equations

1. Provide the name of each of the following partial differential equations. Also list a few physical phenomena described by each:

$$\nabla^2 u = 0 \quad (\text{i})$$

$$\nabla^2 u = f(\mathbf{r}) \quad (\text{ii})$$

$$\nabla^2 u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t} \quad (\text{iii})$$

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} \quad (\text{iv})$$

$$\nabla^2 F + k^2 F = 0 \quad (\text{v})$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi \quad (\text{vi})$$

Which two equations above both reduce to equation (v) upon assuming time-harmonic solutions? *[ANSWER]*

Equation (i) is the Laplace equation, which describes the electric potential in a space that does not contain charge, the gravitational potential in a space that does not contain mass, and the velocity potential in an incompressible fluid in a space that does not contain vortices, sources, and sinks, to name a few examples.

Equation (ii) is the Poisson equation, which describes the electric potential in a space that contains charge, the gravitational potential in a space that contains mass, and the velocity potential in an incompressible fluid in a space that contains sources and sinks.

Equation (iii) is the diffusion equation, which describes the temperature as a function of time in a space with no heat sources, or the concentration of a diffusing substance.

Equation (iv) is the wave equation, which is the most beautiful PDE, describing light, sound, water waves, and gravitational waves, to name a few.

Equation (v) is the Helmholtz equation, which is the spatial part of both the wave equation and the diffusion equation.

Equation (vi) is the Schrodinger equation, which describes nonrelativistic quanta. The paraxial wave equation also has the form of the Schrodinger equation, only with the temporal derivative in equation (vi) replaced with a spatial derivative ( $\partial/\partial z$ , for example), and with the Laplacian in equation (vi) replaced with the transverse Laplacian  $\partial^2/\partial x^2 + \partial^2/\partial y^2$ , for example).

2. Solve Laplace's equation in 2D  $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$  where

$$\begin{aligned}
V &= 0 \quad \text{for } y = 0 \\
V &= 0 \quad \text{for } y = a \\
V &= V_0 \quad \text{for } x = -b \\
V &= V_0 \quad \text{for } x = b.
\end{aligned}$$

are the boundary conditions. [ANSWER]

Separating variables gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

Introducing a separation constant gives

$$\frac{X''}{X} = k^2 \quad (\text{i})$$

$$\frac{Y''}{Y} = -k^2 \quad (\text{ii})$$

The sign in front of  $k^2$  determines variable is harmonic and which is exponential. Since the boundary condition on  $Y$  goes to 0, it is chosen to be the harmonic solution. The boundary condition on  $X$  is finite, and thus it chosen to be exponential solution (because exponential solutions cannot satisfy a 0 boundary condition, because exponentials  $\neq 0$ ). Equation (i) gives

$$X = Ae^{kx} + Be^{-kx},$$

and equation (ii) gives

$$Y = C \cos ky + D \sin ky.$$

Thus the general solution is

$$V = XY = (Ae^{kx} + Be^{-kx})(C \cos ky + D \sin ky)$$

Applying the boundary condition  $V = 0$  for  $y = 0$  sets  $C = 0$ . And applying the boundary condition  $V = 0$  for  $y = L$  sets  $kL = n\pi$ , where  $n = 1, 2, 3 \dots$ . Meanwhile, applying the boundary condition  $V = V_0$  at both  $x = -b$  and  $x = b$  requires that  $A = B$ . The solution is therefore

$$V = XY = \sum_{n=1}^{\infty} C_n \cosh(n\pi x/L) \sin(n\pi y/L)$$

where  $\cosh x = (e^x + e^{-x})/2$  has been used. To find the expansion coefficients  $C_n$ , employ the final (or third) boundary condition,  $V = V_0$  at  $x = b$  and invoke the orthogonality of sines is used:

$$\begin{aligned}
V_0 &= \sum_{n=1}^{\infty} C_n \cosh(n\pi b/L) \sin(n\pi y/L) \\
V_0 \sin(m\pi y/L) &= \sum_{n=1}^{\infty} C_n \cosh(n\pi b/L) \sin(n\pi y/L) \sin(m\pi y/L) \\
V_0 \int_0^L \sin(m\pi y/L) dy &= \sum_{n=1}^{\infty} C_n \cosh(n\pi b/L) \int_0^L \sin(n\pi y/L) \sin(m\pi y/L) dy \\
-V_0 \frac{L}{m\pi} \cos(m\pi y/L) \Big|_0^L &= \sum_{n=1}^{\infty} C_n \cosh(n\pi b/L) \frac{L}{2} \delta_{nm} \\
C_n &= -V_0 \frac{1}{\cosh(n\pi b/L)} \frac{2}{n\pi} \left[ \cos n\pi - 1 \right] \\
&= \begin{cases} 0 & n = 0, 2, 4 \dots \\ \frac{4V_0}{n\pi \cosh(n\pi b/L)} & n = 1, 3, 5 \dots \end{cases}
\end{aligned}$$

The solution is thus fully determined.

