Supplemental Notes

Contents

1 Derivation of the HK integral

The HK integral is the solution $p(\boldsymbol{r})$ to the inhomogenous Helmholtz equation,

$$\nabla^2 p(\boldsymbol{r}) + k^2 p(\boldsymbol{r}) = -f(\boldsymbol{r}), \tag{1}$$

where the inhomogenity is described by the distribution function $f(\boldsymbol{r})$. To derive the HK integral, first recall that the free space Green's function $g(\boldsymbol{r}|\boldsymbol{r}_0)$ solves

$$\nabla^2 g(\boldsymbol{r}|\boldsymbol{r}_0) + k^2 g(\boldsymbol{r}|\boldsymbol{r}_0) = -\delta(\boldsymbol{r} - \boldsymbol{r}_0). \tag{2}$$

Also suppose $\chi(\boldsymbol{r})$ is solution to the homogeneous Helmholtz equation:

$$\nabla^2 \chi + k^2 \chi = 0. \tag{3}$$

Since the general solution is the sum of the inhomogeneous and homogeneous solutions, $G(\boldsymbol{r}_0|\boldsymbol{r}) = g(\boldsymbol{r}_0|\boldsymbol{r}) + \chi(\boldsymbol{r})$ is the general solution to equation (2). That is,

$$\nabla^2 G(\boldsymbol{r}|\boldsymbol{r}_0) + k^2 G(\boldsymbol{r}|\boldsymbol{r}_0) = -\delta(\boldsymbol{r} - \boldsymbol{r}_0). \tag{4}$$

Next, equation (4) is multiplied by $p(\boldsymbol{r})$ and subtracted from the product of G and equation (1):

$$G(\boldsymbol{r}|\boldsymbol{r}_0)\nabla^2 p(\boldsymbol{r}) - p(\boldsymbol{r})\nabla^2 G(\boldsymbol{r}|\boldsymbol{r}_0) = -f(\boldsymbol{r})G(\boldsymbol{r}|\boldsymbol{r}_0) + p(\boldsymbol{r})\delta(\boldsymbol{r} - \boldsymbol{r}_0) \tag{5}$$

Now switching the location of the source from $\boldsymbol{r}_0$ to $\boldsymbol{r}$, $f(\boldsymbol{r}_0) \mapsto f(\boldsymbol{r})$, so equation (5) becomes

$$G(\boldsymbol{r}|\boldsymbol{r}_0)\nabla^2 p(\boldsymbol{r}) - p(\boldsymbol{r})\nabla^2 G(\boldsymbol{r}|\boldsymbol{r}_0) = -f(\boldsymbol{r}_0)G(\boldsymbol{r}|\boldsymbol{r}_0) + p(\boldsymbol{r})\delta(\boldsymbol{r} - \boldsymbol{r}_0) \tag{6}$$

Further, since G satisfies reciprocity, $G(\boldsymbol{r}|\boldsymbol{r}_0) = G(\boldsymbol{r}_0|\boldsymbol{r})$. Recall also that $\delta(\boldsymbol{r} - \boldsymbol{r}_0) = \delta(\boldsymbol{r}_0 - \boldsymbol{r})$. Making these transformations to equation (6) yields

$$G(\boldsymbol{r}_0|\boldsymbol{r})\nabla^2 p(\boldsymbol{r}_0) - p(\boldsymbol{r}_0)\nabla^2 G(\boldsymbol{r}_0|\boldsymbol{r}) = -f(\boldsymbol{r}_0)G(\boldsymbol{r}_0|\boldsymbol{r}) + p(\boldsymbol{r}_0)\delta(\boldsymbol{r}_0 - \boldsymbol{r}) \tag{7}$$

Integrating (7) in the '$_0$' coordinates,

$$\iiint \{G(\boldsymbol{r}_0|\boldsymbol{r})\nabla^2 p(\boldsymbol{r}_0) - p(\boldsymbol{r}_0)\nabla^2 G(\boldsymbol{r}_0|\boldsymbol{r})\} dv_0 = \\ \iiint \{ - f(\boldsymbol{r}_0)G(\boldsymbol{r}_0|\boldsymbol{r}) + p(\boldsymbol{r}_0)\delta(\boldsymbol{r}_0 - \boldsymbol{r})\} dv_0$$

