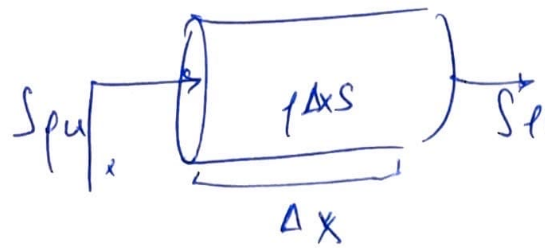
Derivation of exact nonlinear acoustic wave eq. in 1D.



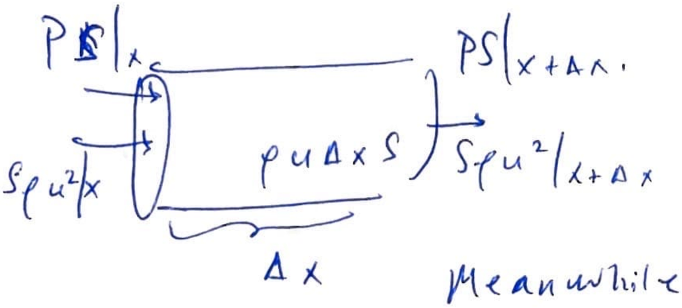
$$S\rho u|_x - S\rho u|_{x+\Delta x} = \frac{\partial}{\partial t} \rho \Delta x S$$

is the statement of mass cons.

$$\lim_{\Delta x \to 0} \frac{\rho u|_x - \rho u|_{x+\Delta x}}{\Delta x} = \frac{\partial \rho}{\partial t}$$

$$-\frac{\partial \rho u}{\partial x} = \frac{\partial \rho}{\partial t}$$

Continuity eq: (1) $\boxed{\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0.}$



State equation. $P = P(\rho)$.
Take $dP/d\rho = c^2$ and thus
$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial \rho}\frac{\partial \rho}{\partial x} = \frac{dP}{d\rho}\frac{\partial \rho}{\partial x} = c^2 \frac{\partial \rho}{\partial x}$.

Meanwhile, $PS|_x - PS|_{x+\Delta x} + S\rho u^2|_x - S\rho u^2|_{x+\Delta x}$
$= \frac{\partial}{\partial t}(\rho u \Delta x S)$.

In lim $\Delta x \to 0$.

$$-\frac{\partial P}{\partial x} - \frac{\partial \rho u^2}{\partial x} = \frac{\partial \rho u}{\partial t}.$$

$$0 = u\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial \rho u^2}{\partial x}.$$

$$0 = u\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} + \rho\frac{\partial u^2}{\partial x} + u^2\frac{\partial \rho}{\partial x}.$$

$$= \underbrace{u[-u\, \partial \rho/\partial x}_{\text{cancel}} \pm \rho u\, \partial u/\partial x] + \rho\frac{\partial u}{\partial t} + \underbrace{\frac{\partial P}{\partial x}}_{c^2 \frac{\partial \rho}{\partial x}\text{ (state).}} + \rho\frac{\partial u^2}{\partial x} + \underbrace{u^2\frac{\partial \rho}{\partial x}}_{\text{cancel}}$$

$$0 = -\rho u\frac{\partial u}{\partial x} + \rho\frac{\partial u}{\partial t} + c^2\frac{\partial \rho}{\partial x} + \rho\frac{\partial u^2}{\partial x} *$$

$$0 = -\rho u\frac{\partial u}{\partial x} + 2\rho u\frac{\partial u}{\partial x} + \rho\frac{\partial u}{\partial t} + c^2\, \partial \rho/\partial x$$

Thus $$\boxed{0 = + \cdot \rho u \frac{\partial u}{\partial x} + \rho \frac{\partial u}{\partial t} + c^2 \, \partial \rho / \partial x.} \quad \text{----}(2)$$

is the exact momentum equation.

On the next two pages, the equation of exact nonlinear wave motion is derived (lossless fluid).

Then, the result is specialized to an adiabatic gas. Let us derive the adiabatic gas law from the 1st Law of thermodynamics, & the equipartition thm.

$$\Delta U = Q + W. = W \quad \text{by def. of adiabatic.}$$

$$U = \frac{f}{2} NkT \quad \leftarrow \text{equipartition thm.}$$

f = deg. of freedom.

$$dU = \frac{f}{2} NkdT.$$

Setting this equal to the work done (by the gas), gives

$$\frac{f}{2} NkdT = -PdV. = -\frac{NkT}{V} dV.$$

Integrate

$$\frac{f}{2} \cancel{Nk} \frac{dT}{T} = -\cancel{Nk} \frac{dV}{V}$$

$$\frac{f}{2} \ln (T/T_0) = -\ln (V/V_0).$$

$$\cancel{\frac{f}{2} T/T_0 = + V_0/V.}$$

$$\ln (T/T_0)^{f/2} = \ln V_0/V.$$

$$VT^{f/2} = V_0 T_0^{f/2}$$

Now substitute in $PV = NkT$ to eliminate T. ie, $\left(\frac{PV}{Nk}\right)^{f/2} = T^{f/2}$

$$(PV)^{f/2}\, V = V_0 (P_0 V_0)^{f/2}$$

$$P^{f/2}\, V^{\frac{f}{2}}\, V = V_0 P_0^{f/2} V_0^{f/2}$$

$$P^{f/2}\, V^{(f+2)/2} = V_0^{\frac{f+2}{2}} P_0^{f/2}.$$

Raise both sides to power $2/f$.

$$P V^{\frac{f+2}{2}} = P_0 V_0^{\frac{f+2}{2}}$$

$$P V^{\gamma} = P_0 V_0^{\gamma}$$

But $\rho \propto 1/V$. Thus

$$P/P_0 = (\rho/\rho_0)^{\gamma}.$$

Now on to the derivation

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + c^2 \frac{\partial \rho}{\partial x} = 0 \quad \text{exact momentum} \qquad (2)$$

in notes

If $\rho = \rho(u)$, then the derivatives of ρ become

$$\boxed{\begin{aligned} \frac{\partial \rho}{\partial t} &= \frac{d\rho}{du} \frac{\partial u}{\partial t} \quad \text{and} \\ \frac{\partial \rho}{\partial x} &= \frac{d\rho}{du} \frac{\partial u}{\partial x}. \end{aligned}}$$

The continuity equation is (exactly).