3. Solve the 2D Laplace equation  $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$  where now

$$\begin{aligned}
V &= 0 \quad \text{for } y = 0 \\
V &= f(x) \quad \text{for } y = H \\
\frac{\partial V}{\partial x} &= 0 \quad \text{for } x = 0 \\
\frac{\partial V}{\partial x} &= 0 \quad \text{for } x = L
\end{aligned}$$

are the boundary conditions. Then solve the problem for  $f(x) = V_0 x/L$ . [ANSWER]

Separating variables leads to

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

Introduce the separation constant  $k^2$ . Now one has to choose whether to set  $k^2$  equal to  $\frac{X''}{X}$  or  $\frac{Y''}{Y}$ . The one that equals  $k^2$  will have decay/growth solutions, and the other one (that equals  $-k^2$ ) will have harmonic solutions. Since  $X'$  goes to 0 at both boundaries,  $\frac{X''}{X} = -k^2$ , and thus  $\frac{Y''}{Y} = k^2$ . This is a good rule of thumb: the solutions that are zero at two boundaries must be harmonic. This choice gives the general solution

$$V = XY = (A \cos kx + B \sin kx)(Ce^{ky} + De^{-ky}).$$

Now the boundary conditions are invoked. Since  $\frac{\partial V}{\partial x} = 0$  for  $x = 0$ ,  $B = 0$ , and since  $\frac{\partial V}{\partial x} = 0$  for  $x = L$ ,  $\sin kL = 0$ , which means that  $kL = n\pi$ , where  $n = 0, 1, 2, \dots$ . Meanwhile, since  $V = 0$  at  $y = 0$ ,  $C = -D$ . Thus the general solution is

$$V(x, y) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}.$$

Now the final boundary condition at  $y = H$  is applied, which allows for the determination of  $A_n$  using the orthogonality of cosines:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} \\
\int_0^L f(x) \cos \frac{m\pi x}{L} dx &= \sum_{n=0}^{\infty} \sinh \frac{n\pi H}{L} A_n \int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx \\
\Rightarrow A_n &= \frac{2}{L \sinh(n\pi H/L)} \int_0^L f(x) \cos \frac{n\pi x}{L} dx
\end{aligned}$$

Therefore, the coefficients  $A_n$  above, along with the general solution for  $V(x, y)$ , solves the Laplace equation for the given boundary conditions.

The coefficients are calculated for  $f(x) = V_0 x/L$ :

$$\begin{aligned}
A_n &= \frac{V_0}{L} \frac{2}{L \sinh(n\pi H/L)} \int_0^L x \cos \frac{n\pi x}{L} dx \\
&= \begin{cases} -\frac{4V_0}{n^2 \pi^2 \sinh(n\pi H/L)} & \text{for } n = 1, 3, 5 \dots \\ 0 & \text{for } n = 0, 2, 4 \dots \end{cases}
\end{aligned}$$

4. Solve the 1D diffusion equation  $\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$  where

$$\begin{aligned}
u &= 0 \quad \text{for } x = -L \\
u &= 0 \quad \text{for } x = L \\
u &= \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \text{at } t = 0
\end{aligned}$$

are the boundary and initial conditions. [ANSWER]

Pick and orthogonal set of basis functions,  $u = X(x)T(t)$ , giving

$$\nu^2 \frac{X''}{X} = \frac{T'}{T}$$

Next, introduce the separation constant  $-k$  and set it equal to  $X''/X$ , because harmonic spatial solutions are expected. This gives

$$\begin{aligned}
X'' + \frac{k^2}{\nu^2} X &= 0 \\
\Rightarrow X &= A \cos k\nu x + B \sin k\nu x
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
T' &= -k^2 T \\
\Rightarrow T &= e^{-k^2 t}
\end{aligned}$$