Applying the sifting property of the delta function on the right-hand-side, writing $G(\boldsymbol{r}_0|\boldsymbol{r})\nabla^2 p(\boldsymbol{r}_0) - p(\boldsymbol{r}_0)\nabla^2 G(\boldsymbol{r}_0|\boldsymbol{r}) = \boldsymbol{\nabla}_0 \cdot (G(\boldsymbol{r}_0|\boldsymbol{r})\boldsymbol{\nabla} p(\boldsymbol{r}_0) - p(\boldsymbol{r}_0)\boldsymbol{\nabla} G(\boldsymbol{r}_0|\boldsymbol{r}))$, and solving for $p(\boldsymbol{r}_0)$,

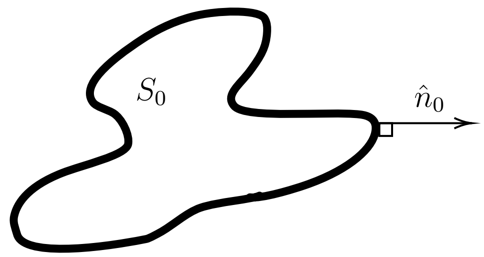

Figure 1: Closed surface S_0 subjected to a source condition at the surface and containing no sources in the enclosed volume.

$$p(\boldsymbol{r}) = \iiint f(\boldsymbol{r}_0)G(\boldsymbol{r}_0|\boldsymbol{r})dv_0 + \iiint \boldsymbol{\nabla}_0 \cdot \{G(\boldsymbol{r}_0|\boldsymbol{r})\boldsymbol{\nabla} p(\boldsymbol{r}_0) - p(\boldsymbol{r}_0)\boldsymbol{\nabla} G(\boldsymbol{r}_0|\boldsymbol{r})\}dv_0$$

Utilizing the divergence theorem on the left-hand-side, writing the gradients as $\frac{\partial}{\partial n_0}$, and utilizing $G(\boldsymbol{r}_0|\boldsymbol{r}) = G(\boldsymbol{r}|\boldsymbol{r}_0)$ (mainly to match Dr. Hamilton's notes),

$$p(\boldsymbol{r}) = \iiint f(\boldsymbol{r}_0)G(\boldsymbol{r}_0|\boldsymbol{r})dv_0 + \oiint \{G(\boldsymbol{r}|\boldsymbol{r}_0)\tfrac{\partial}{\partial n_0}p(\boldsymbol{r}_0) - p(\boldsymbol{r}_0)\tfrac{\partial}{\partial n_0}G(\boldsymbol{r}|\boldsymbol{r}_0)\}dS_0 \quad \text{(HK integral)}$$

This is the Helmholtz-Kirchoff integral, which is used in section (2) to derive the Rayleigh integral.

2 Derivation of the first Rayleigh integral

Consider a closed surface S_0 that is subjected to a velocity source condition on the boundary and contains no sources within the enclosed volume, as illustrated in figure (1). Since there are no sources in the enclosed volume, the volume integral term of the (HK integral) vanishes, leaving

$$p(\boldsymbol{r}) = \oiint \{G(\boldsymbol{r}|\boldsymbol{r}_0)\tfrac{\partial}{\partial n_0}p(\boldsymbol{r}) - \underline{p(\boldsymbol{r})\tfrac{\partial}{\partial n_0}G(\boldsymbol{r}|\boldsymbol{r}_0)}\}dS_0 \tag{8}$$

