Wave Phenomena

These are my class notes from Wave Phenomena (ME 384N, spring 2024), taught by Prof. Mark F. Hamilton in the Walker Department of Mechanical Engineering at UT Austin. My notes mostly follow the material presented in class. I rearranged a few topics for conceptual consistency and made a few additions in dark blue. The quotations of physicists introducing each topic were also my addition. A PDF version of this website is available here.

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Additional handouts

Mathematical preliminary

> *It seems to be one of the fundamental features of nature that fundamental physical laws are described in terms of a mathematical theory of great beauty and power, needing quite a high standard of mathematics for one to understand it. You may wonder: Why is nature constructed along these lines? One can only answer that our present knowledge seems to show that nature is so constructed. We simply have to accept it. One could perhaps describe the situation by saying that God is a mathematician of a very high order, and He used very advanced mathematics in constructing the universe. Our feeble attempts at mathematics enable us to understand a bit of the universe, and as we proceed to develop higher and higher mathematics we can hope to understand the universe better.*

–Paul Dirac

Fourier transforms, Hankel transforms, and Dirac delta functions are discussed before introducing wave physics. Important equations are boxed, and my personal comments are in deep blue.

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Linear systems theory

We begin by discussing the time transform. Fourier transforms are linear operations and hence lend themselve to problems that are linear. The temporal Fourier transform (FT) pair is given by

$$X(\omega) = \mathcal{F}_t\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{i\omega t}dt \tag{1}$$

$$x(t) = \mathcal{F}_t^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{-i\omega t}d\omega \tag{2}$$

The response of a linear system is illustrated schematically below:

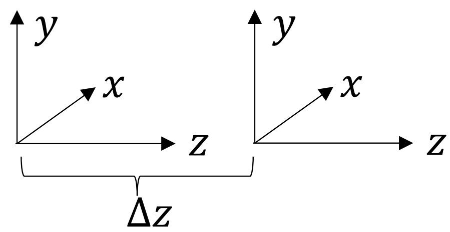
$$x(t) \xrightarrow{\mathcal{F}_t\{x(t)\}} X(\omega) \xrightarrow{H(\omega)X(\omega)} Y(\omega) \xrightarrow{\mathcal{F}_t^{-1}\{Y(\omega)\}} y(t)\,,$$

where $H(\omega)$ is the so-called transfer function.

I find it helpful to remember that "$\mathcal{F}$" corresponds to the operator $\int_{-\infty}^{\infty} \ldots dt$, while an upper case letter corresponds to the transformed lower case function.

Example: wave propagation from position $x, y, z = z_0$ to $x, y, z = z_0 + \Delta z$

As a sneak-peek for what is to come, we outline (informally) how use the Fourier transform (both temporal and spatial, discussed further in the following sections) to propagate from position $x, y, z = z_0$ to a parallel plane $x, y, z = z_0 + \Delta z$ following the general "recipe" provided above.



First take the Fourier time transform $\mathcal{F}_t$ to obtain p_ω:

$$p(x, y, z, t_0) \xrightarrow{\mathcal{F}_t\{p(x,y,z,t_0)\}} p_\omega(\omega, x, y, z_0) .$$

Then take the 2D spatial Fourier transform to obtain $P_\omega(k_x, k_y, z_0)$ (note capital letters are used for quantities in k-space):

$$p_\omega(\omega, x, y, z_0) \xrightarrow{\mathcal{F}_{xy}\{p_\omega(x,y,z_0)\}} P_\omega(k_x, k_y, z_0) .$$

Next, apply the transfer function to $P_\omega(k_x, k_y, z_0)$ to advance spatially.

$$P_\omega(k_x, k_y, z_0) \xrightarrow{H(k_x,k_y)P_\omega(k_x,k_y,z_0)} P_\omega(k_x, k_y, z_0 + \Delta z) .$$

Finally, take the inverse 2D spatial FT to return from k-space to physical space.

$$P_\omega(k_x, k_y, z_0 + \Delta z) \xrightarrow{\mathcal{F}_{xy}^{-1}\{P_\omega(k_x,k_y,z_0+\Delta z)\}} p_\omega(x, y, z + \Delta z) .$$

The above step shall be the stopping point in this class (since we will work largely in k-space, i.e., our interest is in solving the Helmholtz equation, not the wave equation). If desired, however, one can take the inverse time Fourier transform, viz., $\mathcal{F}_t^{-1}\{p_\omega(x, y, z + \Delta z)\}$ to obtain $p(x, y, z + \Delta z, t)$.

What is the transfer function? We will find that

$$H(k_x, k_y) = e^{ik_z \Delta z}, \quad k_z = \sqrt{\frac{\omega^2}{c_0^2} - k_x^2 - k_y^2} .$$

How should one correctly interpret p_ω in the equations above? We will represent the time dependence as

$$p(\mathbf{r}, t) = p_\omega(\mathbf{r}) e^{-i\omega t} = |p_\omega(r)| e^{i\phi_\omega(\mathbf{r})} e^{-i\omega t}$$

and interpret $|p_\omega(\mathbf{r})|$ as the peak pressure of $p(t)$.

Do not try to identify p_ω from P_ω, as illustrated in the following example: Consider the pressure wave $p(t) = p_0 \cos \omega_0 t$. Since there is no spatial dependence, p_ω should be identified as p_0. Meanwhile, the time FT is

$$\begin{aligned} \mathcal{F}_t\{p_0 \cos \omega_0 t\} &= \int_{-\infty}^{\infty} p_0 \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} e^{i\omega t} dt \\ &= \frac{p_0}{2} \int_{-\infty}^{\infty} (e^{i\omega_0 t} + e^{-i\omega_0 t}) e^{i\omega t} dt \\ &= \pi p_0 [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \,, \end{aligned}$$

where one might be tempted to define p_ω as πp_0.

1D Fourier transform

For this discussion, consider the 1D spatial Fourier transform:

$$F(k_x) = \mathcal{F}_x\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-ik_x x} dx \tag{1}$$

$$f(x) = \mathcal{F}_x^{-1}\{F(k_x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x) e^{ik_x x} dk_x \,. \tag{2}$$

The sign convention corresponds to the $e^{-i\omega t}$ time dependence such that

$$f(x) e^{-i\omega t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x) e^{i(k_x x - \omega t)} dk_x$$

is a forward-traveling wave.

Some authors, e.g., Goodman let $k_x = 2\pi f_x$ such that the Fourier transforms are symmetric.

$$\begin{aligned} \mathcal{F}_x^G\{f(x)\} &= \int_{-\infty}^{\infty} f(x) e^{-i2\pi f_x x} dx \equiv F^G(k_x) \\ \mathcal{F}_x^{-1,G}\{F(k_x)\} &= \int_{-\infty}^{\infty} F^G(k_x) e^{i2\pi f_x x} df_x \,, \end{aligned}$$

where $dk_x = 2\pi df_x$. We will avoid this in favour of Eqs. Eqs. (1) and (2). Simply beware of these alternate definitions.

Many more properties of Eqs. (1) and (2) are given in [Papoulis's table] with the following substitutions:

$$\begin{aligned} t &\mapsto x \\ \omega &\mapsto k_x \\ j &\mapsto i \,. \end{aligned}$$

Example: Fourier transform of a rectangle function

Define

$$\operatorname{rect} x = \begin{cases} 1, & |x| \le 1/2 \\ 0, & |x| > 1/2 \end{cases},$$

and let

$$f(x) = A \operatorname{rect}(x/L) .$$

Take the Fourier transform:

$$\begin{aligned}
\mathcal{F}_x\{f(x)\} &= A \int_{-L/2}^{L/2} e^{-ik_x x} dx \\
&= \frac{A}{-ik_x}(e^{-ik_x L/2} - e^{ik_x L/2}) \\
&= \frac{2A}{k_x} \sin(k_x L/2) \\
&= AL \frac{\sin(k_x L/2)}{k_x L/2} = AL j_0(k_x L/2) . \\
&= AL, \quad k_x = 0 .
\end{aligned}$$

That is to say, the Fourier transform evaluated at $k_x = 0$ is the area under the curve:

$$F(k_x = 0) = \int_{-\infty}^{\infty} f(x) dx .$$

2D spatial Fourier transform

The two-dimensional (2D) spatial FT will prove to be useful:

$$\begin{aligned}
F(k_x, k_y) = \mathcal{F}_{xy}\{f(x,y)\} &= \iint_{-\infty}^{\infty} f(x,y) e^{-i(k_x x + k_y y)} dx dy \\
f(x,y) = \mathcal{F}_{xy}^{-1}\{F(k_x,k_y)\} &= \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} F(k_x,k_y) e^{i(k_x x + k_y y)} dk_x dk_y
\end{aligned}$$

Wavenumber space is sometimes referred to as "k-space." Note that k-space itself arises in the first place from taking a time FT (of the wave equation). Beware of the different sign conventions used in physics and engineering.

In class, the "xy" subscripts of $\mathcal{F}$ were dropped for convenience, but I shall retain them in the online notes for clarity.

$$F(k_x, k_y) = \mathcal{F}_{xy}\{f(x, y)\} = \iint_{-\infty}^{\infty} f(x, y) e^{-i(k_x x + k_y y)} dxdy$$
$$f(x, y) = \mathcal{F}_{xy}^{-1}\{F(k_x, k_y)\} = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} F(k_x, k_y) e^{i(k_x x + k_y y)} dxdy$$

Note that the operators commute:

$$\begin{aligned} \mathcal{F}_{xy}\{f(x, y)\} &= \mathcal{F}_y\{\mathcal{F}_x[f(x, y)]\} \\ &= \mathcal{F}_x\{\mathcal{F}_y[f(x, y)]\}, \end{aligned}$$

and if $f(x, y) = g(x)h(y)$ then $\mathcal{F}\{f(x, y)\} = G(k_x)H(k_y)$.

Finally, note that a vectorial form of the 2D spatial Fourier transform is often utilized. Let

$$\begin{aligned} \boldsymbol{\rho} &= (x, y) = x\mathbf{e}_x + y\mathbf{e}_y \\ \boldsymbol{\kappa} &= (k_x, k_y) = k_x\mathbf{e}_x + k_y\mathbf{e}_y, \end{aligned}$$

so $\boldsymbol{\rho} \cdot \boldsymbol{\kappa} = k_x x + k_y y$. Then the 2D spatial Fourier transform pair reads

$$\begin{aligned} F(\boldsymbol{\kappa}) &= \int f(\boldsymbol{\rho}) e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}} d\boldsymbol{\rho} \\ f(\boldsymbol{\rho}) &= \int F(\boldsymbol{\kappa}) e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}} d\boldsymbol{\kappa}. \end{aligned}$$

The different uses of the letter "f" might be confusing. To reiterate: F refers to the transformed "version" of f, whereas $\mathcal{F}$ is the integral transformation. It helps to loosely think, "f is the uncooked meal, F is the cooked meal, and $\mathcal{F}$ is the microwave."

Transform theorems

A full list of some transformation theorems is provided here. Some will be proven as homework problems. The theorems in that document are in terms of the 2D spatial Fourier transform. Here, we prove two theorems for the 1D spatial Fourier transform relating to **similarity** and **differentiation**. They can straightforwardly generalized to the 2D case.

Similarity.

If $\mathcal{F}_x\{f(x)\} = F(k_x)$, then what is $\mathcal{F}_x\{f(ax)\}$? To find out, we simply apply the definition of the 1D spatial Fourier transform:

$$\mathcal{F}_x\{f(ax)\} = \int_{-\infty}^{\infty} f(ax) e^{-ik_x x} dx$$

To evaluate the integral, the substitution $u = ax$ (and hence $dx = du/a$) is used. Note that the sign of a determines which integral is to be taken:

$$\mathcal{F}_x\{f(ax)\} = \begin{cases} \int_{-\infty}^{\infty} f(u) e^{-ik_x u/a} \frac{du}{a}, & a > 0 \\ \int_{\infty}^{-\infty} f(u) e^{-ik_x u/a} \frac{du}{a}, & a < 0 \end{cases}$$

To eliminate the conditional statement above, simply consider the magnitude of a:

$$\begin{aligned}\mathcal{F}_x\{f(ax)\} &= \int_{-\infty}^{\infty} f(u)e^{-ik_x u/a}\frac{du}{|a|}, \quad \forall a \\ &= \frac{1}{|a|}\int_{-\infty}^{\infty} f(u)e^{-ik_x u/a}\,du\,.\end{aligned}$$

Recalling Eq. (1), $F(k_x) = \mathcal{F}_x\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-ik_x x}dx$, the integral above is identified as $F(k_x/a)$. This is done by corresponding u above as x in Eq. (1), and corresponding k_x/a above as k_x in Eq. (1). Thus,

$$\boxed{\mathcal{F}_x\{f(ax)\} = \frac{1}{|a|}F(k_x/a)\,.}$$

Differentiation.

