The Evanescent Spectrum of a Spherical Wave

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In Prof. Hamilton's Wave Phenomena class, we showed that the angular spectrum of a spherical wave (i.e., its decomposition into plane waves) is

$$\frac{e^{ikr}}{r} = \frac{i}{2\pi}\iint_0^\infty \frac{e^{ik_z|z|}}{k_z} e^{i(k_x x + k_y y)} dk_x dk_y, \qquad k_z = \sqrt{k^2 - k_x^2 - k_y^2}. \tag{1}$$

As the field is axisymmetric about the z-axis, this is equivalent to

$$\frac{e^{ikr}}{r} = i\int_0^\infty \frac{e^{ik_z|z|}}{k_z} J_0(\kappa\rho)\kappa d\kappa \tag{2}$$

where

$$\kappa = \sqrt{k_x^2 + k_y^2}, \qquad \rho = \sqrt{x^2 + y^2}, \qquad k_z = \sqrt{k^2 - \kappa^2}, \qquad r = \sqrt{\rho^2 + z^2}. \tag{3}$$

We were asked: *What is the interpretation of the evanescent spectrum where $\kappa > k$?* I may not have a totally satisfying answer, but I have some thoughts...

To gain intuition we examine the field along the positive z axis, where (2) reduces to

$$\frac{e^{ikz}}{z} = i\int_0^\infty \frac{e^{ik_z z}}{k_z} \kappa d\kappa. \tag{4}$$

We split this integral into the propagating and evanescent components:

$$\frac{e^{ikz}}{z} = i\int_0^k \frac{e^{i\sqrt{k^2-\kappa^2}z}}{\sqrt{k^2-\kappa^2}} \kappa d\kappa + \int_k^\infty \frac{e^{-\sqrt{\kappa^2-k^2}z}}{\sqrt{\kappa^2-k^2}} \kappa d\kappa. \tag{5}$$

Both of these integrals are evaluated with u-substitution, yielding

$$\mathscr{I}_P = i\int_0^k \frac{e^{i\sqrt{k^2-\kappa^2}z}}{\sqrt{k^2-\kappa^2}} \kappa d\kappa = \frac{e^{ikz}-1}{z}, \qquad \mathscr{I}_E = \int_k^\infty \frac{e^{-\sqrt{\kappa^2-k^2}z}}{\sqrt{\kappa^2-k^2}} \kappa d\kappa = \frac{1}{z}. \tag{6}$$

In the limit $z \to 0$ the propagating components yield the finite value $\mathscr{I}_P \to ik$. It seems the propagating components are unable to reproduce the singularity at $z = 0$, and therefore the evanescent components $\mathscr{I}_E$ are needed.

This is similar to what Brekhovskikh wrote ([1], p. 231):

It is impossible to obtain the field which would have the required singularity as $R \to 0$ by superposition of ordinary plane waves only... We have waves propagating in the horizontal plane (the xy plane) with a wavelength approaching zero and simultaneously attenuating in the vertical direction with an attenuation coefficient approaching infinity. At $x = y = 0$ the superposition of an infinite number of these waves gives an infinite value for the field.

References

[1] L. M. Brekhovskikh, *Waves in Layered Media*, 2nd ed. (1980).