Spatial Fourier Transform Theorems[1]

ME/EE 384N-8, Wave Phenomena, Spring 2024

Definitions:

$$F(k_x, k_y) = \mathcal{F}\{f(x,y)\} = \iint_{-\infty}^{\infty} f(x,y) e^{-i(k_x x + k_y y)}\, dxdy$$
$$f(x,y) = \mathcal{F}^{-1}\{F(k_x,k_y)\} = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} F(k_x,k_y) e^{i(k_x x + k_y y)}\, dk_x dk_y$$

Fourier integral theorem:

$$\mathcal{F}^{-1}\Big\{\mathcal{F}\{f(x,y)\}\Big\} = \mathcal{F}\left\{\mathcal{F}^{-1}\{f(x,y)\}\right\} = f(x,y)$$

Linearity:

$$\mathcal{F}\{af(x,y) + bg(x,y)\} = a\mathcal{F}\{f(x,y)\} + b\mathcal{F}\{g(x,y)\}$$

Similarity: If $\mathcal{F}\{f(x,y)\} = F(k_x,k_y)$, then

$$\mathcal{F}\{f(ax,by)\} = \frac{1}{|ab|} F\left(\frac{k_x}{a}, \frac{k_y}{b}\right)$$

Differentiation:

$$\mathcal{F}\left\{\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial y^m} f(x,y)\right\} = (ik_x)^n (ik_y)^m F(k_x,k_y)$$
$$\mathcal{F}^{-1}\left\{\frac{\partial^n}{\partial k_x^n}\frac{\partial^m}{\partial k_y^m} F(k_x,k_y)\right\} = (-ix)^n(-iy)^m f(x,y)$$

Shifting: If $\mathcal{F}\{f(x,y)\} = F(k_x,k_y)$, then

$$\mathcal{F}\{f(x-a, y-b)\} = F(k_x,k_y) e^{-i(k_x a + k_y b)}$$
$$\mathcal{F}^{-1}\{F(k_x - \alpha, k_y - \beta)\} = f(x,y) e^{i(\alpha x + \beta y)}$$

Rayleigh (Parseval):

$$\iint_{-\infty}^{\infty} |f(x,y)|^2\, dxdy = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} |F(k_x,k_y)|^2\, dk_x dk_y$$

[1] After J. W. Goodman, *Fourier Optics* (Roberts & Co., Englewood, 2005); see Appendix A.

Convolution: If

$$f(x,y) ** g(x,y) = \iint_{-\infty}^{\infty} f(x',y')g(x-x',y-y')\,dx'dy'$$

$$F(k_x,k_y) ** G(k_x,k_y) = \iint_{-\infty}^{\infty} F(k_x',k_y')G(k_x-k_x',k_y-k_y')\,dk_x'dk_y'$$

then

$$\mathcal{F}\{f(x,y) ** g(x,y)\} = F(k_x,k_y)G(k_x,k_y)$$

$$\mathcal{F}^{-1}\{F(k_x,k_y) ** G(k_x,k_y)\} = 4\pi^2 f(x,y)g(x,y)$$

Autocorrelation function and power spectrum: If

$$R(x,y) = \iint_{-\infty}^{\infty} f(x',y')f^*(x'-x,y'-y)\,dx'dy'$$

then

$$\mathcal{F}\{R(x,y)\} = |F(k_x,k_y)|^2$$