X_k = current position

$\mathbb{X}_k$ = reference position

motion: $x_k = \hat{x}_k(\mathbb{X}_m, t)$

Inverse motion: $\mathbb{X}_k = \mathbb{X}_k(x_m, t)$.

$u_k = x_k - \mathbb{X}_k$ = displacement

Lagrangian displacement: (mat'l)

$$u_k = \hat{x}_k(\mathbb{X}_M, t) - \mathbb{X}_k. = \hat{u}_k(\mathbb{X}_m, t).$$

Lagrangian velocity

$$v_k = \frac{\partial}{\partial t}\hat{u}_k(\mathbb{X}_m, t) = \hat{v}_k(\mathbb{X}_m, t). \qquad (1).$$

Lagrangian acceleration

$$a_k = \frac{\partial}{\partial t}\hat{v}_k(\mathbb{X}_m, t) = \hat{a}_k(\mathbb{X}_m, t).$$

Eulerian displacement (spatial).

$$u_k = x_k - \mathbb{X}_k(x_m, t) = u_k(x_m, t).$$

Eulerian velocity.

$$= u_k(\hat{x}_m(\mathbb{X}_m, t), t).$$

$$v_k = \frac{\partial}{\partial t}u_k(x_m, t) = \frac{\partial u_k}{\partial t} + \frac{\partial u_k}{\partial x_m}\frac{\partial \hat{x}_m}{\partial t}$$

But by equation (1) $\frac{\partial \hat{x}_m}{\partial t} = v_m$: Thus,

$$v_k = \frac{\partial u_k}{\partial t} + \frac{\partial u_k}{\partial x_m} v_m \qquad \text{etc.}$$

$$= \frac{\partial u_k}{\partial t} + \frac{\partial u_k}{\partial x_m}\left(\frac{\partial u_m}{\partial t} + \frac{\partial u_k}{\partial x_m} v_m\right)$$

Similarly, Eulerian acceleration:

$$a_k = \frac{\partial v_k(x_m, t)}{\partial t} = \frac{\partial v_k}{\partial t} + \frac{\partial v_k}{\partial x_m}\frac{\partial \hat{x}_m}{\partial t} \quad \text{by (1).}$$

$$= \frac{\partial v_k}{\partial t} + \frac{\partial v_k}{\partial x_m} v_m. \quad \text{similar regression.}$$

Linear elasticity: no difference between Lagrangian & Eulerian equations for disp, velocity, acceleration.