In particular, let us consider the Fourier transform of the n^{th} derivative of $f(x)$:

$$\mathcal{F}_x\left\{\frac{\partial^n}{\partial x^n}f(x)\right\},$$

which we can write as

$$\mathcal{F}_x\left\{\frac{\partial^n}{\partial x^n}\mathcal{F}_x^{-1}[F(k_x)]\right\}$$

because $\mathcal{F}_x^{-1}[F(k_x)] = f(x)$. Invoking Eq. (2), the Fourier transform is written explicitly:

$$\mathcal{F}_x\left\{\frac{\partial^n}{\partial x^n}f(x)\right\} = \mathcal{F}_x\left\{\frac{\partial^n}{\partial x^n}\frac{1}{2\pi}\int_{-\infty}^{\infty}[F(k_x)]e^{ik_x x}dk_x\right\}$$

The derivative is brought inside the integral, and the integrand is evaluated:

$$\mathcal{F}_x\left\{\frac{\partial^n}{\partial x^n}f(x)\right\} = \mathcal{F}_x\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}(ik_x)^n F(k_x)e^{ik_x x}dk_x\right\}$$

Again by Eq. (2), the inverse spatial Fourier transform is identified:

$$\mathcal{F}_x\left\{\frac{\partial^n}{\partial x^n}f(x)\right\} = \mathcal{F}_x\{\mathcal{F}_x^{-1}[(ik_x)^n F(k_x)]\}$$

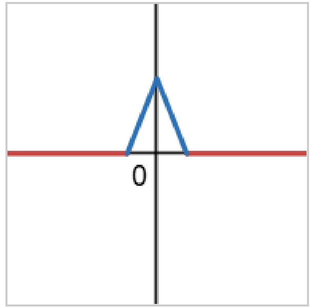
Since the forward and inverse transform are inverses, we obtain

$$\boxed{\mathcal{F}_x\left\{\frac{\partial^n}{\partial x^n}f(x)\right\} = (ik_x)^n F(k_x)\,.}$$

Example: Fourier transform of triangle function

Define

$$\text{tri}(x) = \Lambda(x) \equiv \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}.$$

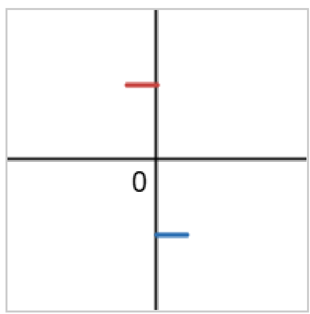


In this example, we evaluate the Fourier transform of the function

$$f(x) = A\Lambda(x/L)$$

rather cleverly. We will use the derivative theorem to obtain the Fourier transform of this function (rather than take the FT directly). In particular, the derivative of $f(x) = A\Lambda(x/L)$ is

$$f'(x) = \frac{A}{L}\left[\text{rect}\left(\frac{x + L/2}{L}\right) - \text{rect}\left(\frac{x - L/2}{L}\right)\right].$$



Earlier, we found that the Fourier transform of the rectangle function is

$$\mathcal{F}_x\left\{\text{rect}\,\frac{x}{L}\right\} = L\frac{\sin(k_x L/2)}{k_x L/2}.$$

The shifting property is used (which will be proved as a homework problem):

$$\mathcal{F}_x\{f(x-a)\} = F(k_x)e^{-ik_x a}.$$

So the Fourier transform of the *derivative* of the triangle function is

$$\mathcal{F}_x\{f'(x)\} = A\frac{\sin(k_x L/2)}{k_x L/2}(e^{ik_x L/2} - e^{-ik_x L/2}).$$

But $e^{ik_x L/2} - e^{-ik_x L/2} = 2i\sin(k_x L/2)$. So we have (after normalization)

$$\mathcal{F}_x\{f'(x)\} = ik_x AL\left[\frac{\sin(k_x L/2)}{k_x L/2}\right]^2.$$

However, this is the Fourier transform of the *derivative* of the triangle function. Using the derivative theorem for $n = 1$,

$$\mathcal{F}_x\left\{\frac{\partial}{\partial x}f(x)\right\} = ik_xF(k_x)\,,$$

we obtain the 1D spatial Fourier transform of the triangle function:

$$F(k_x) = \mathcal{F}_x\{f(x)\} = AL\left[\frac{\sin(k_xL/2)}{k_xL/2}\right]^2.$$

As a sanity check, evaluate the FT at the origin of k-space and see if we obtain the area under $f(x)$. Indeed, $F(k_x) = AL$, which is indeed the area under the triangle function.

The angular spectrum

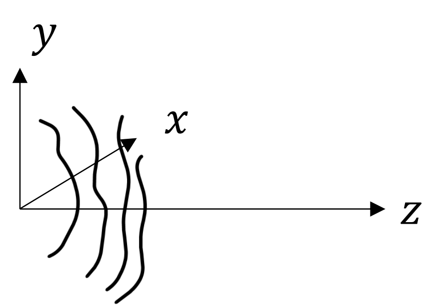
The above discussion has been purely mathematical. Now let us apply this mathematics to the study of wave phenomena. Consider a time-harmonic acoustic pressure field:

$$p(x,y,z,t) = p_\omega(x,y,z)e^{-i\omega t}\,.$$

Thus the wave equation becomes

$$\nabla^2 p_\omega + k^2 p_\omega = 0\,. \tag{1}$$

Let p_ω be an arbitrary field propagating in the $+z$ direction. At a given instant, it passes through the plane $z = 0$.



Let us characterize the field in the plane $z = 0$ as

$$p_\omega(x,y,0) = p_0(x,y)\,.$$

The 2D spatial Fourier transform of this source condition is

$$P_0(k_x,k_y) = \iint_{-\infty}^{\infty} p_0(x,y)e^{-i(k_xx+k_yy)}dxdy\,, \tag{2}$$

so

$$p_0(x,y) = \iint_{-\infty}^{\infty} P_0(k_x,k_y) e^{i(k_x x + k_y y)} dk_x dk_y \,. \tag{3}$$

Note that a general plane-wave solution of Eq. (1) is

$$p_\omega(x,y,z) = Ae^{i(k_x x + k_y y + k_z z)} = Ae^{i\mathbf{k}\cdot\mathbf{r}} \,,$$

where $k = k_x\mathbf{e}_x + k_y\mathbf{e}_y + k_z\mathbf{e}_z$ and $|\mathbf{k}| = \omega/c_0$. Thus

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} \,.$$

Thus Eq. (3) is a superposition of plane waves passing through the plane $z = 0$ with complex amplitudes

$$A(k_x,k_y) = P_0(k_x,k_y)\frac{dk_x dk_y}{4\pi^2} \,.$$

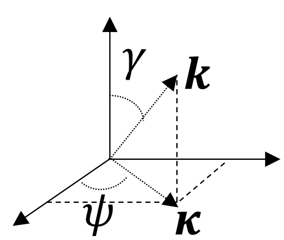
Each plane wave in the summation is uniquely determined by the pair (k_x,k_y). Now write

$$\begin{aligned} \mathbf{k} &= k_x\mathbf{e}_x + k_y\mathbf{e}_y + k_z\mathbf{e}_z \\ &= \boldsymbol{\kappa} + k_z\mathbf{e}_z \,. \end{aligned}$$

Now we introduce spherical coordinates, where the meaning of the z axis is preserved between the Cartesian and spherical coordinates:

$$\begin{aligned} k_x &= k\sin\gamma\cos\psi = \kappa\cos\psi \\ k_y &= k\sin\gamma\sin\psi = \kappa\sin\psi \\ k_z &= k\cos\gamma \,, \end{aligned}$$

where γ is the polar angle in k-space, while ψ is the azimuthal angle in k-space.

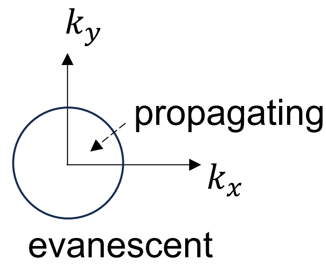


I do not know what to label the axes in the figure above. In future sections, you will see that they are labeled (x,y,z), which I abhor, because these are spatial coordinates with dimensions of length, while the vectors $\mathbf{k}$ and $\boldsymbol{\kappa}$ have dimensions of inverse length.

The ordered pair (k_x,k_y) is uniquely defined by (ψ,γ), and hence

$$P_0(k_x,k_y) = \text{ angular spectrum} = P_0(\gamma,\psi) \,.$$

Note that only components for which $k_x^2 + k_y^2 < k^2$ can propagate. That is, outside a circle of radius $|\mathbf{k}|$ $= \omega/c_0$, the wave field is evanescent:



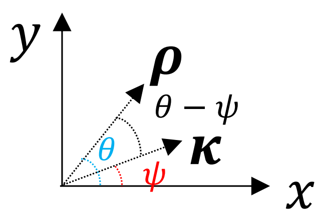
Hankel transform

For axisymmetric cases, the Hankel transform (as apposed to the 2D spatial Fourier transform) proves to be useful.

Suppose $f(x, y) = f(\sqrt{x^2 + y^2}) = f(\rho)$. Then, the 2D spatial Fourier transform becomes

$$\mathcal{F}_\rho\{f(\rho)\} = \int_0^{2\pi}\int_0^{\infty} f(\rho)e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\,\rho\,d\rho\,d\phi$$

where sometimes the differential area element $\rho\,d\rho\,d\phi$ is denoted $d\boldsymbol{\rho}$. The relationship between ρ and κ is shown below:



Since $\boldsymbol{\kappa}$ is fixed, we may align $\boldsymbol{\kappa}$ along the x-axis (i.e., set $\psi = 0$) for the case that $f \neq f(\phi)$. Then

$$\boldsymbol{\kappa}\cdot\boldsymbol{\rho} = \kappa\rho\cos\phi\,.$$

and

$$\mathcal{F}_\rho\{f(\rho)\} = \int_0^{\infty} f(\rho)\rho d\rho\int_0^{2\pi} e^{-ik\rho\cos\phi}d\phi\,,$$

and the polar integral is an integral identity of the 0th order Bessel function, $2\pi J_0(\kappa\rho)$. Thus the transform becomes

$$\mathcal{F}_\rho\{f(\rho)\} = 2\pi\int_0^{\infty} f(\rho)J_0(\kappa\rho)\,\rho\,d\rho = F(\kappa)$$

Define the Hankel transform:

$$F_H(\kappa) \equiv \mathcal{H}_\rho\{f(\rho)\} = \int_0^{\infty} f(\rho)J_0(\kappa\rho)\,\rho\,d\rho \tag{1}$$

Thus

$$F(\kappa) = 2\pi F_H(\kappa)\,, \tag{2}$$

which is a critical (and annoying) difference between the Fourier and Hankel transforms.

Similarly, the inverse 2D spatial Fourier transform of an axisymmetric function k-space, $F(\kappa)$, is

$$f(\rho) = \mathcal{F}_\rho^{-1}\{F(\kappa)\} = \frac{1}{4\pi^2}\iint F(\kappa)e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\,\kappa\,d\kappa\,d\psi$$

Align $\boldsymbol{\rho}$ with the k_x axis. Thus in this case

$$\boldsymbol{\kappa}\cdot\boldsymbol{\rho} = \kappa\rho\cos\psi\,.$$

So

$$\begin{aligned} f(\rho) &= \frac{1}{2\pi}\int_0^\infty F(\kappa)J_0(\kappa\rho)\kappa\,d\kappa \\ &= \int_0^\infty F_H(\kappa)J_0(\kappa\rho)\kappa\,d\kappa\,, \end{aligned}$$

where the second equality follows from Eq. (2). So

$$f(\rho) = \mathcal{H}_\rho^{-1}\{F_H(\kappa)\} = \int_0^\infty F_H(\kappa)J_0(\kappa\rho)\kappa d\kappa\,. \tag{3}$$

Eqs. (1) and (3) are standard definitions of forward and inverse Hankel transforms, related to Fourier transforms by Eq. (2).