Thus

$$XT = (A \cos \kappa \nu x + B \sin \kappa \nu x) e^{-k^2 t} .$$

Applying the first to BCs gives  $A \cos(-k\nu L) + B \sin(-k\nu L) = 0$  and  $A \cos(k\nu L) + B \sin(k\nu L) = 0$ . The odd and even properties of cosine and sine can be used in the first equation to obtain a full-rank linear system of equations:

$$A \cos(k\nu L) - B \sin(k\nu L) = 0$$

$$A \cos(k\nu L) + B \sin(k\nu L) = 0$$

Adding the two equations gives the condition  $k = (2n - 1)\pi/2\nu L$ , and subtracting the two equations gives the condition  $k = \pi n/\nu L$ , where  $n = 1, 2, 3 \dots$ . Thus the general solution is in terms of two infinite series:

$$u = \sum_{n=1}^{\infty} \left\{ A_n \cos \frac{(2n-1)\pi x}{2L} \exp \left[ -[2n-1]\pi/2 t \right] + B_n \sin \frac{n\pi x}{L} \exp \left[ (-n\pi/L)^2 t \right] \right\}$$

What a terrible looking equation. Anyhow, denoting the Heaviside step function as  $H(x)$ , the initial condition is invoked:

$$H(x) = \sum_{n=1}^{\infty} \left\{ A_n \cos \frac{(2n-1)\pi x}{2L} + B_n \sin \frac{n\pi x}{L} \right\}$$

Orthogonality can be used to find  $A_n$  and  $B_n$ :

$$\int_{-L}^L H(x) \cos \frac{(2n-1)\pi x}{2L} dx = \sum_{n=1}^{\infty} A_n \int_{-L}^L \cos \frac{(2n-1)\pi x}{2L} \cos \frac{(2m-1)\pi x}{2L} dx$$

$$\int_0^L \cos \frac{(2m-1)\pi x}{2L} dx = LA_m$$

$$A_n = \frac{2}{\pi(2n-1)} \sin \frac{(2n-1)\pi x}{2L} \Big|_{x=0}^{x=L}$$

$$= \frac{2}{\pi(2n-1)} \sin \frac{(2n-1)\pi}{2}$$

$$= \frac{2(-1)^{n-1}}{\pi(2n-1)},$$

and

$$\int_{-L}^L H(x) \sin \frac{n\pi x}{L} dx = \sum_{n=1}^{\infty} B_n \int_{-L}^L \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L} dx$$

$$\int_0^L \sin \frac{n\pi x}{L} dx = LB_n$$

$$B_n = -\frac{1}{n\pi} \cos \frac{n\pi x}{L} \Big|_{x=0}^{x=L}$$

$$= -\frac{1}{n\pi} [\cos(n\pi) - 1] = -\frac{1}{n\pi} [(-1)^n - 1]$$

$$= \begin{cases} 0, & n = 0, 2, 4 \dots \\ \frac{2}{n\pi}, & n = 1, 3, 5 \dots \end{cases}$$

Inserting the calculated values for  $A_n$  and  $B_n$  into the general solution above gives the complete solution.

Note, however, that it is easier to shift the original coordinates by  $L$ , i.e.,  $x + L = x'$ , so that the boundary conditions are

$$\begin{aligned}
u &= 0 \quad \text{for } x' = 0 \\
u &= 0 \quad \text{for } x' = 2 \\
u &= \begin{cases} 1, & x' \geq L \\ 0, & x' < 2L \end{cases} \quad \text{at } t = 0.
\end{aligned}$$

Then one must simply remember to shift the coordinates back, i.e.,  $x = x' - L$ , when presenting the final solution.

5. Solve the Schrodinger equation for a potential that is 0 inside a cube of side length  $l$ , and infinite at the boundaries. Discuss the degeneracy of modes. [ANSWER]

Let  $\Psi = XYZT$ . Separating variables leads to

$$-\frac{\hbar^2}{2m} \left( \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} \right) = i\hbar \frac{T'}{T}$$

The time dependence is easily solved, introducing the separation constant  $E$ :

$$\begin{aligned}
i\hbar \frac{T'}{T} &= E \\
\Rightarrow T &= \exp(-iEt/\hbar)
\end{aligned}$$

Meanwhile the spatial dependence is solved by writing

$$\begin{aligned}
\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} &= -\frac{2mE}{\hbar^2} \\
\frac{X''}{X} &= -k_x^2 \\
\frac{Y''}{Y} &= -k_y^2 \\
\frac{Z''}{Z} &= -k_z^2.
\end{aligned}$$

Since the boundary condition is 0 at  $X, Y, Z = 0$  and  $L$ , one throws out the cosine solutions and obtains

$$k_x = \frac{n\pi}{L}, \quad k_y = \frac{m\pi}{L}, \quad k_z = \frac{l\pi}{L}.$$

Thus the eigenenergies are

$$E_n = \frac{\hbar^2 \pi^2}{2mL^2} \sqrt{n^2 + m^2 + l^2}$$

There is no nice expression for the degeneracy in terms of the indices.

