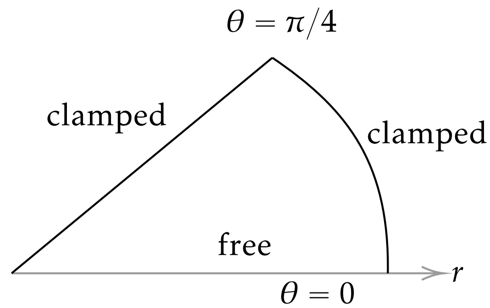
Part 1. Solve the 2D wave equation in $\eta(r,\theta)$ for a pie-shaped drumhead subtending 45°, where one edge, say $\theta = 0$, is free, while the curved boundary at $r = a$ and the other edge at $\theta = 45^\circ$ are clamped. Find the eigenfrequencies and identify the lowest one.



Start with the general solution:

$$\eta = J_m(kr)\begin{Bmatrix}\cos m\theta \\ \sin m\theta\end{Bmatrix}\begin{Bmatrix}\cos \omega t \\ \sin \omega t\end{Bmatrix}$$

Applying the boundary condition at $\theta = 0$ gives

$$\left.\frac{\partial \eta}{\partial \theta}\right|_{\theta=0} = 0$$

$$-A_m m \sin 0 + B_m m \cos 0 = 0 \quad \Longrightarrow \quad B_m = 0$$

Applying the boundary condition at $\theta = \pi/4$ gives

$$A_m \cos(m\pi/4) = 0 \quad \Longrightarrow \quad m\pi/4 = \frac{\pi}{2}(2l-1),\, l = 1, 2, 3, \ldots$$

or $m = 4l - 2 = 2, 6, 10, \ldots$. Meanwhile, applying the radial boundary results in

$$J_m(ka) = 0 \quad \Longrightarrow \quad k_{mn} = \alpha_{mn}/a$$

Thus the solution is

$$\boxed{\eta(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{m=2,6,10\ldots}^{\infty} J_m(\alpha_{mn} r/a)\cos(m\theta)(C_{mn}\cos\omega_{mn}t + D_{mn}\sin\omega_{mn}t),} \qquad \text{(i)}$$

and the eigenfrequencies are

$$\boxed{f_{mn} = \frac{c_0 \alpha_{mn}}{2\pi a}, \quad m = 2, 6, 10, \ldots.}$$

The lowest eigenfrequency corresponds to $m = 2$ and $n = 1$:

$$f_{21} = \frac{5.136 c_0}{2\pi a}.$$

Part 2. Suppose there is an initial displacement on the membrane of η_0, which would correspond to a "plucked" initial condition. Also assume there is no initial velocity:

$$\eta(r,\theta,0) = \begin{cases} \eta_0, & \theta = [0, \pi/4), \, r = [0,a] \\ 0, & \theta = [\pi/4, 2\pi) \end{cases}$$

$$\dot{\eta}(r,\theta,0) = 0$$

Apply the initial condition to the eigenfunctions given by equation (i) to find C_{mn} and D_{mn}. Leave the relevant expansion coefficients in integral form, and evaluate the integral for $m = 0$ only. Compare the coefficients for $m = 0$ to Blackstock's equation (11.B-13) on page 400 for a circular drumhead clamped at $r = a$ subject to the analogous "plucked" initial condition.

Taking the derivative of equation (i) and setting it equal to 0 at $t = 0$ (the second initial condition above) shows that $D_{mn} = 0$. Meanwhile, setting $t = 0$ to equation (i) and applying the initial displacement boundary condition gives

$$\sum_{n=1}^{\infty} \sum_{m=2,6,10...}^{\infty} C_{mn} J_m(\alpha_{mn} r/a) \cos(m\theta) = \begin{cases} \eta_0, & \theta = [0, \pi/4), \, r = [0,a] \\ 0, & \theta = [\pi/4, 2\pi) \end{cases}.$$

The coefficient C_{mn} is found by using two orthogonality relations: the cosine orthogonality relation for $m, q > 1$,

$$\int_0^{\pi} \cos m\theta \cos q\theta \, d\theta = \frac{\pi}{2}\delta_{mq},$$

and the Bessel orthogonality,

$$\int_0^a J_m(\alpha_{mn} r/a) J_m(\alpha_{mn'} r/a) r dr = \frac{a^2}{2}[J_m'(\alpha_{mn})]^2 \delta_{nn'}.$$

They can be applied in any order. Here the cosine orthogonality is applied first by multiplying both sides by $\cos q\theta \, d\theta$ and integrating from $\theta = 0$ to $\theta = \pi$:

$$\sum_{n=1}^{\infty} \sum_{m=2,6,10...}^{\infty} C_{mn} J_m(\alpha_{mn} r/a) \int_0^{\pi} \cos(m\theta)\cos(q\theta) d\theta = \eta_0 \int_0^{\pi/4} \cos(q\theta) d\theta$$

$$\sum_{n=1}^{\infty} C_{mn} J_m(\alpha_{mn} r/a) = \frac{2}{\pi}\eta_0 \sin m\theta \Big|_0^{\pi/4}$$

$$\sum_{n=1}^{\infty} C_{mn} J_m(\alpha_{mn} r/a) = \frac{2}{\pi}\eta_0 (-1)^m, \quad m = 2, 6, 10, \ldots.$$

The equation above is now multiplied by $J_m(\alpha_{mn'} r/a) r dr$ and integrated from $r = 0$ to $r = a$:

$$\sum_{n=1}^{\infty} C_{mn} \int_0^a J_m(\alpha_{mn} r/a) J_m(\alpha_{mn'} r/a) r dr = \frac{2}{\pi}\eta_0 (-1)^m \int_0^a J_m(\alpha_{mn} r/a) r dr$$

The Bessel orthogonality relation is applied, leading to an integral equation for the expansion coefficients,

$$C_{mn} = \frac{4\eta_0(-1)^m}{\pi a^2[J'_m(\alpha_{mn})]^2}\int_0^a J_m(\alpha_{mn}r/a)rdr \tag{1}$$

I think the integral above can be taken analytically for only the $m = 0$ case, in which case

$$\begin{aligned} C_{0n} &= \frac{4\eta_0}{\pi\alpha_{0n}}\frac{J_1(\alpha_{0n})}{[J'_0(\alpha_{0n})]^2} \\ &= \frac{4\eta_0}{\pi\alpha_{0n}J_1(\alpha_{0n})} \\ &= \frac{2}{\pi}\frac{2\eta_0}{\alpha_{0n}J_1(\alpha_{0n})} \end{aligned}$$

The quantity

$$\frac{2\eta_0}{\alpha_{0n}J_1(\alpha_{0n})}$$

is Blackstock's expansion coefficient for the plucked circular drumhead. Apparently, wedge's modes are decreased by a factor of $2/\pi \simeq 0.64$ for the $m = 0$ family of modes.