IntelliChoice SAT Math Camp

Complex Numbers

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1 Introduction to Complex Numbers

You are most familiar with *real numbers*. There is another type of number: *imaginary numbers*. Imaginary numbers are the square roots of negative numbers.

Some examples of real numbers include $7, -1, \pi$, and $\sqrt{2}$

Some examples of imaginary numbers include $\sqrt{-5}, \sqrt{-1}, \sqrt{-\pi}$, and $\sqrt{-2}$.

Mathematicians like to define $i \equiv \sqrt{-1}$. This is convenient because we can factor out $\sqrt{-1}$ from any imaginary number. For example, $\sqrt{-5} = \sqrt{-1}\sqrt{5} = i\sqrt{5}$. This is mainly done because "i" is easier to write than "$\sqrt{-1}$."

1. Something interesting happens when you square i. Try it below:

$$i^2 = (\sqrt{-1})^2 =$$

Apparently, there is a pathway between imaginary and real numbers.

2.

$$i^3 = (i^2)i =$$

Hint: Use the what you found out in the previous question.

3.

$$i^4 = (i^2)^2 =$$

4.

$$i^5 = (i^4)i =$$

The pattern repeats. $i^5 = i$, $i^6 = i^2$, $i^7 = i^3$, $i^8 = i^4$,... So if you know i^2, i^3, and i^4, you know i^n, where n is any integer! The SAT people like to ask questions like:

5.

$$i^{37} =$$

Just decompose this into $(i^4)^4 i$

6.

$$i^{102} =$$

7.

$$i^{59} =$$

8.

$$i^{81} =$$

9.

$$i^{-107} =$$

Remember $\alpha^{-1} \equiv \frac{1}{\alpha}$

10.

$$i^{512} =$$

11.

$$i^{-10001} =$$

This all might seem made-up to you now; you may question the utility of these imaginary numbers. Their real utility will become apparent once you study calculus. There is a beautiful equation that connects arguably the four most important numbers in all of math: e (Euler's number), π, -1, and i:

$$e^{i\pi} = -1$$

In a few years, I'm sure you will feel that this statement is trivial.

2 Rationalizing the Denominator

Both real and imaginary numbers belong under the umbrella of *complex numbers*, which have the form $\alpha + i\beta$. α is the real part, and β is the imaginary part. Mathematicians like to call $\alpha + i\beta = z$

Suppose you want to express a real number. Then you would let α be that real number while letting $\beta = 0$. Suppose you want to express an imaginary number. Then you would let $\alpha = 0$ and β be that imaginary number.

Things become interesting when there are both real and imaginary components to z. The only question the SAT will ask involving these cases of $\alpha \neq 0$, $\beta \neq 0$ will be simplifying a complex number in the form $\frac{1}{\alpha+i\beta}$. The way to do it is by multiplying by $\frac{\alpha-i\beta}{\alpha-i\beta}$, which is equivalent to multiplying by "1."

This is technique is called "rationalizing the denominator." It works every time on the SAT.

Simplify the following:

12.

$$\frac{3}{2+2i}$$

Hint: Multiply numerator and denominator by $2 - 2i$.

13.

$$\frac{-6}{4+8i}$$

14.

$$\frac{-\pi}{\pi-9i}$$

15.

$$\frac{-10}{100-10i}$$

16.

$$\frac{-\sin 30^\circ}{2\sin 60^\circ - 9i}$$

17.
$$\frac{-\sin 60^\circ}{2-\cos 30^\circ i}$$

18.
$$\frac{\tan(\pi/4)}{\sin\pi - 9i}$$

19.
$$\frac{1-2i}{2-1i}$$

20.
$$\frac{10-2i}{2-100i}$$

3 A Useful Analogy

Just as we don't like complex numbers in the denominator, we usually don't like square roots in denominators, either. For example, mathematicians usually find it irritating to write something in the form of

$$\frac{1}{\sqrt{\beta}}$$

The method to get the square root out of the denominator is similar to what we did in section (2) of this worksheet. We multiply by $\frac{\sqrt{\beta}}{\sqrt{\beta}}$, another fancy form of the number 1.

If you have something in the form of

$$\frac{1}{\alpha + \sqrt{\beta}}$$

then multiply by $\frac{\alpha - \sqrt{\beta}}{\alpha - \sqrt{\beta}}$, another fancy form of 1.

21.

$$\frac{4}{\sqrt{2}}$$

22.

$$\frac{1 - \sqrt{5}}{\sqrt{5}}$$

23.

$$\frac{\pi}{4 + \sqrt{17}}$$

24.

$$\frac{\pi + \pi^2}{\pi^4 - \sqrt{\pi^3}}$$