6. List the eigenfunctions that solve the Helmholtz equation in Cartesian coordinates, cylindrical coordinates, and spherical coordinates. Given this, which equations in problem (1) of this section are solved spatially? What are the time dependences for each of these equations? [ANSWER]

The eigenfunctions that solve the Helmholtz equation in Cartesian coordinates are exponentials of (almost always) imaginary arguments (i.e., sines and cosines), which describe spatially harmonic waves. However, note that exponentials of real arguments are also eigenfunctions of the Helmholtz equation in Cartesian coordinates. These correspond to decay and (less often) growth (evanescent waves). This is an important distinction between the Helmholtz equation and its paraxial approximation in acoustics and optics: the paraxial approximation does not contain evanescent waves.

The eigenfunctions that solve the Helmholtz equation in cylindrical coordinates consist of radial, polar, and axial functions. The radial eigenfunctions are Bessel and Neumann functions  $J_n$  and  $N_n$ . The polar eigenfunctions are Legendre polynomials  $P_n$ . The axial eigenfunctions are spatial harmonics (sines and cosines).

The eigenfunctions that solve the Helmholtz equation in spherical coordinates consist of radial, polar, and azimuthal functions. The radial eigenfunctions are spherical Bessel and spherical Neumann functions  $j_n$  and  $n_n$ . The polar eigenfunctions are the associated Legendre functions  $P_n^m$ . The azimuthal eigenfunctions are spatial harmonics (sines and cosines).

Given these eigenfunctions, the spatial part of equations (iii) (the diffusion equation), (iv) (the wave equation) are solved. Also, if  $V$  in equation (vi) (the Schrodinger equation) is 0 in a box, sphere, or cylinder and  $\infty$  at the boundaries, then equation (vi) reduces to equation (iii), and thus shares the same eigenfunctions as the Helmholtz equation. For example, a quantum particle in infinite spherical well has the same eigenfunctions as sound in a spherical enclosure, or temperature in a hollow sphere.

Further, it is interesting to think of the *time* eigenfunctions of the diffusion equation, the wave equation, and the Schrodinger equation. All share the same *spatial* eigenfunctions, but the *time* eigenfunctions for the wave equation are harmonic, while those for the diffusion equation are exponential decay/growth. Meanwhile, what are the *time* eigenfunctions for the Schrodinger equation (regardless of  $V$ ? They would be exponential decay/growth if it were not for the  $i$  in the Schrodinger equation, which makes them *harmonic*. Thus the Schrodinger equation is *physically* more like a wave equation than a diffusion equation, but is *mathematically* more like a diffusion equation than a wave equation.

On this point, one should note that the Schrodinger equation and heat diffusion equations do not obey time-reversal symmetry, while the wave equation does.

7. ✪ One solution to the 1D wave equation  $\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0$  is

$$p = \begin{Bmatrix} e^{ax} \\ e^{-ax} \end{Bmatrix} \begin{Bmatrix} e^{act} \\ e^{-act} \end{Bmatrix} \quad (\text{a})$$

What a bizarre-looking solution! If you do not believe it is a solution to the wave equation, you can check it for yourself by setting the separation constant equal to  $X''/X = a^2$ , which gives  $T''/T = (\alpha c)^2$ ! *Is this solution a wave?* Meanwhile, a solution to the 1D diffusion equation  $\frac{\partial^2 p}{\partial x^2} - \kappa^2 \frac{\partial p}{\partial t} = 0$  is

$$p = \begin{Bmatrix} \cos kx \\ \sin kx \end{Bmatrix} e^{-(k/\kappa)^2 t} = \begin{Bmatrix} e^{ikx} \\ e^{-ikx} \end{Bmatrix} e^{-(k/\kappa)^2 t} \quad (\text{b})$$

Is *this* solution a wave? [ANSWER]

Equation a describes solutions that grow exponentially in space and time. At first glance, this is no wave! We are not used to such bizarre behaviours, i.e., waves that grow and decay exponentially in space and time.