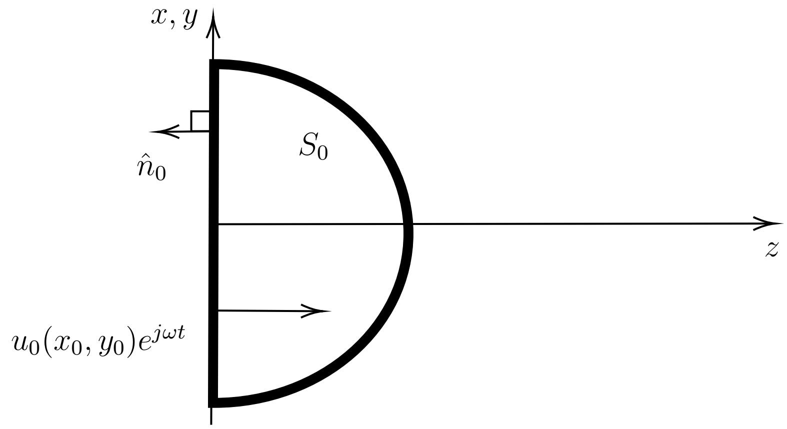

Figure 2: Specialization of figure (1) to a vibrating velocity source at $z = 0$. Note that the unit normal vector $\hat{n}_0$ points in the opposite direction as the z-axis. The arc of the hemisphere is sufficiently far away such that the pressure goes to 0 there. Therefore, the surface at $z = 0$ is the only contribution to the integral. *See Sommerfeld radiation condition*

The boundary of interest is a rigid plane at $z = 0$, portions of which vibrate in the z-direction, as shown in figure (2). A velocity source condition is defined:

$$u(x, y, z = 0, t) = u_0(x_0, y_0)e^{-i\omega t} \tag{9}$$

Equation (9) can be incorporated into the second factor of equation (8) using the momentum equation, $\frac{\partial p(\boldsymbol{r}_0)}{\partial n_0} = -\frac{\partial p(x_0,y_0)}{\partial z_0} = -\rho_0 \frac{\partial u(x,y,z=0,t)}{\partial t}$, which simplifies to

$$\begin{aligned} \frac{\partial p(x_0,y_0)}{\partial z_0} &= \rho_0 \frac{\partial u_0(x_0,y_0)e^{-i\omega t}}{\partial t} \\ &= -i\omega\rho_0 u_0(x_0, y_0) \qquad \text{(second factor)} \end{aligned}$$

Next, it is desirable to choose a Green's function that makes the underlined term in equation (8) vanish (i.e., $\frac{\partial G}{\partial n_0} = 0$ on S_0). Denoting

$$g_{\pm}(\boldsymbol{r}|\boldsymbol{r}_0) = \frac{e^{ikR_{\pm}}}{4\pi R_{\pm}},$$

where

$$R_{\pm} = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z \mp z_0)^2},$$

one such choice is

$$G(\boldsymbol{r}|\boldsymbol{r}_0) = g_+(\boldsymbol{r}|\boldsymbol{r}_0) + g_-(\boldsymbol{r}|\boldsymbol{r}_0) \tag{10}$$

Upon evaluating this Green's function at the location of the source, $z_0 = 0$ the first factor of equation (8) becomes

$$\begin{aligned} G(\boldsymbol{r}|\boldsymbol{r}_0)\Big|_{z_0=0} &= 2g(\boldsymbol{r}|\boldsymbol{r}_0) \\ &= \frac{e^{ikR}}{4\pi R} + \frac{e^{ikR}}{4\pi R} \\ &= \frac{e^{ikR}}{2\pi R} \end{aligned} \tag{first factor}$$

where $R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}$. Substituting the (second factor) and the (first factor) into equation (8) gives

$$\begin{aligned} p(\boldsymbol{r}) &= \oiint (-i\omega\rho_0 u_0(x_0, y_0)) \frac{e^{ikR}}{2\pi R} \, \mathrm{d}S_0 \\ &= -i\frac{\rho_0 c_0 k}{2\pi} \oiint \frac{u_0(x_0, y_0)e^{ikR}}{R} \, \mathrm{d}S_0 \end{aligned} \tag{first Rayleigh integral}$$

3 Derivation of the second Rayleigh integral

Consider a closed surface S_0 that is subjected to a pressure source condition on the boundary and contains no sources within the enclosed volume, as illustrated in figure (3). Since there are no sources in the enclosed volume, the volume integral term of the (HK integral) vanishes,