Example: Fourier transform of the circ function

Define

$$\mathrm{circ}\,(\rho) = \begin{cases} 1 & \rho \le 1 \\ 0 & \rho > 1 \end{cases}$$

and let $f(\rho) = A\,\mathrm{circ}\,(\rho/a)$. Then the *Hankel* transform of $f(\rho)$ is

$$\mathcal{H}_\rho\{A\,\mathrm{circ}\,(\rho/a)\} = F_H(\kappa) = A\int_0^a J_0(\kappa\rho)\rho d\rho\,.$$

Noting that $\int_0^t J_0(t)\,t\,dt = xJ_1(x)$, the integral above is evaluated by letting $t = \kappa\rho$ and hence $dt = \kappa d\rho$, $\rho d\rho = tdt/\kappa^2$. Thus

$$\begin{aligned} F_H(\kappa) &= \frac{A}{k^2}\int_0^\kappa J_0(t)\,t\,dt \\ &= A\frac{a}{\kappa}J_1(ka) \\ &= A\frac{a^2}{2}\frac{2J_1(\kappa a)}{\kappa a} \end{aligned}$$

Note that $\frac{2J_1(\kappa a)}{\kappa a} = 1$ for $\kappa a = 0$. However we are interested in the Fourier transform

$$\mathcal{F}_\rho\{A\,\mathrm{circ}\,(\rho/a)\} = F(\kappa) = 2\pi F_H(\kappa) = A\pi a^2 \frac{2J_1(\kappa a)}{\kappa a}\,.$$

As a sanity check, note that this equals the volume, $A\pi a^2$ for $\kappa = 0$ ($k_x = k_y = 0$).

Dirac delta functions

One definition of the Dirac delta function is

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)\,. \tag{1}$$

$\delta(x)$ is a generalized function, also called a "functional," which is something that operates on an integrand. It can be thought of as an integral mapping from $f(x)$ to $f(0)$.

An alternative definition is two-fold:

$$\delta(x) = 0, \quad x \neq 0 \tag{2}$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1\,. \tag{3}$$

It will be of interest to represent a δ function as a limit of an ordinary function $\mu(x,\epsilon)$.

$$\delta(x) = \lim_{\epsilon\to 0} \mu(x,\epsilon)\,.$$

To construct $\mu(x,\epsilon)$, start with another function without an ϵ, $\zeta(x)$, having unit area

$$\int_{-\infty}^{\infty} \zeta(x)dx = 1\,.$$

Then let

$$\mu(x,\epsilon) = \frac{1}{\epsilon}\zeta(x/\epsilon)\,.$$

Let us prove that

$$\delta(x) = \lim_{\epsilon\to 0} \mu(x,\epsilon)$$

for an arbitrary $\mu(x,\epsilon)$. Substituting $\mu(x,\epsilon) = \frac{1}{\epsilon}\zeta(x/\epsilon)$ into the definition of the delta function given by Eq. (1) gives

$$\begin{aligned} f(0) &= \int_{-\infty}^{\infty} f(x) \lim_{\epsilon\to 0} \mu(x,\epsilon)dx \\ &= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} f(x)\frac{1}{\epsilon}\zeta(x/\epsilon)dx\,. \end{aligned}$$

Now introduce a variable substitution $x = \epsilon y$. Thus $dx = \epsilon\, dy$. The limits remain unchanged (though there is a subtle point here):

$$
\begin{aligned}
f(0) &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(\epsilon y)\frac{1}{\epsilon}\zeta(y)\epsilon dy \\
&= f(0)\int_{-\infty}^{\infty}\zeta(y)dy
\end{aligned}
$$

But since $\int_{-\infty}^{\infty}\zeta(x)dx = 1$, the above is simply equal to $f(0)$. Thus Eq. (1) is satisfied.

Example: construct delta function out of rectangle function

It is desired to construct a delta function out of

$$
\zeta(x) = \text{rect}\,(x)\;.
$$

Following the recipe above, identify $\mu(x,\epsilon) = \frac{1}{\epsilon}\,\text{rect}\,(x/\epsilon)$, and thus identify the delta function to be

$$
\delta(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\,\text{rect}\,(x/\epsilon)\;.
$$

Example: construct delta function out of sinc function

Now it is desired to construct a delta function out of a sinc function, which has the form $\sin(x)/x$. Note that

$$
\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \pi\;.
$$

Since it is desired for ζ to come out to 1 when integrated over x from $-\infty$ to $+\infty$, we let $\zeta(x) = \sin(x)/(\pi x)$. Then

$$
\begin{aligned}
\mu(x,\epsilon) &= \frac{1}{\epsilon}\frac{\sin(x/\epsilon)}{\pi x/\epsilon} \\
&= \frac{\sin(x/\epsilon)}{\pi x}\;.
\end{aligned}
$$

Thus the delta function is

$$
\delta(x) = \lim_{\epsilon\to 0}\frac{\sin(x/\epsilon)}{\pi x}\;.
$$

To verify this, we can insert our delta function into definition and evaluate the integral (letting $x = \epsilon y$):

$$
\begin{aligned}
\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(x)\frac{\sin(x/\epsilon)}{\pi x}dx &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(\epsilon y)\frac{\sin(y)}{\pi\epsilon y}\epsilon\, dy \\
&= f(0)\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(y)}{y}dy = f(0)\;.
\end{aligned}
$$

It is found that the delta function based on sinc indeed satisfies the definition of the delta function.

Using the result of the previous example, the Fourier transform of the complex exponential can be taken rather cleverly. Start with the previous result, $\delta(x) = \lim_{\epsilon\to 0} \frac{\sin(x/\epsilon)}{\pi x}$, write sine using complex exponentials (e.g., $\sin a = (e^{ia} - e^{-ia})/2i$), cleverly write $e^{ix/\epsilon} - e^{-ix/\epsilon} = \int_{-1/\epsilon}^{1/\epsilon} ix e^{ik_x x} dk_x$, and take the limit:

$$\begin{aligned}\delta(x) &= \lim_{\epsilon\to 0} \frac{\sin(x/\epsilon)}{\pi x} \\ &= \lim_{\epsilon\to 0} \frac{1}{i2\pi x}(e^{ix/\epsilon} - e^{-ix/\epsilon}) \\ &= \lim_{\epsilon\to 0} \frac{1}{i2\pi x} \int_{-1/\epsilon}^{1/\epsilon} ix e^{ik_x x} dk_x \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik_x x} dk_x \,.\end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} e^{ik_x x} dk_x = 2\pi\delta(x)\,.$$

(Alternatively, we have $\int_{-\infty}^{\infty} e^{-ik_x x} dk_x = 2\pi\delta(x)$.) And for the dual space,

$$\int_{-\infty}^{\infty} e^{-ik_x x} dx = 2\pi\delta(k_x)\,.$$

Example: Fourier integral theorem

Using the above result, the Fourier integral theorem, i.e., the Fourier transform of the inverse Fourier transform of a function gives the function itself, is now proven:

$$\begin{aligned}\mathcal{F}_x^{-1}\{\mathcal{F}_x[f(x)] &= \mathcal{F}_x^{-1}\{F(k_x)\}\} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x') e^{-ik_x x'} dx' \right] e^{ik_x x} dk_x \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x') \left[\int_{-\infty}^{\infty} e^{-ik_x (x-x')} dk_x \right] dx' \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x') 2\pi\delta(x-x') dx' \\ &= f(x)\,.\end{aligned}$$

Thus $\mathcal{F}_x^{-1}\{\mathcal{F}_x[f(x)]\} = f(x)$.

A few properties of the delta function are now discussed:

Derivative property

Denoting $\delta'(x) = d\delta/dx$, a property involving the derivative of the delta function can be derived by integration by parts:

$$\int_{-\infty}^{\infty} f(x)\delta'(x)dx = \int_{-\infty}^{\infty} f(x)\delta(x) - \int_{-\infty}^{\infty} f'(x)\delta(x)dx = -f'(0)$$

In general, then,

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x)\,dx = (-1)^n f^{(n)}(0)}$$

Normal mode expansion of the delta function

Let

$$\{\phi_n(x)\} = \text{ complete orthogonal set}$$

so

$$\int \phi_n(x)\phi_m(x)dx = \delta_{mn}$$

We can expand any function $f(x)$ in terms of a complete orthogonal set, viz., $f(x) = \sum_n a_n\phi_n(x)$. Let $f(x) = \delta(x - x_0)$:

$$\delta(x - x_0) = \sum_n a_n\phi_n(x)\,. \tag{4}$$

Multiply by ϕ_m and take integral of both sides:

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(x - x_0)\phi_m(x)dx &= \sum_n \int_{-\infty}^{\infty} a_n\phi_n(x)\phi_m(x)dx \\ &= \sum_n a_n\delta_{nm} = a_m\end{aligned}$$

Using the sifting property on the left-hand side, the expansion coefficients are $a_n = \phi_n(x_0)$. Thus Eq. (4) becomes

$$\boxed{\delta(x - x_0) = \sum_n \phi_n(x)\phi_n(x_0)\,.}$$

Example: normal modes on fixed-fixed string

The modes of a string are given by

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin(n\pi x/L)\,, \quad \text{where} \quad \int_0^L \phi^2 dx = 1$$

Since $\delta(x - x_0) = \sum_n \phi_n(x)\phi_n(x_0)$, the delta function in this case is

$$\delta(x - x_0) = \frac{2}{L}\sum_n \sin(n\pi x/L)\sin(m\pi x/L)\,.$$

Note that the dimensions of a delta function are always the inverse of the differential element. More properties of the delta function are available here.

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Fourier acoustics

> *Primary causes are unknown to us; but are subject to simple and constant laws, which may be discovered by observation, the study of them being the object of natural philosophy.*

–Joseph Fourier

With the prerequisite mathematics developed, the Helmholtz equation is now solved with remarkable ease using 2D spatial Fourier transforms. These solutions are broadly referred to as *Fourier acoustics*. These solutions can also be applied to the study of waves in solids.

The first two sections on pressure and velocity sources are also covered here. The treatment is basically the same, though those notes are slightly more succinctly. (Note that hats are used instead of capital letters for Fourier-transformed quantities in those notes.)

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Pressure source

Begin with the Helmholtz equation

$$\nabla^2 p_\omega + k^2 p_\omega = 0\,. \tag{1}$$

Given field in the source plane $p_\omega(x, y, 0)$, find $p_\omega(x, y, z > 0)$. Start by taking the 2D spatial Fourier transform of Eq. (1):

$$\mathcal{F}_{xy}\left\{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)p_\omega + k^2 p_\omega\right\}$$

Using the derivative property and calling $P = P(k_x, k_y, z)$, the above becomes

$$\frac{d^2P}{dz^2} + k_z^2 P = 0\,, \quad k_z = \sqrt{k^2 - k_x^2 - k_y^2}\,.$$

For forward propagation, the above 2nd order ODE reads

$$P(k_x, k_y, z) = P(k_x, k_y, 0)e^{ik_z z} .$$

To return to physical space, the inverse 2D spatial Fourier transform is taken:

$$p_\omega(x, y, z) = \mathcal{F}_{xy}^{-1}\{P(k_x, k_y, 0)e^{ik_z z}\} .$$

Since $P(k_x, k_y, 0) = \mathcal{F}_{xy}\{p(k_x, k_y, 0)\}$, the entire field can be written in terms of the source condition:

$$\boxed{p_\omega(x, y, z) = \mathcal{F}_{xy}^{-1}\{\mathcal{F}_{xy}\{p_\omega(x, y, 0)\}e^{ik_z z}\} ,} \tag{2}$$

where the z component of the wavenumber is

$$\begin{aligned} k_z &= \sqrt{k^2 - k_x^2 - k_y^2} \\ &= \begin{cases} \text{positive real} & k_x^2 + k_y^2 < k^2 \\ \text{positive imaginary} & k_x^2 + k_y^2 > k^2 . \end{cases} \end{aligned}$$

Note that the *entire* angular spectrum (all k_x, k_y) is required to reproduce the exact field.