However, a wave need not be harmonic to be a wave (which can be proved by the closure of the Fourier series). A second glance at equation (a) shows that it is in fact of the form  $f(x \pm ct)$ , which we know solves the wave equation. Physically, equation (a) is disturbance that propagates at a finite speed  $c$  whose waveform is exponential. Thus is equation (a) indeed describes wave motion.

Meanwhile, equation (b) is not a solution to the wave equation because it cannot be written in the form  $f(x \pm ct)$ . Spatially, the solutions are harmonic (wave-like), but that is not a sufficient criterion for a physical wave.

8. List the eigenfunctions that solve the Laplace equation in Cartesian coordinates, cylindrical coordinates, and spherical coordinates. [ANSWER]

The eigenfunctions of that solve the Laplace equation in Cartesian coordinates are equivalent those that solve the Helmholtz equation in Cartesian, i.e., exponentials of real and imaginary arguments.

Two of the three eigenfunctions that solve the Laplace equation in cylindrical coordinates are also identical to those that solve the Helmholtz equation in Cartesian coordinates: Bessel and Neumann functions in  $r$  and sines and cosines in  $\theta$ . However, the eigenfunctions of the Laplace equation for  $z$  are exponentials of real arguments, while those of the Helmholtz equation are exponentials of imaginary arguments.

Similarly, two of the three eigenfunctions that solve the Laplace equation in spherical coordinates are also equivalent to those that solve the Helmholtz equation in Spherical coordinates: Legendre polynomials in  $\cos \theta$  and sines and cosines in  $\phi$  (Blackstock's  $\psi$ ). However, the eigenfunctions of the Laplace equation for  $r$  are  $r^l$  and  $r^{-(l+1)}$ , while those of the Helmholtz equation are spherical Bessel and Neumann functions  $j_n$  and  $n_n$ .

9. Solve the Laplace equation  $\nabla^2 V = 0$  (assuming polar symmetry) where  $V = V_0(\theta)$  on the boundary of a sphere of radius  $a$ , where

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

is Laplace's equation in spherical coordinates with no azimuthal dependence. Does the coordinate  $z = \cos \theta$  actually correspond to the  $z$  axis? [ANSWER]

Let  $V = R(r)\Theta(\theta)$  and substitute into the above:

$$\frac{\partial}{\partial r}(r^2 R' \Theta) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}(\sin \theta R \Theta') = 0.$$

Divide by  $R\Theta$ :

$$\frac{1}{R} \frac{\partial}{\partial r}(r^2 R') + \frac{1}{\Theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}(\sin \theta \Theta') = 0.$$

The variables have been separated in the two terms above. A separation constant  $l(l+1)$  is introduced. This is cheating, but it is done because we anticipate the solution to the polar equation to be Legendre polynomials, which converge when the separation constant is chosen:

$$\begin{aligned} \frac{1}{\Theta} \frac{1}{\sin \theta} \frac{d}{d\theta}(\sin \theta \Theta') &= -l(l+1) \\ \frac{d}{d\theta}(\sin \theta \Theta') &= -l(l+1) \sin \theta \Theta \end{aligned}$$

That is a bizarre-looking equation may look more familiar if a change of variable  $z = \cos \theta$  is made. *This relation holds only for a unit sphere, so no, in reality  $z$  does not correspond to the Cartesian  $z$  coordinate.* Then,  $\frac{dz}{d\theta} = -\sin \theta$ , and the derivatives with respect to  $\theta$  are

$$\begin{aligned} \frac{d}{d\theta} &= \frac{d}{dz} \frac{dz}{d\theta} = -\sin \theta \frac{d}{dz} \\ \frac{d^2}{d\theta^2} &= \sin^2 \theta \frac{d^2}{dz^2} \end{aligned}$$

The differential equation becomes

$$-\sin \theta \frac{d}{dz} \left( \sin^2 \theta \frac{d\Theta}{dz} \right) = -l(l+1) \sin \theta \Theta$$

Dividing the equation above by  $-\sin \theta$  and writing  $\sin^2 \theta = 1 - \cos^2 \theta$  gives

$$\frac{d}{dz} \left( [1 - z^2] \frac{d\Theta}{dz} \right) = l(l+1) \Theta.$$

Expanding the derivatives and writing everything on the left-hand side gives

$$(1 - z^2) \frac{d^2 \Theta}{dz^2} - 2z \frac{d\Theta}{dz} + l(l+1) \Theta = 0$$