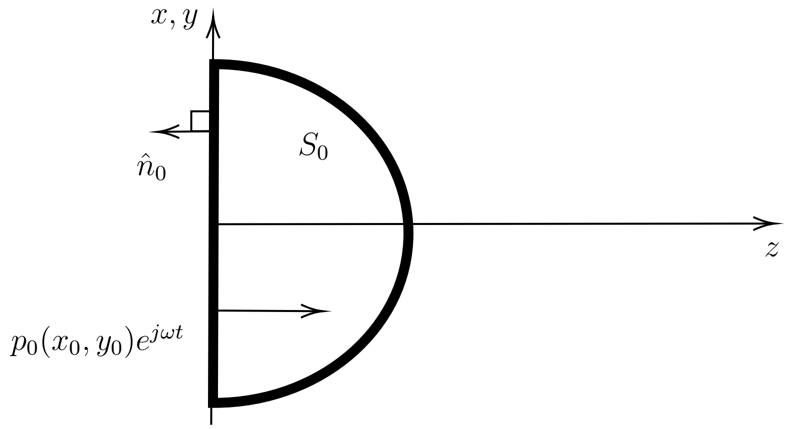

Figure 3: Specialization of figure (1) to a vibrating rigid pressure source at $z = 0$.

leaving

$$p(\boldsymbol{r}) = \oiint \left\{ \underline{G(\boldsymbol{r}|\boldsymbol{r}_0)\frac{\partial}{\partial n_0}p(\boldsymbol{r})} - p(\boldsymbol{r})\frac{\partial}{\partial n_0}G(\boldsymbol{r}|\boldsymbol{r}_0) \right\} dS_0 \tag{11}$$

The boundary of interest is a pressure source at $z = 0$ that vibrates in the z-direction. A pressure source condition is defined:

$$p(x, y, z = 0, t) = p_0(x_0, y_0)e^{-i\omega t} \tag{12}$$

Equation (12) is incorporated into the first factor of equation (11) by simply suppressing the $e^{-i\omega t}$ time dependence.

Next, it is desirable to choose a Green's function that makes the underlined term in equation (11) vanish (i.e., $G = 0$ on S_0). Denoting $g_{\pm}(\boldsymbol{r}|\boldsymbol{r}_0)$ as in section (2), one such choice is

$$G(\boldsymbol{r}|\boldsymbol{r}_0) = g_+(\boldsymbol{r}|\boldsymbol{r}_0) - g_-(\boldsymbol{r}|\boldsymbol{r}_0) \tag{13}$$

Upon evaluating this choice of the Green's function at $z_0 = 0$, the location of the source, $G(\boldsymbol{r}|\boldsymbol{r}_0)$ in the underlined term vanishes as desired:

$$G(\boldsymbol{r}|\boldsymbol{r}_0)\Big|_{z_0=0} = \frac{e^{ikR}}{4\pi R} - \frac{e^{ikR}}{4\pi R} = 0$$

where $R = \sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}$ as in section (2).

Meanwhile, the second factor of equation (8) becomes

$$\begin{aligned}
\frac{\partial}{\partial n_0} G(\boldsymbol{r}|\boldsymbol{r}_0)\Big|_{z_0=0} &= -\frac{1}{4\pi}\frac{\partial}{\partial z_0}\left[\frac{e^{ik\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right. \\
&\qquad \left.- \frac{e^{ik\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right]\Bigg|_{z_0=0} \\
&= \frac{1}{2\pi}\left(ike^{ikR} z R^{-2} + e^{ikR} z R^{-3}\right) \\
&= -\frac{z}{2\pi}\frac{e^{ikR}}{R}\left(-\frac{ik}{R} + \frac{1}{R^2}\right) && \text{(second factor)}
\end{aligned}$$

where again $R = \sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}$. Substituting the (second factor) and the (first factor) into equation (8) gives

$$\begin{aligned}
p(x,y,z) &= \oiint p_0(x_0,y_0)\frac{z}{2\pi}\frac{e^{ikR}}{R}\left(-\frac{ik}{R} + \frac{1}{R^2}\right) \mathrm{d}S_0 \\
&= \frac{z}{2\pi}\oiint p_0(x_0,y_0)\left(-\frac{ik}{R} + \frac{1}{R^2}\right)\frac{e^{ikR}}{R} \mathrm{d}S_0 && \text{(second Rayleigh integral)}
\end{aligned}$$