For axisymmetric sources, Eq. (2) becomes

$$\begin{aligned} p_\omega(r, z) &= \mathcal{H}^{-1}\{\mathcal{H}\{p_\omega(\rho, 0)\}e^{ik_z z}\} \\ k_z &= \sqrt{k^2 - \kappa^2} , \quad \kappa^2 = k_x^2 + k_y^2 . \end{aligned}$$

Example: Bessel beams

This topic was originally presented after the Fraunhofer approximation/before the Fresnel approximation were discussed. However, it serves as a good example of Fourier acoustics used for an axisymmetric pressure source.

Bessel (nondiffracting) beams were "discovered" in optics in the 80s [Durnan, JOSA A, **4**, 651 (1987)].

Consider the pressure source

$$p_\omega(x, y, 0) = p_\omega(\rho) = p_0 J_0(\alpha\rho) ,$$

where α is a constant, not a direction cosine. The solution is found by taking

$$p(\rho, z) = \mathcal{H}^{-1}\{\mathcal{H}\{p_0(\rho)\}e^{ik_z z}\} ,$$

where $k_z = \sqrt{k^2 - \kappa^2}$. The Hankel transform is found by looking at the delta function handout:

$$\begin{aligned} \mathcal{H}\{p_0(\rho)\} &= p_0 \int_0^\infty J_0(\alpha\rho)J_0(\kappa\rho)\rho\, d\rho \\ &= \frac{p_0}{\alpha}\delta(\kappa - \alpha) . \end{aligned}$$

Thus the pressure field is

$$\begin{aligned} p_0(\rho, z) &= \frac{p_0}{\alpha} \int_0^\infty \delta(\kappa - \alpha) e^{i(k^2-\kappa^2)^{1/2}} J_0(\kappa\rho)\kappa\, d\kappa \\ &= p_0 J_0(\alpha\rho) e^{i(k^2-\alpha^2)^{1/2} z} \\ &\propto J_0(\alpha\rho) \text{ for all } z\,, \end{aligned}$$

i.e., there is no diffraction in the field. Note that the beam propagates if α is less than k. The phase speed is

$$c_{\text{ph}} = \frac{\omega}{k_z} = \frac{\omega}{\sqrt{k^2 - \alpha^2}} = \frac{c_0}{\sqrt{1 - (\alpha/k)^2}}\,.$$

The glaring issue with the realization of such a non-diffracting beam is that

$$J_0(\alpha\rho) \propto \frac{1}{\sqrt{\rho}}\,, \quad \alpha\rho \gg 1\,,$$

so the source power is

$$W = \oint \mathbf{I} \cdot d\mathbf{S} = \frac{p_0^2}{2\rho_0 c_0} \int_0^\infty J_0^2(\alpha\rho)\pi\rho\, d\rho \to \infty\,.$$

That is to say, an infinite amount of energy is required to generate such a beam! For practical (local) realization of a Bessel beam, the source needs to be truncated before a and 0 outside, i.e.,

$$p_0(\rho) = p_0 J_0(\alpha\rho)\,, \quad \rho \le a\,.$$

Velocity source

For a velocity source (which is more practical), we should recall the linearized Newton's law:

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p = 0\,,$$

or for time-harmonic waves, $-i\omega\rho_0\mathbf{u}_\omega + \nabla p_\omega = 0$. Solving for $\mathbf{u}$ gives

$$\mathbf{u}_\omega = \frac{\nabla p_\omega}{ik\rho_0 c_0}\,.$$

The 2D spatial Fourier transform of the left-hand side of the above is simply

$$\mathcal{F}_{xy}\{\mathbf{u}_\omega(x, y, z)\} = \mathbf{U}(k_x, k_y, z)\,.$$

Meanwhile, for the right-hand side, note that by the derivative property of the Fourier transform,

$$\begin{aligned}\mathcal{F}_{xy}\Big\{\nabla p_\omega(x,y,z)\Big\} &= \mathcal{F}_{xy}\Bigg\{\Big(\mathbf{e}_x\partial_x + \mathbf{e}_y\partial_y + \mathbf{e}_z\partial_z\Big)p_\omega\Bigg\} \\ &= i(k_x\mathbf{e}_x + k_y\mathbf{e}_y)P(k_x,k_y,z) + \mathbf{e}_z\mathcal{F}_{xy}\left\{\frac{\partial p_\omega}{\partial z}\right\}.\end{aligned}$$

But since $P(k_x,k_y,z) = P(k_x,k_y,0)e^{ik_z z}$, the last term of the above, $\mathbf{e}_z\mathcal{F}_{xy}\{\partial p_\omega/\partial z\}$, is

$$\begin{aligned}\mathbf{e}_z\mathcal{F}_{xy}\left\{\frac{\partial p_\omega}{\partial z}\right\} &= \mathbf{e}_z\mathcal{F}_{xy}\left\{\frac{\partial}{\partial z}\mathcal{F}_{xy}^{-1}[P(k_x,k_y,0)e^{ik_z z}]\right\} \\ &= \mathbf{e}_z\mathcal{F}_{xy}\{\mathcal{F}_{xy}^{-1}ik_z[P(k_x,k_y,0)e^{ik_z z}]\} \\ &= \mathbf{e}_z ik_z P(k_x,k_y,0)e^{ik_z z} \\ &= \mathbf{e}_z ik_z P(k_x,k_y,z)\,.\end{aligned}$$

So

$$\begin{aligned}\mathcal{F}_{xy}\{\nabla p_\omega(x,y,z)\} &= i(k_x\mathbf{e}_x + k_y\mathbf{e}_y)P(k_x,k_y,z) + \mathbf{e}_z ik_z P(k_x,k_y,z) \\ &= i\mathbf{k}P(k_x,k_y,z)\,,\end{aligned}$$

where

$$\mathbf{k} = k_x\mathbf{e}_x + k_y\mathbf{e}_y + (k^2 - k_x^2 - k_y^2)^{1/2}\mathbf{e}_z = \mathbf{k}(k_x,k_y)\,.$$

Since $\mathbf{u}_\omega = \frac{\nabla p_\omega}{ik\rho_0 c_0}$, its 2D spatial Fourier transform is

$$\mathbf{U}(k_x,k_y,z) = \frac{1}{\rho_0 c_0}\frac{\mathbf{k}}{k}P(k_x,k_y,z)\,. \tag{3}$$

Assuming that $\mathbf{u}_\omega(x,y,0) = u_0(x,y)\mathbf{e}_z$, Eq. (3) evaluated in the source plane $z = 0$ is

$$\begin{aligned}\mathbf{U}_0(k_x,k_y) &= \mathcal{F}_{xy}\{u_0(x,y)\}\mathbf{e}_z = U_0(k_x,k_y)\mathbf{e}_z \\ &= \frac{1}{\rho_0 c_0}\frac{k_z\mathbf{e}_z}{k}P_0(k_x,k_y)\end{aligned}$$

Solving for the 2D spatial Fourier transform of the source pressure gives

$$P_0(k_x,k_y) = \rho_0 c_0\frac{k}{k_z}U(k_x,k_y)\,.$$

Since $p_\omega(x,y,z) = \mathcal{F}_{xy}^{-1}\{P_0(k_x,k_y)e^{ik_z z}\}$, the full field due to a velocity source is

$$\boxed{p_\omega(x,y,z) = \rho_0 c_0\mathcal{F}_{xy}^{-1}\left\{U_0(k_x,k_y)\frac{k}{k_z}e^{ik_z z}\right\}}\,,$$

where $P_0(k_x,k_y) = \mathcal{F}_{xy}\{p_0(x,y)\}$. Note that there is a singularity at $k_z = 0$, corresponding to the perimeter of the radiation circle.

Intensity

How does one calculate intensity using Fourier acoustics? Recall that

$$\begin{aligned}\mathbf{I} &= \langle p\mathbf{u}\rangle \\ &= \langle \Re\{p_\omega e^{-i\omega t}\}\Re\{\mathbf{u}_\omega e^{-i\omega t}\}\rangle \\ &= \frac{1}{2}\Re\{p_\omega^* \mathbf{u}_\omega\} = \frac{1}{2}\Re\{p_\omega \mathbf{u}_\omega^*\}\,,\end{aligned}$$

where

$$\begin{aligned} p_\omega(x,y,z) &= \mathcal{F}_{xy}^{-1}\{P(k_x,k_y,z)\} \\ \text{and}\quad \mathbf{u}_\omega(x,y,z) &= \mathcal{F}_{xy}^{-1}\{\mathbf{U}(k_x,k_y,z)\} = \mathcal{F}_{xy}^{-1}\{U_x\mathbf{e}_x + U_y\mathbf{e}_y + U_z\mathbf{e}_z\}\,.\end{aligned}$$

Code

The whole code is nondimensionalized. The z axis is nondimensionalized by z_0, the Rayleigh distance, sometimes called the "collumnation distance" or "diffraction length" $ka^2/2 \sim S/\lambda$ (surface area to wavelength). The transverse axis is nondimensionalized by the source radius a ($R = r/a$), as is the wavenumber ($K = ka$). The code is set up to handle a velocity source, but it can easily be modified to handle pressure sources by implementing the discussion above.

The code itself contains all of these definitions. This sheet contains a full explanation and discussion of the code.

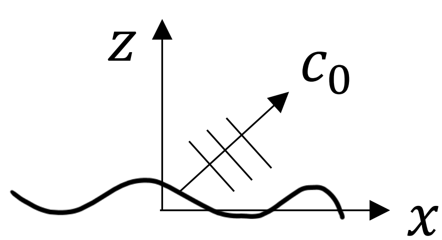
Radiation from plate due to 1D bending wave

The bending wave has the form

$$\begin{aligned}\mathbf{u}(x,y,0,t) &= u_0 e^{i(k_b x - \omega t)}\mathbf{e}_z \qquad k_b = \omega/c_b \\ &= u_0(x)e^{-i\omega t}\end{aligned}$$

where the normal particle velocity on the surface is

$$u_0(x) = u_0 e^{ik_b}\,.$$



To calculate the field in the fluid medium, first take the 2D spatial Fourier transform of the source condition,

$$\begin{aligned} U_0(k_x, k_y) &= \mathcal{F}_{xy}\{u_0(x)\} \\ &= u_0 \iint_{-\infty}^{\infty} e^{-i(k_x-k_b)x-ik_y y} dx\, dy \\ &= 4\pi^2 u_0 \delta(k_x - k_b)\delta(k_y)\,. \end{aligned}$$

Now convert to a pressure source using the result from the section on velocity sources.

$$\begin{aligned} P(k_x, k_y) &= \rho_0 c_0 \frac{k}{k_z} U_0(k_x, k_y) \\ &= 4\pi^2 \rho_0 c_0 u_0 \frac{k}{k_z} \delta(k_x - k_b)\delta(k_y)\,. \end{aligned}$$

The pressure field is found by the standard Fourier acoustics procedure:

$$\begin{aligned} p_\omega(x, y, z) &= \mathcal{F}_{xy}^{-1}\{P_0(k_x, k_y) e^{ik_z z}\} \\ &= 4\pi^2 \rho_0 c_0 u_0 k\, \mathcal{F}_{xy}^{-1}\{\delta(k_x - k_b)\delta(k_y) e^{ik_z z}/k_z\} \\ &= \rho_0 c_0 u_0 k \iint_{-\infty}^{\infty} \delta(k_x - k_b)\delta(k_y) \frac{e^{i(k^2-k_x^2-k_y^2)^{1/2}}}{\sqrt{k^2 - k_x^2 - k_y^2}} e^{ik_x x + ik_y y} dk_x dk_y \\ &= \frac{\rho_0 c_0 u_0 k}{\sqrt{k^2 - k_b^2}} e^{ik_b x + i(k^2-k_b^2)^{1/2} z}\,. \end{aligned}$$

If $k > k_b$ (or $c_b > c_0$), then the wave propagates into the fluid, while if $k < k_b$ (or $c_b < c_0$), then the wave is evanescent in the fluid.

