Insert $\phi = A(x_3)\, e^{j(\omega t - kx_1)}$ into $\nabla^2 \phi - \frac{1}{c_L^2}\ddot{\phi} = 0.$

Note that $\nabla^2\phi = \frac{\partial^2}{\partial x_1^2} A(x_3)\, e^{j(\omega t - kx_1)} + \frac{\partial^2}{\partial x_3^2} A(x_3)\, e^{j(\omega t - kx_1)}$

$= -k^2 A(x_3) e^{j(\omega t - kx_1)} + A''(x_3)\, e^{j(\omega t - kx_1)}.$

And note that $\ddot{\phi} = -\omega^2 \phi$. Thus the wave equation become

$-k^2 A(x_3) + A''(x_3) + \frac{\omega^2}{c_L^2} A(x_3) = 0.$

$$\boxed{A''(x_3) + \left(\frac{\omega^2}{c_L^2} - k^2\right) A(x_3) = 0}$$

Identify $\frac{\omega^2}{k^2} = c_R^2$.

$0 = A''(x_3) + k^2\left[\frac{c_R^2}{c_L^2} - 1\right] A(x_3)$

Similarly inserting $\psi = B(x_3)\, e^{j(\omega t - kx_1)}$ into $\nabla^2\psi - \frac{1}{c_T^2}\ddot{\psi} = 0.$ gives

$$\boxed{B''(x_3) + \left(\frac{\omega^2}{c_T^2} - k^2\right) B(x_3) = 0}$$

$0 = B''(x_3) + k^2\left[\frac{c_R^2}{c_T^2} - 1\right] B(x_3).$

The solutions of these ODEs are harmonic but we flip sign of $\sqrt{\ }$: $A(x_3) = A\, e^{\pm kq x_3}$, where $q = \sqrt{1 - c_R^2/c_L^2}$

$B(x_3) = B\, e^{\pm ks x_3}$, where $s = \sqrt{1 - c_R^2/c_T^2}$.

to get exp. decay and growth solutions.

Thus the general solution for the potential functions is

$\phi(\vec{x}) = A\, e^{\pm kq x_3}\, e^{j(\omega t - kx_1)}.$

$\psi(\vec{x}) = B\, e^{\pm ks x_3}\, e^{j(\omega t - kx_1)}.$

But we pick only the exponential decay solutions because exp growth is not physical here.