4 Accounting for the focusing

Figure (4) shows the distances of consideration when accounting for the curvature of the transducer. Since the transducer is described by a section of a spherical surface centered at point $z_\mathrm{T} = F$, all the points on its surface satisfy

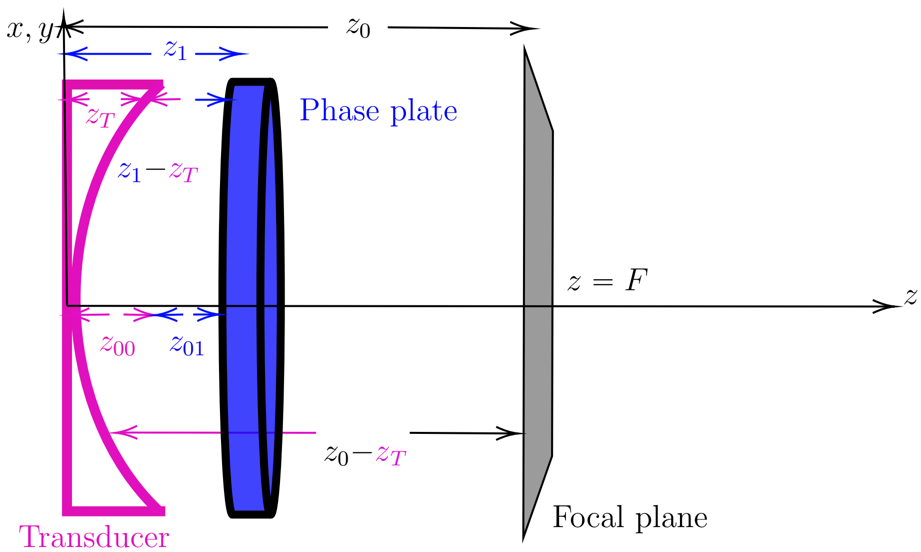

Figure 4: The more detailed schematic above shows the distances of consideration when accounting for the curvature of the transducer.

$$x_{\mathrm{T}}^2 + y_{\mathrm{T}}^2 + (z_{\mathrm{T}} - F)^2 = F^2$$

Solving the above for z_{T},

$$z_{\mathrm{T}} = \frac{2F - \sqrt{4F^2 - 4(x_{\mathrm{T}}^2 + y_{\mathrm{T}}^2)}}{2}$$

The distance from a point on the transducer $(x_{\mathrm{T}}, y_{\mathrm{T}}, z_{\mathrm{T}})$ to a point on the phase plate (x_1, y_1, z_1) is therefore given by $R_{\mathrm{T}} = \sqrt{(x_1 - x_{\mathrm{T}})^2 + (y_1 - y_{\mathrm{T}})^2 + (z_1 - z_{\mathrm{T}})^2}$, where z_{T} is as defined above.

5 Sanity check

To validate the focusing modeled in section (4), the phase plate is removed from the model, leaving a focused circular piston and a focal plane. The Fresnel approximation allows for an analytical solution to this problem. Evaluated at the focal plane $z = F$, the complex pressure solution is [1]

$$p(\sigma, F) = -ik\rho_0 c_0 \frac{e^{ikF}}{F} e^{ik\sigma^2/2F} \left(\frac{a^2 u_0}{2} \frac{2J_1(ka\sigma/F)}{ka\sigma/F} \right) \tag{14}$$

where $\sigma = \sqrt{x_0^2 + y_0^2}$ and where the term in parentheses is the Hankel transform of the source function $u_0(\sigma)$. For ease of comparison, a one-dimensional result is desired. Setting $y_0 = 0$, equation (14) becomes

$$p(x_0, 0, F) = -ik\rho_0 c_0 \frac{e^{ikF}}{F} e^{ikx_0^2/2F} \left(\frac{a^2 u_0}{2} \frac{2J_1(kax_0/F)}{kax_0/F} \right) \tag{15}$$