For additional insight, convert pressure to the particle velocity:

$$\begin{aligned} \mathbf{u}_\omega(x, y, z) &= \nabla p_\omega / ik\rho_0 c_0 \\ &= u_0 \frac{k_b \mathbf{e}_x}{\sqrt{k^2 - k_b^2}} e^{ik_b x + i\sqrt{k^2-k_b^2} z}\,. \end{aligned}$$

The intensity can then be calculated:

$$\begin{aligned} \mathbf{I} &= \frac{1}{2}\Re\{p_\omega \mathbf{u}_\omega^*\} \\ &= \frac{\rho_0 c_0 u_0^2}{2(1 - k_b^2/k^2)} \left[\frac{k_b}{k}\mathbf{e}_x + \sqrt{1 - k_b^2/k^2}\,\mathbf{e}_z\right] \quad k_b < k \\ &= \frac{\rho_0 c_0 u_0^2}{2(1 - k_b^2/k^2)} \frac{k}{k_b} e^{-2(k_b^2-k^2)^{1/2} z} \quad k_b > k\,. \end{aligned}$$

In the second case, there is no power radiated in the z direction.

As a sanity check, what if $c_b \to \infty$? Then one should obtain a plane wave in the z direction. Indeed, for $k_b \to 0$,

$$p_\omega(x,y,z) = \rho_0 c_0 u_0 e^{ikz} .$$

Generalization to elastic waves in isotropic solids

Let ξ be the particle displacement. It can be separated into a irrotational and rotational components,

$$\xi = \xi_l + \xi_t ,$$

where ξ_l is the longitudinal (compressional) wave displacement ($\nabla \times \xi_l = \mathbf{0}$), and where ξ_t is the shear (transverse) wave displacement ($\nabla \cdot \xi_t = 0$). These two displacements above satisfy wave equations of their own. This was shown without proof, but the derivation is worked out from first principles here.

Compressional waves

The compressional wave equation is

$$\nabla^2 \xi_l = \frac{1}{c_l^2} \frac{\partial \xi_l}{\partial t^2}$$

where the longitudinal sound speed is

$$c_l = \sqrt{\frac{K + 4\mu/3}{\rho}} ,$$

where K is the bulk modulus and μ is the shear modulus. By the Helmholtz decomposition theorem, ξ_ℓ is defined as the gradient of a scalar potential ϕ,

$$\xi_l = \nabla \phi ,$$

giving

$$\nabla^2 \phi = \frac{1}{c_l} \frac{\partial^2 \phi}{\partial t^2} .$$

Assuming ϕ is time harmonic results in the Helmholtz equation, which is the same as for waves in fluids:

$$\nabla^2 \phi + k^2 \phi = 0 .$$

Thus we can swap out p for ϕ in Fourier theory presented earlier, and one can obtain the full compressional wave field:

$$\phi_\omega(x,y,z) = \mathcal{F}_{xy}^{-1}\{\mathcal{F}_{xy}[\phi_\omega(x,y,0)]e^{ik_z z}\} .$$

Shear waves

The wave equation for shear waves is vectorial, i.e.,

$$\nabla^2 \xi_t = \frac{1}{c_t^2}\frac{\partial \xi_t}{\partial t^2} \qquad c_t = \sqrt{\mu/\rho} \tag{1}$$

The other condition noted above is that $\nabla \cdot \xi_t = 0$, which is often utilized to to defined terms of the vector potential $\xi_t = \nabla \times \psi$ (analogous to how the longitudinal displacement was defined in terms of a scalar potential above). However, even with the vector potential, we would still have three components, so let us follow Landau and Lifshitz and simply use the Cartesian components of ξ_t,

$$\xi_t = \xi_x \mathbf{e}_x + \xi_y \mathbf{e}_y + \xi_z \mathbf{e}_z ,$$

such that Eq. (1) becomes

$$\nabla^2 \xi_x + k_t^2 \xi_x = 0 \tag{2}$$
$$\nabla^2 \xi_y + k_t^2 \xi_y = 0 \tag{3}$$
$$\nabla^2 \xi_z + k_t^2 \xi_z = 0 . \tag{1}$$

Note that in the equations above, only two components are independent, because the third is related by $\nabla \cdot \xi_t = 0$, which in Cartesian coordinates reads

$$\frac{\partial \xi_x}{\partial x} + \frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z} = 0 .$$

Eqs. (2) and (3) are solved separately using Fourier acoustics:

$$\xi_x(x, y, z) = \mathcal{F}_{xy}^{-1}\{\mathcal{F}_{xy}\{\xi_x(x, y, 0)\}e^{ik_z z}\} .$$

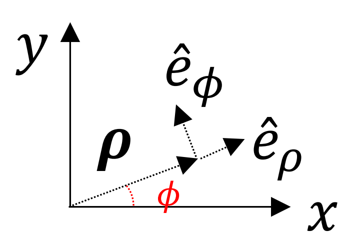
Example: Axisymmetric shear waves

Torsional waves

In torsional waves, the transverse displacement is given by

$$\xi_t(\rho, \phi, z = 0) = \xi_\phi(\rho)\mathbf{e}_\phi ,$$

where the coordinates and unit vectors are shown below:



Then

$$\begin{aligned}\xi_x &= -\xi_\phi(\rho)\sin\phi \\ \xi_y &= \xi_\phi(\rho)\cos\phi \\ \xi_z &= 0\end{aligned}$$

and

$$\nabla\cdot\xi_t = \frac{1}{\rho}\frac{\partial(\rho\xi_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial\xi_\phi}{\partial\phi} + \frac{\partial\xi_z}{\partial z}$$

Since $\xi_t(\rho,\phi,z=0) = \xi_\phi(\rho)\mathbf{e}_\phi$, the above equation reduces to $\nabla\cdot\psi = 0$. Meanwhile, the Laplacian in cylindrical coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\,.$$

Thus the Laplacian of the wave variable in Eqs. (2) and (3) read

$$\begin{aligned}\nabla^2\xi_x &= \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right)(-\xi_\phi\sin\phi) \\ \nabla^2\xi_y &= \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right)(\xi_\phi\cos\phi)\,,\end{aligned}$$

and both Eqs. (2) and (3) yield

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)(\xi_\phi\cos\phi)$$

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2}\right)\xi_\phi + \frac{\partial^2\xi_\phi}{\partial z^2} + k_t^2\xi_\phi \tag{5}$$

The above equation is now solved using Hankel transforms. Define the nth order Hankel transform pair:

$$\begin{aligned}F_H^n(\kappa) &= \mathcal{H}^n\{f(\rho)\} = \int_0^\infty f(\rho)J_n(\kappa\rho)\rho d\rho \\ f(\rho) &= \mathcal{H}^{-1}{}_n\{F_H^n(\kappa)\} = \int_0^\infty F_H^{(n)}(\kappa)J_n(\kappa\rho)\kappa d\kappa\,.\end{aligned}$$

Because of the identity

$$\int_0^\infty J_n(\kappa\rho)J_n(\kappa\rho')\kappa d\kappa = \frac{\delta(\rho-\rho')}{\rho}$$

and

$$\mathcal{H}^{-1}{}_n\left\{\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{n^2}{\rho^2}\right)f(\rho)\right\} = -k^2F_H{}^n(\kappa)$$

So Eq. (5) becomes

$$\frac{d^2\hat{\xi}_\phi}{dz^2} + (k_t^2 - \kappa^2)\hat{\xi}_\phi = 0$$

Thus the solution for the torsional wave field is

$$\xi_\phi(\rho, z) = \mathcal{H}^{-1}{}_1\{\mathcal{H}_1[\xi_\phi(\rho, 0)]e^{ik_z z}\}$$

Example: Uniform circular motion:

$$\begin{aligned}
\xi_\phi(\rho) &= \xi_0 \frac{\rho}{a} \,\text{circ}\,(\rho/a) \\
\hat{\xi}(\kappa) &= \mathcal{H}_1\{\xi_\phi(\rho)\} \\
&= \frac{\xi_0}{a}\int_0^a \rho J_1(\kappa\rho)\rho d\rho, \quad t = \kappa\rho \\
&= \frac{\xi_0}{\kappa^3 a}\int_0^{\kappa a} J_1(t)t^2 dt
\end{aligned}$$

Note that

$$\int_0^x J_{n-1}(t)t^n dt = x^n J_n(x)$$

so

$$\begin{aligned}
\hat{\xi}_\phi(\kappa) &= \xi_0 \frac{a}{\kappa} J_2(\kappa a) \\
&= 0 \quad \kappa - 0,
\end{aligned}$$

because $J_2(x) = x^2/8 + \mathcal{O}(x^4)$ The directivity is thus proportional to

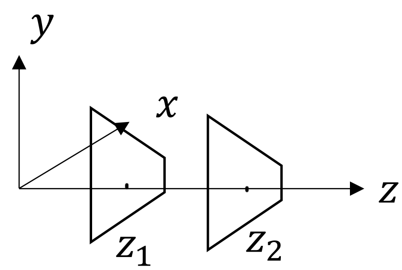
$$\left|\frac{J_2(\kappa a)}{\kappa a}\right|,$$

and thus there is a zero field on the propagation axis.

Nearfield acoustical holography (NAH)

This topic was originally presented after focused sources were discussed. See Williams pgs. 89-92.

Consider two planes, z_1 = constant and z_2 = constant, where $z_2 > z_1$. The purpose of nearfield acoustical holography (NAH) is to calculate the field near the source given the field far away. NAH can be used to investigate structural integrity of a vehicle, for example.



Given $p_\omega(x, y, z_1)$, then the field at z_2 is

$$p_\omega(x, y, z_2) = \mathcal{F}_{xy}^{-1}\{\mathcal{F}_{xy} p_\omega(x, y, z_1) e^{ik_z(z_2 - z_1)}\}, \tag{1}$$

where the projection of the wavenumber in the z direction is, for propagating and evanescent waves, respectively,

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}, \quad k_x^2 + k_y^2 \le k^2 \tag{2a}$$

$$= i\sqrt{k_x^2 + k_y^2 - k^2}, \quad k_x^2 + k_y^2 > k^2 \tag{2b}$$

Eq. (1) is inverted to calculate the pressure field in the plane z_1 = constant:

$$p_\omega(x, y, z_1) = \mathcal{F}_{xy}^{-1}\{\mathcal{F}_{xy}[p_\omega(x, y, z_2)] e^{-ik_z(z_2 - z_1)}\}, \tag{3}$$

Thus as $z_2 - z_1$ increases (z_1 moved farther to the left with z_2 fixed), the evanescent waves *grow*. Thus Eq. (3) is not a *physical* solution to the wave equation. Normally, z_2 and z_1 are defined as

$$z_2 = z_H = \text{ hologram plane}$$
$$z_1 = z_I = \text{ image plane},$$

and the z component of $\mathbf{u}$ is normally desired at $z = z_I$. Let the normal component of the particle velocity in the image plane be

$$u_0(x, y, z_I) = \mathbf{u}(x, y, z_I) \cdot \mathbf{e}_z .$$

Then the Fourier transform of the normal velocity field in the image plane is

$$U_0(k_x, k_y, z_I) = \frac{1}{\rho_0 c_0} \frac{k_z}{k} P_0(k_x, k_y, z_I) .$$

Eq. (3) can thus converted to give the partical velocity in the image plane, z_1 = constant:

$$u_0(x, y, z_I) = \mathcal{F}_{xy}^{-1}\{\mathcal{F}_{xy}[p_\omega(x, y, z_H)] G_H(k_x, k_y, z_H - z_I)\},$$

where the propagator above is

$$G_H(k_x, k_y, z_H - z_I) = \frac{1}{\rho_0 c_0} \frac{k_z}{k} \begin{cases} e^{-i(k^2 - k_x^2 - k_y^2)^{1/2}(z_H - z_I)}, & k_x^2 + k_y^2 \le k^2 \\ e^{(k_x^2 + k_y^2 - k^2)^{1/2}(z_H - z_I)}, & k_x^2 + k_y^2 > k^2 \end{cases}$$

The engineering challenge is to have $|z_H - z_I| \lesssim \lambda$ to measure the evanescent waves, without which resolution is limited to $\sim \lambda$, because of the restriction that $k_x^2 + k_y^2 \le k^2$ for propagating waves, or

$$\frac{1}{\lambda_{x,y}} \leq \frac{1}{\lambda^2} \implies \lambda_{x,y}^2 \geq \lambda^2 .$$

Thus λ is the minimum resolvable spatial scale if only propagating waves are measured in the hologram plane z_H = constant.