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0. \qquad (1)$$

on notes.

(1) & (2) become

$$\begin{cases} \dfrac{d\rho}{du} \dfrac{\partial u}{\partial t} + u \dfrac{d\rho}{du} \dfrac{\partial u}{\partial x} + \rho \dfrac{\partial u}{\partial x} = 0. \\ \rho \dfrac{\partial u}{\partial t} + \rho u \dfrac{\partial u}{\partial x} + c^2 \dfrac{d\rho}{du} \dfrac{\partial u}{\partial x} = 0. \end{cases}$$

& group by derivatives of u.

$$\begin{cases} \dfrac{d\rho}{du} \dfrac{\partial u}{\partial t} + \left[u \dfrac{d\rho}{du} + \rho\right] \dfrac{\partial u}{\partial x} = 0 \\ \rho \dfrac{\partial u}{\partial t} + \left[\rho u + c^2 \dfrac{d\rho}{du}\right] \dfrac{\partial u}{\partial x} = 0. \end{cases}$$

In matrix form;

$$\underbrace{\begin{bmatrix} d\rho/du & u \dfrac{d\rho}{du} + \rho \\ \rho & \rho u + c^2 d\rho/du \end{bmatrix}}_{\det \to 0.} \begin{bmatrix} \partial u/\partial t \\ \partial u/\partial x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{d\rho}{du}\left[\rho u + c^2 \frac{d\rho}{du}\right] - \rho u \frac{d\rho}{du} + -\rho^2 = 0. \checkmark$$

$$\cancel{\frac{d\rho}{du}\rho u} + \left(\frac{d\rho}{du}\right)^2 c^2 - \cancel{\rho u \frac{d\rho}{du}} - \rho^2 = 0.$$

$$\left(\frac{d\rho}{du}\right)^2 c^2 = \rho^2.$$

Thus $\boxed{\pm \rho/c = \frac{d\rho}{du}}$. (3)

Thus, by the momentum eq. (2).

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + c^2 \frac{\partial \rho}{\partial x} = 0 \quad \text{becomes}$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + c^2 \left(\frac{d\rho}{du}\right) \frac{\partial u}{\partial x} = 0.$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} \pm c^2 \frac{\rho}{c} \partial u/\partial x = 0.$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} \pm \rho c \frac{\partial u}{\partial x} = 0.$$

$$\rho \frac{\partial u}{\partial t} + (\rho u \pm \rho c) \frac{\partial u}{\partial x} = 0.$$

$$\boxed{\frac{\partial u}{\partial t} + (u \pm c) \frac{\partial u}{\partial x} = 0.}$$ Poisson, 1808.

For isentropic gas, $c^2 = \frac{\partial P}{\partial \rho} = \frac{P_0 \gamma}{\rho_0} \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1}$

$(P = P_0 (\rho/\rho_0)^\gamma)$

$$c^2 = c_0^2 (\rho/\rho_0)^{\gamma - 1}$$

$$c = c_0 (\rho/\rho_0)^{\frac{\gamma-1}{2}}.$$

$$\frac{dc}{d\rho} = \frac{c_0}{\rho_0} \frac{\gamma-1}{2} \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-3}{2}}$$

$$= \frac{c_0}{\cancel{\rho_0}} \left(\frac{\gamma-1}{2}\right) \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} \left(\frac{\rho}{\cancel{\rho_0}}\right)^{-1} = \frac{\gamma-1}{2} c/\rho.$$

Therefore, therefore.

$$\frac{dc}{d\rho} = \frac{\gamma-1}{2} \frac{c}{\rho}.$$

Meanwhile, from (3)

$$du = \frac{c}{\rho} d\rho. = \frac{c}{\rho} \frac{d\rho}{dc} dc$$

$$= \frac{c}{\rho} \frac{\rho}{c} \frac{2}{\gamma-1} dc$$

$$du = \frac{2\,dc}{\gamma-1}.$$

Integrate this relation; viz., $\int_0^u du' = \int_{c_0}^{c} \frac{2\,dc}{\gamma-1}$

$$u = \frac{2(c-c_0)}{\gamma-1}.$$

Thus $\frac{u(\gamma-1)}{2} = c - c_0$

$$c_0 + \frac{\gamma-1}{2} u = c$$

Put it back into eq of Poisson, 1808:

$$\frac{\partial u}{\partial t} + \left[u \pm \left(c_0 + \frac{\gamma-1}{2} u\right)\right] \frac{\partial u}{\partial x} = 0.$$

$$\frac{\partial u}{\partial t} + \left[c_0 + \frac{\gamma+1}{2} u\right] \frac{\partial u}{\partial x} = 0.$$

$$\boxed{\frac{\partial u}{\partial t} + (c_0 + \beta u) \frac{\partial u}{\partial x} = 0}$$

exact nonlinear progressive wave eq. for isentropic gas.