The magnitude and phase of the analytical solution above is compared to the magnitude and phase of the numerical solution given by the first Rayleigh integral:

$$p(x_0, 0, F) = -i\frac{\rho_0 c_0 k}{2\pi} \oiint_S \frac{V e^{ikR}}{R} \, \mathrm{d}S \tag{16}$$

where $R = \sqrt{(x_0 - x_\mathrm{T})^2 + (y_0 - y_\mathrm{T})^2 + (z_0 - z_\mathrm{T})^2}$. The comparison of equations (15) and (16) is discussed in figure (5)

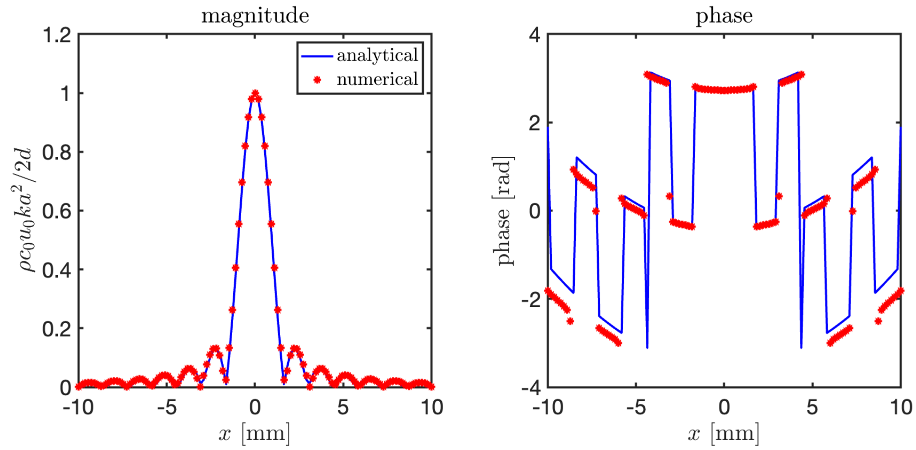

Figure 5: The plots above compare the analytical and numerical pressure magnitude (left) and phase (right). To facilitate the comparison, one-dimensional results are used by setting $y_0 = 0$. The x-axis from -10 mm to 10 mm is plotted above, but the analysis applies to the two-dimensional results because the solution is symmetric about the z-axis. There is strong agreement between the numerically and analytically found pressure magnitude, showing that the model presented in section (4) is valid. The numerically found phase represents the exact solution; the analytically found phase accumulates error farther from the origin because it is the result of the Fresnel approximation.

6 Fourier acoustics

To solve the problem of an arbitrary pressure or velocity source vibrating in the z-direction, the source's surface can be treated as an infinite number of point sources, and the pressure due to the surface is the integral of their contributions over the surface. This approach, developed in sections (1)-(3), amounts to computing n double summations m times, where n is the number of points chosen in the source plane and m is the number of points chosen in the observation plane. The computation time is therefore quadratic in the number the points chosen. The results presented in section (2) of the project report required ~ 1.75 hours to run.

Fourier acoustics offers an exact numerical solution that is computationally much more efficient than the approach followed in section (2) of the report. First start with the definition of the 2D spatial Fourier transform:

$$\hat{f}(k_x, k_y) = F\{f(x,y)\} = \iint_{-\infty}^{\infty} f(x,y) e^{j(k_x x + k_y y)}\, \mathrm{d}x\, \mathrm{d}y \qquad \text{(spatial Fourier transform)}$$

with the inverse Transform defined as:

$$f(x,y) = F^{-1}\{\hat{f}(k_x, k_y)\} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \hat{f}(k_x, k_y) e^{-j(k_x x + k_y y)}\, \mathrm{d}k_x\, \mathrm{d}k_y$$