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Rayleigh integrals

> *The only merit of which I personally am conscious was that of having pleased myself by my studies, and any results that may be due to my researches were owing to the fact that it has been a pleasure for me to become a physicist.*
>
> –John Strutt, 3rd Baron Rayleigh

Fourier acoustics is now used to derive Rayleigh's famous diffraction integrals. The plane wave decomposition (angular spectrum) of a spherical wave is first needed to derive these integrals.

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Angular specturm of a spherical wave

Note that this topic is covered in Brekhovskikh, pages 228-229.

Let $f(x, y, z) = e^{ikr}/r$ be the functional dependence of the radiation due to a point source at $r = 0$ (i.e., the amplitude factor has been neglected), where $r = \sqrt{x^2 + y^2 + z^2}$ and $\rho = \sqrt{x^2 + y^2}$. To obtain the plane wave decomposition (angular spectrum) of the spherical wave in all space, first calculate the 2D spatial Fourier transform of the wave at the source $f(x, y, 0)$:

$$\begin{aligned} F_0(\boldsymbol{\kappa}) = F_0(k_x, k_y) = \mathcal{F}_{xy}\{f(x, y, 0)\} &= \iint_{-\infty}^{\infty} \frac{e^{ik\rho}}{\rho} e^{-i(k_x x + k_y y)}\, dx\, dy \\ = \mathcal{F}_{xy}\{f(x, y, 0)\} &= \iint_{-\infty}^{\infty} \frac{e^{ik\rho}}{\rho} e^{-i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}}\, \rho\, d\rho\, d\phi \end{aligned}$$

As before, due to axisymmetry, take $\boldsymbol{\kappa} = \kappa \mathbf{e}_x$, giving

$$\boldsymbol{\kappa} \cdot \boldsymbol{\rho} = \kappa \rho \cos\phi\,.$$

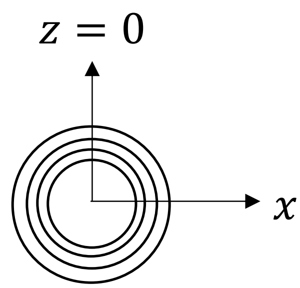
So

$$F_0(\kappa) = \int_0^{2\pi} d\phi \int_0^{\infty} e^{i(k - \kappa\cos\phi)\rho} d\rho\,.$$

This is a tricky integral to take, since the integrand of the ρ integral is oscillatory. To evaluate the integral, let $k = k + i\epsilon$ and then take $\epsilon \to 0$.

$$
\begin{aligned}
F_0(\kappa) &= \int_0^{2\pi} d\phi \frac{e^{i(k-\kappa\cos\phi)}e^{-\epsilon\rho}}{i(k-\kappa\cos\phi)-\epsilon}\Bigg|_{\rho=0}^{\rho=\infty} \\
&= -\int_0^{2\pi} \frac{d\phi}{i(k-\kappa\cos\phi)\rho-\epsilon} \\
&= i\int_0^{2\pi} \frac{d\phi}{k-\kappa\cos\phi} = i\frac{2\pi}{\sqrt{k^2-\kappa^2}} = \frac{i2\pi}{k_z}\,.
\end{aligned}
$$

Note that e^{ikr}/r is symmetric about the plane $z = 0$, as shown below:



With the Fourier transform of the source condition taken, the full field can therefore be calculated for all z (both positive and negative) using the standard Fourier acoustics procedure:

$$
\begin{aligned}
f(x,y,z) = e^{ikr}/r &= \mathcal{F}_{xy}^{-1}\{F_0(\kappa)e^{ik_z|z|}\} \\
&= \mathcal{F}_{xy}^{-1}\left\{\frac{i2\pi}{k_z}e^{ik_z|z|}\right\}.
\end{aligned}
$$

Taking the 2D spatial Fourier transform of both sides above gives the plane wave decomposition of a spherical wave:

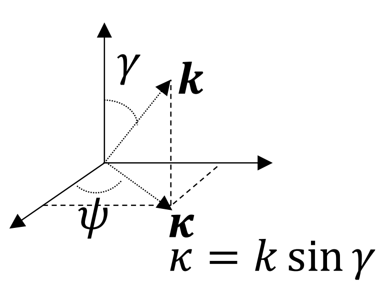
$$
\boxed{\mathcal{F}_{xy}\{e^{ikr}/r\} = i2\pi\frac{e^{ik_z|z|}}{k_z}\,.}
$$

Since $F(\kappa) = 2\pi F_H(\kappa)$, the above can conveniently be written in terms of the Hankel transform $\mathcal{H}_1$,

$$
\mathcal{H}_1\{e^{ikr}/r\} = i\frac{e^{ik_z|z|}}{k_z}\,.
$$

Aside: evanescent waves in spherical radiation

Recall that the wavevector **k** is related to its projection in the x-y plane by



Thus the Fourier transform of the source condition is

$$\begin{aligned} F_0(\kappa) &= \frac{i2\pi}{\sqrt{k^2-\kappa^2}} \\ &= \frac{i2\pi}{k\sqrt{1-\sin^2\gamma}} \\ &= \frac{i2\pi}{k\cos\gamma}\,, \quad \gamma \le 90^\circ\,. \end{aligned}$$

The meaning of the evanescent waves in the plane wave decomposition of a spherical wave is not clear. For more insight into the presence of the evanescent waves in the spectrum of a converging spherical wave, see these notes by Jackson S. Hall.

Derivation of first Rayleigh integral using Fourier acoustics

This is *not* how Rayleigh derived his integral. For that, see Prob. 6 here.

Given an arbitrary normal velocity distribution

$$\mathbf{u} = (x,y,0,t) = u_0(x,y)e^{-i\omega t}\mathbf{e}_z\,.$$

Follow the recipe for the calculation of the field. The 2D spatial Fourier transform of the source condition, and its mapping to a pressure source, is

$$\begin{aligned} U_0(k_x,k_y,0) &= \mathcal{F}_{xy}\{u_0(x,y)\}\,, \\ P_0(k_x,k_y,0) &= \rho_0 c_0 \frac{k}{k_z} U(k_x,k_y,0)\,. \end{aligned}$$

Then the solution to the Helmholtz equation is

$$\begin{aligned} p_\omega(x,y,z) &= \mathcal{F}_{xy}^{-1}\{P_0(k_x,k_y)e^{ik_z z}\} \\ &= \rho_0 c_0 k\, \mathcal{F}_{xy}^{-1}\{U_0(k_x,k_y)e^{ik_z z}/k_z\} \\ &= \rho_0 c_0 k\, \mathcal{F}_{xy}^{-1}\{U_0(k_x,k_y)\} ** \mathcal{F}_{xy}^{-1}\{e^{ik_z z}/k_z\} \\ &= \rho_0 c_0 k\, u(x,y) ** \frac{e^{ikr}}{i2\pi r}\,, \end{aligned}$$

where $r = \sqrt{x^2+y^2+z^2}$, and where the result from the previous section, $\mathcal{F}_{xy}\{e^{ikr}/r\} = i2\pi e^{ik_z|z|}/k_z$, has been used to evaluate the third line above. Writing the convolution explicitly and denoting

$$R = \sqrt{(x-x_0)^2+(y-y_0)^2+z^2}$$

results in the so-called *Rayleigh integral of the first kind* of acoustics:

$$p_\omega(x,y,z) = -\frac{ik\rho_0 c_0}{2\pi} \iint_{-\infty}^{\infty} u_0(x_0,y_0) \frac{e^{ikR}}{R} dx_0 dy_0 \,.$$

Example: Point velocity source (baffled monopole):

The source condition for a point velocity source is

$$u_0(x,y) = Q\delta(x)\delta(y)\,,$$

where Q is a volume velocity. Insertion of this source condition into the Rayleigh integral gives

$$\begin{aligned} p_\omega(r) &= -\frac{ik\rho_0 c_0 Q}{2\pi} \iint_{-\infty}^{\infty} \delta(x_0)\delta(y_0) \frac{e^{ikR}}{R} dx_0\, dy_0 \\ &= -\frac{ik\rho_0 c_0 Q}{2\pi} \frac{e^{ikr}}{r} = -i\omega Q \frac{\rho_0 e^{ikr}}{2\pi r}\,. \end{aligned}$$

This is the expression for a monopole radiating in a half-space. Multiply by $1/2$ for the expression for the radiation due to a monopole in free space.

Derivation of second Rayleigh integral using Fourier acoustics

Note that this is called the first Rayleigh integral of optics, because it is more common in optics to calculate the electric field beyond the source in terms of the electric field at the source.

Now consider a *pressure* source

$$p(x,y,0,t) = p_0(x,y)e^{-i\omega t}\,.$$

Then take its 2D spatial Fourier transform:

$$P_0(k_x,k_y) = \mathcal{F}_{xy}\{p_0(x,y)\}\,.$$

The full field is given by

$$\begin{aligned} p_\omega(x,y,z) &= \mathcal{F}_{xy}^{-1}\{P_0(k_x,k_y)e^{ik_z z}\} \\ &= p_0(x,y) * *\mathcal{F}_{xy}^{-1}\{e^{ik_z z}\} \end{aligned}$$

Note that

$$\begin{aligned} \mathcal{F}_{xy}^{-1}\{e^{ik_z z}\} &= \mathcal{F}_{xy}^{-1}\{ik_z e^{ik_z z}/ik_z\} \\ &= \frac{\partial}{\partial z}(-e^{ikr}/2\pi r) \end{aligned}$$

where the first line holds by multiplication and division by 1, and where second line holds because $\mathcal{F}_{xy}\{e^{ikr}/r\} = (i2\pi/k_z)e^{ik_z|z|}$ and thus $-e^{ikr}/2\pi r = -\mathcal{F}_{xy}^{-1}\{(i/k_z)e^{ik_z|z|}\} = \mathcal{F}_{xy}^{-1}\left\{\frac{1}{ik_z}e^{ik_z|z|}\right\}$, so

$$\mathcal{F}_{xy}^{-1}\{e^{ik_z|z|}\} = \mathcal{F}_{xy}^{-1}\left\{\frac{\partial}{\partial z}\left[\frac{1}{ik_z}e^{ik_z|z|}\right]\right\} = \frac{\partial}{\partial z}\mathcal{F}_{xy}^{-1}\left\{\left[\frac{1}{ik_z}e^{ik_z|z|}\right]\right\}$$
$$= \frac{\partial}{\partial z}(-e^{ikr}/2\pi r)\,.$$

So the pressure field can be written as a convolution:

$$p_\omega(x,y,z) = p_0(x,y) ** \frac{\partial}{\partial z}(-e^{ikr}/2\pi r)\,.$$

Writing the convolution explicitly results in

$$p_\omega(x,y,z) = -\frac{1}{2\pi}\iint_{-\infty}^{\infty} p_0(x_0,y_0)\frac{\partial}{\partial z}\frac{e^{ikR}}{R}dx_0dy_0\,.$$

Noting that

$$\frac{\partial}{\partial z}(e^{ikR}/R) = (ik - 1/R)\frac{e^{ikR}}{R}\frac{\partial R}{\partial z}$$

and

$$\frac{\partial R}{\partial z} = \frac{\partial}{\partial z}\sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}$$
$$= \frac{2z}{2R} = z/R\,,$$

the partial derivative in the above integral becomes

$$\frac{\partial}{\partial z}(e^{ikR}/R) = (ik - 1/R)\frac{e^{ikR}}{R}\frac{z}{R}$$
$$= ikz(1 - 1/ikR)\frac{e^{ikR}}{R^2}\,.$$

The integral above then becomes the second Rayleigh integral of acoustics,

$$\boxed{p_\omega(x,y,z) = -\frac{ikz}{2\pi}\iint_{-\infty}^{\infty} p_0(x_0,y_0)(1 - 1/ikR)\frac{e^{ikR}}{R^2}dx_0dy_0\,.}$$

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Fraunhofer approximation

> *The number of different optical phenomena has become in our time so great that caution must be taken so as to avoid being deceived, and also to refer the phenomena to the simple laws.*
>
> –Joseph von Fraunhofer

The Fraunhofer approximation (far-field approximation) of the first Rayleigh integral is derived. A discussion on beamwidth follows. An engineering application of beam steering is then presented. Reflection problems involving diffracting beams and "beam displacement" are also discussed. The section concludes with a short discussion on arrays.