This is the canonical form of Legendre's equation. See the first question of the ODE section above for the solution. Meanwhile, the radial equation is

$$\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = Rl(l+1)$$

The general solution to this equation is (just memorize it):

$$R(r) = A_l r^l + \frac{B_l}{r^{l+1}}$$

Thus the general solution to Laplace's equation in spherical coordinates with no azimuthal dependence is

$$V = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

To satisfy the boundary condition that  $V = V_0(\theta)$  at  $r = a$ , first the  $B_l = 0$  because those terms diverge at the origin. Then set

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos \theta) = V_0(\theta).$$

The coefficient  $A_l$  can be solved using the orthogonality of cosines:

$$A_l = \frac{2l+1}{2a^l} \int_0^\pi V_0(\theta) P_l(\cos \theta) \sin \theta d\theta.$$

You can probably guess that solution to Laplace's equation (with no symmetry assumed) is

$$V = \sum_{m=0}^{m=l} \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l^m(\cos \theta) (C_m \cos m\phi + D_m \sin m\phi).$$

where  $\phi$  is the azimuthal angle.

10. What is the solution to Poisson's equation  $\nabla^2 u = f(\mathbf{r})$  in terms of the appropriate Green's function,  $G(\mathbf{r}|\mathbf{r}')$ ? *[ANSWER]*

The solution is

$$u(\mathbf{r}) = \int_V G(\mathbf{r}|\mathbf{r}') f(\mathbf{r}') dV'$$

where  $G(\mathbf{r}|\mathbf{r}')$  is the sum of the free space Green's function and the solution to Laplace's equation.

11. Solve Laplace's equation  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$  for a semi-infinite plate satisfying the following boundary conditions:

$$\begin{aligned} u(x, \infty) &= 0 \\ u(0, y) &= 0 \\ u(x, 0) &= \begin{cases} T_0 & \text{for } x \in (0, a] \\ 0 & \text{for } x > 1 \end{cases} \end{aligned}$$

Note that

$$\begin{cases} f(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty g(k) \sin kx dk \\ g(k) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx dx \end{cases}$$

is the sine Fourier transform pair. *[ANSWER]*

Write the solution  $u$  as the product of basis functions  $XY$ . The PDE becomes

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Note that the eigenfunctions  $Y$  must vanish at  $y = \infty$ . Anticipating exponential solutions for  $Y$ , set  $\frac{Y''}{Y} = k^2$ . Thus  $Y \propto e^{-ky}$ , where the growth solutions have been tossed because they are unphysical. Meanwhile,  $\frac{X''}{X} = -k^2$ , so  $X = A \cos kx + B \sin kx$ . Since  $X(0) = 0$ , the  $\cos kx$  term cannot be included, so  $A = 0$ . This leaves the general solution

$$u(x, y) = B e^{-ky} \sin kx.$$

Now the boundary condition at  $y = 0$  needs to be used to find the coefficient  $B$ . However,  $u$  at the boundary changes value from  $T_0$  to 0 at  $x = a$ . Thus a continuous variable is needed to satisfy the boundary condition (rather than a discrete variable):

$$u(x, y) = \int_0^{\infty} B(k) e^{-ky} \sin(kx) dk.$$

At  $y = 0$ , the solution is

$$u(x, 0) = \int_0^{\infty} B(k) \sin(kx) dk. \quad (i)$$

Now, finding the coefficients  $B(k)$  amounts to taking an inverse Fourier transform. Note that Fourier sine transform pair is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(k) \sin kx dk$$

$$g(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin kx dx.$$

The pair is written so that the first equation above has the form of equation (i):

$$u(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left[ \sqrt{\frac{\pi}{2}} B(k) \right] \sin(kx) dk$$

$$B(k) = \frac{2}{\pi} \int_0^{\infty} u(x) \sin(kx) dx.$$

The second equation above is the equation for the expansion coefficients, upon letting  $u(x)$  be  $T_0$  for  $x \leq a$  and 0 otherwise:

$$B(k) = \frac{2T_0}{\pi} \int_0^1 \sin kx dx$$

$$= \frac{2T_0}{\pi k} (1 - \cos k).$$

Thus the integral solution of the PDE is

$$u(x, y) = \frac{2T_0}{\pi} \int_0^{\infty} \frac{1 - \cos k}{k} \sin(kx) e^{-ky} dk.$$

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