(inverse spatial Fourier transform)

Now consider a pressure wave propagating along the z-axis. The spatial part can be described in complex-exponential form,

$$\hat{p} = C_1 e^{jk_z z} + C_2 e^{-jk_z z}$$

the Fourier transform of which is

$$\hat{p}(k_x, k_y, z) = \hat{p}_0(k_x, k_y) e^{-jk_z z}$$

where

$$\hat{p}_0(k_x, k_y) = \hat{p}(k_x, k_y, 0) = F\{p(x, y, 0)\}$$

In order to propagate the pressure field forward, a factor of $e^{-jk_z z}$ is included in the spatial Fourier transform of the pressure field. Thus

$$p(x,y,z) = F^{-1}\{\hat{p}_0(k_x, k_y) e^{-jk_z z}\} = F^{-1}\{F\{p(x,y,0)\} e^{-jk_z z}\} \qquad \text{(Free-space propagation)}$$

The model set up as follows. The first Fourier transform is performed to propagate pressure produced by the the velocity source towards the phase plate:

$$p(x, y, z) = \rho_0 c_0 F^{-1}\left\{\frac{k}{k_z}\hat{u}_0(k_x, k_y)e^{-jk_z z}\right\}$$

Next, the phase is factored into the pressure field to create the $e^{jl\varphi}$ vorticity. where φ is $\arctan y/x$. With the pressure field now given beyond the phase plate, the propagation of that field towards arbitrary distance away from the plate can then be performed using the free-space propagation equation.

7 Integral representation of Bessel function

A given integral representation of the Bessel function is [4]

$$J_n(\eta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ni\varphi - i\eta\sin\varphi}\,\mathrm{d}\varphi \tag{17}$$

We want to show that equation (17) is equivalent to the integral at hand (times the coefficient $e^{in\pi/2}$ which tacks on a real or complex unit):

$$\frac{e^{in\pi/2}}{2\pi}\int_{0}^{2\pi} e^{ni\varphi - i\eta\cos\varphi}\,\mathrm{d}\varphi. \tag{18}$$

Start by substituting $\varphi \mapsto \varphi - \frac{\pi}{2}$ into equation (18).

$$\begin{aligned}\frac{e^{in\pi/2}}{2\pi}\int_{0}^{2\pi} e^{ni\varphi - i\eta\cos\varphi}\,\mathrm{d}\varphi &= \frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2} e^{ni\varphi - i\eta\sin\varphi}\,\mathrm{d}\varphi \\ &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi} e^{ni\varphi - i\eta\sin\varphi}\,\mathrm{d}\varphi + \frac{1}{2\pi}\int_{\pi}^{3\pi/2} e^{ni\varphi - i\eta\sin\varphi}\,\mathrm{d}\varphi\end{aligned}$$

Substituting $\varphi \mapsto \varphi + \frac{\pi}{2}$ into the second integral,

$$\frac{e^{in\pi/2}}{2\pi}\int_0^{2\pi} e^{ni\varphi-i\eta\cos\varphi}\,\mathrm{d}\varphi = \frac{1}{2\pi}\int_{-\pi/2}^{\pi} e^{ni\varphi-i\eta\sin\varphi}\,\mathrm{d}\varphi + \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} e^{ni\varphi-i\eta\sin\varphi}\,\mathrm{d}\varphi$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ni\varphi-i\eta\sin\varphi}\,\mathrm{d}\varphi$$

It is seen that $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ni\varphi-i\eta\sin\varphi}\,\mathrm{d}\varphi = \frac{e^{in\pi/2}}{2\pi}\int_0^{2\pi} e^{ni\varphi-i\eta\cos\varphi}\,\mathrm{d}\varphi$, and therefore

$$J_n(\eta) = \frac{e^{in\pi/2}}{2\pi}\int_0^{2\pi} e^{ni\varphi-i\eta\cos\varphi}\,\mathrm{d}\varphi.$$

References

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