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Derivation

The Fraunhofer approximation will now be derived from the first Rayleigh integral,

$$p_\omega(x,y,z) = -\frac{ik\rho_0 c_0}{2\pi}\iint_{-\infty}^{\infty} u(x_0,y_0)\frac{e^{ikR}}{R}dx_0dy_0\,. \tag{1}$$

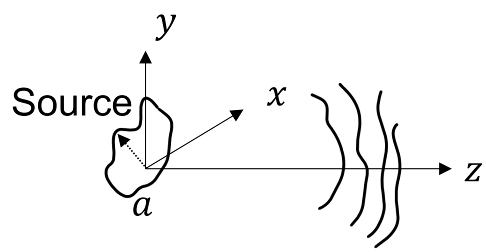
Begin by considering the source-to-listener distance R, and factor out $r = \sqrt{x^2+y^2+z^2}$:

$$\begin{aligned} R &= \sqrt{(x-x_0)^2+(y-y_0)^2+z^2} \\ &= \sqrt{r^2-2(xx_0+yy_0)+(x_0^2+y_0^2)} \\ &= r\left(1-2\frac{xx_0+yy_0}{r^2}+\frac{x_0^2+y_0^2}{r^2}\right)^{1/2}. \end{aligned}$$

Recall the first-order binomial expansion, $(1+\epsilon)^n \simeq 1+n\epsilon$. Binomially expanding the above accordingly gives

$$R = r - \frac{xx_0+yy_0}{r} + \frac{x_0^2+y_0^2}{2r} + \mathcal{O}(1/r^3)\,.$$

Now consider a finite velocity source of characteristic size a:



The contribution to the field from the velocity source is negligible for $x^2 + y^2 \geq a^2$. Thus the integrand in Eq. (1) is negligible for $x_0^2 + y_0^2 \geq a^2$. So if

$$k\frac{x_0^2 + y_0^2}{2r} = \mathcal{O}(ka^2/2r) \ll 1$$

then $r \gg \frac{1}{2}ka^2$. Thus the exponential factor in the Eq. (1) becomes

$$e^{ikR} \simeq \exp\left(ikr - ik\frac{xx_0 + yy_0}{r}\right). \tag{2}$$

Define

$$z_0 = \frac{1}{2}ka^2 = \pi a^2/\lambda \sim \frac{S}{\lambda}$$

as the Rayleigh distance. (For the case of a circular piston, the Rayleigh distance is exactly the ratio of the surface area of the source to the wavelength.)

Meanwhile the big R is approximated as r. That is to say, the difference in amplitude between two radiating elements in the source plane is negligible (whereas the phase difference between these two elements has been approximated as the binomial expansion). So, for $r \gg z_0$, Eq. (1) becomes

$$p_\omega(x, y, z) = -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikr}}{r}\iint_{-\infty}^{\infty} u_0(x_0, y_0)e^{-i(kxx_0/r + kyy_0/r)}dx_0 dy_0\,.$$

The above can be written as

$$p_\omega(x, y, z) = -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikr}}{r}\mathcal{F}_{xy}\{u_0(x, y)\}\Big|_{k_x = kx/r\,, k_y = ky/r}\,.$$

In spherical coordinates, this reads

$$\boxed{p_\omega(r, \theta, \phi) = -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikr}}{r}U_0(k\alpha, k\beta)\,,}$$

where

$$\alpha = x/r = \sin\theta\cos\phi$$
$$\beta = y/r = \sin\theta\sin\phi\,.$$

For axisymmetric cases,

$$U_0 = U_0(\kappa)\Big|_{\kappa=k\rho/r} = U_0(k\sin\theta)\,.$$

Let us consider the range of wavenumber components that the approximation spans:

$$\begin{aligned}(k\alpha)^2 + (k\beta)^2 &= k^2\sin^2\theta\,(\cos^2\phi + \sin^2\phi) = k^2\sin^2\theta \\ &\le k^2\,, \quad \theta \le 90^\circ\,.\end{aligned}$$

Thus the Fraunhofer approximation accounts for *only the contributions that propagate*. That is to say, the half-space $0^\circ \le \theta \le 90^\circ$ covers all values of $(k\alpha, k\beta) = (k_x, k_y)$ within the radiation circle.

Example: Axial field

To describe the axial field, set $\theta = 0$ so $x = y = 0$, $\alpha = \beta = 0$, $r = z$. So

$$U_0(0,0) = Q = \text{volume velocity}\,,$$

so the axial far field is

$$\begin{aligned}p(z,t) &= -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikz}}{z}Qe^{-i\omega t} \\ &= \frac{\rho_0}{2\pi z}[-i\omega Q e^{-i\omega(t-z/c_0)}] \\ &\to \rho_0\frac{\dot{Q}(t-z/c_0)}{2\pi z}\,.\end{aligned}$$

The above expression is equivalent to the pressure field radiated by a baffled simple (point) source in an infinite half-space (whereas for free space, a 4 appears in the denominator). Note that the axial waveform is the *time derivative* of the source waveform.

Beamwidth

Consider a source with characteristic radius $\rho \sim a$. The characteristic radius of the angular spectrum is then $\kappa \sim 1/a$, because their product must be a constant. In the previous section, it was shown that in the far field, $p_\omega(\theta) \propto U_0(k\sin\theta)$, so if the κ that maximizes the angular spectrum is $\kappa_{\max} \sim 1/a$, then $k\sin\theta_{\max} \sim 1/a$, or, inverting for the maximum angle,

$$\begin{aligned}\theta_{\max} &\sim \arcsin(1/ka) \\ \theta_{\max} &\sim 1/ka\,, \quad ka \gg 1\end{aligned}$$

This shows that the beamwidth (in radians!) will roughly go as $1/ka$.

For a more precise measure of beamwidth, the half-power angle is used, which is introduced by example below.

Example: Gaussian source

Consider a Gaussian velocity source,

$$u_0(\rho) = u_0 e^{-(\rho/a)^2}\,.$$

The Fourier transform of a Gaussian is a Gaussian:

$$\begin{aligned} U_0(\kappa) &= 2\pi\mathcal{H}\{u_0(\rho)\} = \pi a^2 u_0 e^{-(\kappa a/2)^2} \\ &= \pi a^2 u_0 \exp\left[-\left(\frac{\kappa}{2/a}\right)^2\right] \\ &= \pi a^2 u_0 e^{-\frac{1}{4}(ka\sin\theta)^2} \end{aligned}$$

So in the far field,

$$p_\omega(r,\theta) \propto \frac{1}{r} e^{-\frac{1}{4}(ka\sin\theta)^2}\,.$$

The half-power angle is found by finding the angle at which the power (which is proportional to the square of the pressure) is half its value at maximum:

$$\frac{p_\omega^2(r,\theta_{\mathrm{HP}})}{p_\omega^2(r,0)} \equiv \frac{1}{2} = e^{\frac{1}{2}(ka\sin\theta_{\mathrm{HP}})^2}$$

$$\sin\theta_{\mathrm{HP}} = \frac{\sqrt{2\ln 2}}{ka} = \frac{1.2}{ka}\,, \quad \theta_{\mathrm{HP}} \simeq \frac{1.2}{ka}\,, ka \gg 1\,.$$

Example: Circular piston

Now consider a uniform circular piston velocity source,

$$u_0(\rho) = u_0 \,\mathrm{circ}\,(\rho/a)\,,$$

the Fourier transform of which is

$$\begin{aligned} U_0(\kappa) &= 2\pi\mathcal{H}\{u_0(\rho)\} \\ &= 2\pi u_0 \int_0^a J_0(\kappa\rho)\rho d\rho\,, \quad x = \kappa\rho \\ &= 2\pi \frac{u_0}{\kappa^2}\int_0^a J_0(x) x\, dx \\ &= 2\pi \frac{u_0}{\kappa^2} x J_1(x)\Big|_0^{\kappa a} \\ &= 2\pi u_0 \frac{a}{\kappa} J_1(\kappa a)\,. \end{aligned}$$

From the Fraunhofer approximation, set $\kappa = k\sin\theta$, giving

$$U_0(k \sin\theta) = \pi a^2 u_0 \frac{2J_1(ka\sin\theta)}{ka\sin\theta} \,.$$

(As a sanity check, set $\theta = 0$ and note that $\pi a^2 u_0$ is the volume velocity.) By the Fraunhofer approximation, the pressure field is thus given by

$$\begin{aligned} p_\omega(r,\theta) &= -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikr}}{r}U_0(k\sin\theta) \\ &= -i\rho_0 c_0 u_0 \frac{ka^2}{2r} D(\theta) e^{ikr} \,, \\ \text{where} \quad D(\theta) &= \frac{2J_1(ka\sin\theta)}{ka\sin\theta} \end{aligned}$$

Thus the pressure field can be written in the concise dimensionless form

$$\frac{|p_\omega(r,\theta)|}{\rho_0 c_0 u_0} = \frac{z_0}{r}|D(\theta)| \,, \quad z_0 = ka^2/2 \,.$$

The half-power angle can be found by setting

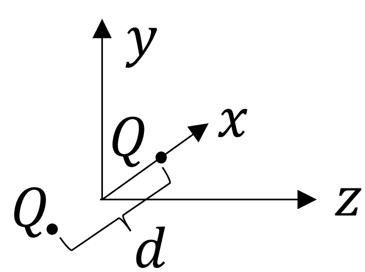
$$|D(\theta_{\text{HP}})|^2 = \frac{1}{2} \,,$$

giving

$$\theta_{\text{HP}} = \frac{1.6}{ka} \,, \quad ka \gg 1 \,.$$

Example: Baffled dipole

Note that "baffled dipole" refers to two monopoles in a half-space, as depicted below:



The velocity source condition is

$$u_0(x,y) = Q\Big[\delta(x - d/2)\delta(y) - \delta(x + d/a)\delta(y)\Big] \,,$$

and the Fourier transform of the source condition is

$$
\begin{aligned}
U_0(k_x, k_y) &= Q \iint_{-\infty}^{\infty} \left[\delta(x - d/2)\delta(y) - \delta(x + d/a)\delta(y)\right] e^{-i(k_x x + k_y y)} dx\, dy \\
&= Q \int_{-\infty}^{\infty} \left[\delta(x - d/2) - \delta(x + d/a)\right] e^{-ik_x x} dx \\
&= Q[e^{-i(k_x d/2)} - e^{i(k_x d/2)}] \\
&= -i2Q \sin(k_x d/2)\,.
\end{aligned}
$$

Then, from the Fraunhofer approximation,

$$
\begin{aligned}
p(r, \theta, \phi) &= -\frac{ik\rho_0 c_0}{2\pi} \frac{e^{ikr}}{r} U_0(k_x = k\alpha\,, k_y = k\beta) \\
&= -\frac{1}{\pi} k\rho_0 c_0 Q \frac{e^{ikr}}{r} \sin\left(\frac{kd}{2} \sin\theta \cos\phi\right).
\end{aligned}
$$

For a *point* dipole, take $d \to 0$, with Qd held fixed, identifying $D = Qd$ = dipole strength:

$$
p(r, \theta, \phi) = \frac{k^2 \rho_0 c_0 D}{2\pi} \frac{e^{ikr}}{r} \sin\theta \cos\phi\,.
$$

To obtain the result of a point dipole in a free field, simply divide the result above by 2.

Beam steering

Beam steering is an engineering application that is commonly used in underwater and biomedical acoustics.

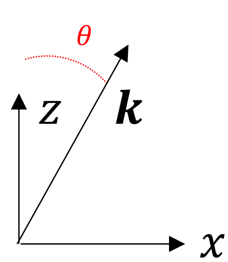
Consider plane waves propagating at an angle θ_0 with respect to z axis in the x-z plane:

$$
p_\omega(x, y, z) = p_0 e^{i\mathbf{k}\cdot\mathbf{r}} = p_0 e^{i(kx \sin\theta_0 + kz\cos\theta_0)}\,,
$$

In the source plane $z = 0$, the pressure field is

$$
p_\omega(x, y, 0) = p_0 e^{ikx\sin\theta_0}\,.
$$

Therefore, to steer an arbitrary source function [either velocity $u_0(x, y)$ or pressure $p_0(x, y)$] in the angle θ_0 with respect to z axis, simply multiply the source function by $e^{ikx\sin\theta_0}$:



Then, since $e^{ikx\sin\theta_0} e^{i\omega t} = e^{-i\omega(t - \frac{x}{c}\sin\theta_0)}$, steering is practically accomplished in the time domain with the time delay transformation

$$t \mapsto t - \frac{x}{c_0}\sin\theta_0 \,.$$

The far field pressure (due to a steered velocity source, for example) is given by the Fraunhofer approximation,

$$\begin{aligned} p_\omega(r,\theta,\phi) &= -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikr}}{r}\mathcal{F}_{xy}\{u_0(x,y)e^{ikx\sin\theta_0}\}\big|_{k_x=k\alpha\ ,k_y=k\beta} \\ &= -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikr}}{r}U_0[k(\alpha-\sin\theta_0),k\beta]\,, \end{aligned}$$

where $\alpha = \sin\theta\cos\theta$ and $\beta = \sin\theta\sin\phi$. The directivity is proportional to

$$U_0[k(\sin\theta\cos\phi - \sin\theta_0), k\sin\theta\sin\phi]\,.$$

Note that the above shfit in the angular spectrum does *not* result in a pure rotation of the field by θ_0. It resembles a rotation only if

1. $\phi \simeq 0$, i.e., close to x-z plane.
2. $\theta, \theta_0 \ll 1$, i.e., close to z axis.

If those conditions are met, then $U_0[k(\sin\theta\cos\phi - \sin\theta_0), k\sin\phi\sin\phi]$ reduces to

$$U_0 \simeq U_0[k(\theta-\theta_0),0]\,.$$

The above transformation of the angular spectrum is usually accurate for the main lobe and for small steering angles.

Example: Steered Gaussian source

The source condition is

$$u_0(x,y) = u_0 e^{-(x^2+y^2)/a^2}\,.$$

The 2D spatial Fourier transform of the source condition is

$$\begin{aligned} U_0(k_x,k_y) &= u_0\mathcal{F}_x\{e^{-x^2/a^2}\}\mathcal{F}_y\{e^{-y^2/a^2}\} \\ &= u_0\sqrt{\pi}a\,e^{-k_x^2a^2/4}\sqrt{\pi}a\,e^{-k_y^2a^2/4} \\ &= \pi a^2 u_0 e^{-(k_x^2+k_y^2)a^2/4} \end{aligned}$$

In terms of the direction cosines, the angular spectrum of the velocity source condition reads

$$\begin{aligned} &U_0[k(\sin\theta\cos\phi - \sin\theta_0, k\sin\theta\sin\phi)] \\ &\quad = \pi a^2 u_0 \exp\left\{-\frac{k^2a^2}{4}(\sin\theta\cos\phi - \sin\theta_0^2)^2 + \sin^2\theta\sin^2\phi\right\}, \end{aligned}$$

whereas a pure rotation corresponds to

$$U_0 = \pi a^2 u_0 e^{-(ka/2)^2\sin^2(\theta-\theta_0)}\,.$$

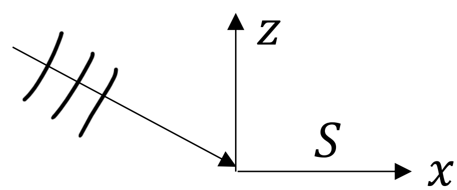
As a sanity check, in the absence of the phase shading, $\theta_0 = 0$, and setting $\phi = 0$, the above reduces to

$$U_0 = \pi a^2 e^{-(ka\sin\theta)^2/4}, \quad \theta_0 = 0 .$$

Reflection problems

Let S be the surface at $z = 0$ with a plane-wave reflection coefficient $R(\theta, \phi)$, and let $p_i(x, y, z)$ be the pressure field incident on S. This surface S may be

- a second fluid
- an elastic half-space
- a plate of thickness h
- a layered medium, etc.



Denoting the 2D spatial Fourier transform of the incident pressure at $z = 0$, and the reflection coefficient as

$$P_{i0} = \mathcal{F}_{xy}\{p_i(x, y, 0)\} \quad \text{and} \quad R(\theta, \phi) = R(k_x, k_y) ,$$

respectively, the reflected field is given by Fourier acoustics:

$$p_r(x, y, z) = \mathcal{F}_x^{-1}\{R(k_x, k_y)P_{i0}(k_x, k_y)e^{ik_z z}\} ,$$

where

$$k_x = k\alpha , \quad k_y = k\beta .$$

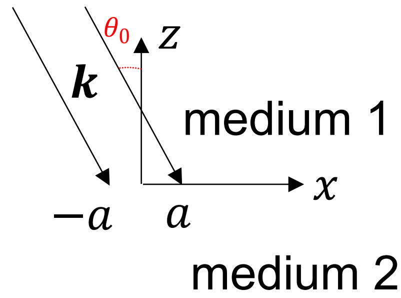
The analysis that follows is a *nonspecular* effect (i.e., not ray theory) that was observed by Goos and Hanchen (1947, 1949) in optics and by Schoch (1950, 1952) and Brekhovskikh (1980) in acoustics.

To simplify the problem, assume a 2D field (no y-dependence). At $z = 0$, the reflected wave is given by

$$p_r(x, 0) = \mathcal{F}_x^{-1}\{R(k_x)P_{i0}(k_x)\} , \tag{1}$$

while the incident beam has an arbitrary dependence on x:

$$p_i(x, 0) = p_0(x)e^{ikx\sin\theta_0} .$$



The spatial Fourier transform of the incident wave $p_i(x,0) = p_0(x)e^{ikx\sin\theta_0}$ is

$$P_{i0}(k_x) = \mathcal{F}_x\{p_i(x,0)\} = P_0(k_x - k_{x0})\,,$$

where $P_0(k_x) = \mathcal{F}_x\{p_0(x)\}$ and where $k_{x0} = k\sin\theta_0$. Eq. (1) becomes

$$p_r(x,0) = \mathcal{F}_x^{-1}\{R(k_x)P_0(k_x - k_{x0})\}\,. \tag{2}$$

For $ka \gg 1$, recall that the directivity (which is proportional to the spatial Fourier transform of the source) is very small (Recall that a directivity of 1 corresponds to a monopole), so

$$\frac{|P_0(k_x)|}{|P_0(0)|} \ll 1 \quad \text{for} \quad |k_x| \ll \frac{1}{a}\,,$$

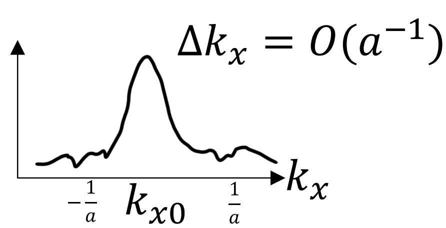
or shifting this relation by k_{x0},

$$\frac{|P_0(k_x - k_{x0})|}{|P_0(k_{x0})|} \ll 1 \quad \text{for} \quad |k_x - k_{x0}| \ll \frac{1}{a}\,.$$

The integrand in Eq. (2) is significant only for

$$|k_x - k_{x0}| = \mathcal{O}(1/a)\,, \tag{3}$$

as shown schematically below:



Now exress the complex-valued reflection coefficient in complex polar form as

$$R(k_x) = M(k_x)e^{i\phi(k_x)}\,, \quad M = |R|\,, \tag{4}$$

and assume that $M(k_x)$ is slowly varying in the regime given by Eq. (3). Expand the phase about $k_x = k_{x0}$:

$$\begin{aligned}\phi(k_x) &= \phi(k_{x0}) + (k_x - k_{x0})\phi'(k_{x0}) + \dots \\ &= \phi_0 + (k_x - k_{x0})\phi_0' + \dots\,,\end{aligned} \tag{5}$$

where

$$\phi_0 = \phi(k_{x0}), \quad \phi_0' = \left.\frac{d\phi}{dk_x}\right|_{k_x=k_{x0}}, \quad \text{etc}.$$

Also let $M_0 = M(k_{x0})$. Substitute Eqs. (4) and (5) into Eq. (1), defining $M_0 e^{i(\phi_0 - k_{x0}\phi_0')} \equiv M_0 e^{i\psi_0}$

$$\begin{aligned} p_r(x,0) &= \mathcal{F}_x^{-1}\{M(k_x)e^{i\phi(k_x)}P_{i0}(k_x)\} \\ &= M_0 e^{i(\phi_0 - k_{x0}\phi_0')}\mathcal{F}_x^{-1}\{e^{ik_x\phi_0'}P_{i0}(k_x)\} \\ &= M_0 e^{i\psi_0}\frac{1}{2\pi}\int_{-\infty}^{\infty} P_{i0}(k_x)e^{ik_x(x+\phi_0')}dk_x \\ &= M_0 e^{i\psi_0} p_i(x + \phi_0', 0) . \end{aligned}$$

Thus the reflected beam is *displaced* by an amount $-\phi_0'$, and the reflected pressure magnitude at the interface is

$$|p_r(x,0)| \simeq M_0 |p_i(x - \Delta, 0)| ,$$

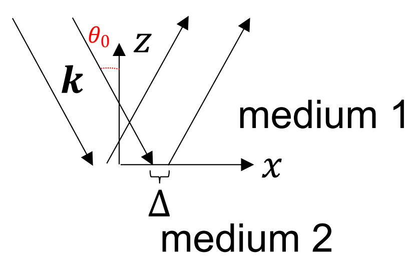
where $\Delta = -\phi_0' = -d\phi/dk_x\big|_{k_x=k_{x0}}$. Recalling that $k_x = k\sin\theta$, the displacement can be written as

$$\Delta = -\frac{1}{dk_x/d\theta}\left.\frac{d\phi}{d\theta}\right|_{\theta=\theta_0} ,$$

or

$$\boxed{\Delta = -\frac{\phi'(\theta_0)}{k\cos\theta_0}} .$$

Thus the amount by which the beam is displaced is proportional to the *derivative of the phase of the reflection coefficient*. This occurs at critical angles, cutoff frequencies, etc. The displaced reflected beam is shown schematically below:



Arrays

See Williams Sec. 2.11.6 for a similar discussion.

Consider a single source $u_1(x, y)$ replicated N times with a weighting factor w_n:

$$
\begin{aligned}
u_0(x,y) &= \sum_{n=1}^{N} w_n u_1(x-x_n, y-y_n) \\
&= \sum_{n=1}^{N} w_n u_1(x,y) ** \delta(x-x_n)\delta(y-y_n) \\
&= u_1(x,y) ** \sum_{n=1}^{N} w_n \delta(x-x_n)\delta(y-y_n)\,.
\end{aligned}
$$

By the convolution theorem, the 2D spatial Fourier transform of $u_0(x,y)$ is

$$
\begin{aligned}
U_0(k_x,k_y) &= U_1(k_x,k_y)A(k_x,k_y) \\
\text{where}\quad A(k_x,k_y) &= \sum_{n}^{N} w_n \mathcal{F}_{xy}\{\delta(x-x_n)\delta(y-y_n)\} \\
&= \sum_{n}^{N} w_n e^{-i(k_x x_n + k_y y_n)}\,.
\end{aligned}
$$

In the far field, therefore, the pressure field due to the array is

$$
p_\omega(r,\theta,\phi) = -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikr}}{r}U_1(k\alpha,k\beta)A(k\alpha,k\beta)\,,
$$

where

$$
\begin{aligned}
A(k\alpha,k\beta) &= \sum_{n=1}^{N} w_n e^{i(kx_n\alpha + ky_n\beta)}\,, \\
\alpha &= \sin\theta\cos\phi \\
\beta &= \sin\theta\sin\phi
\end{aligned}
$$

Now require $r \gg ka_A^2/2$ where a_A is the characteristic radius of the array.

To *steer* the array, replace the weighting factor w_n with

$$
w_n \mapsto w_n e^{ikx_n \sin\theta_0}\,.
$$

Then the angular spectrum becomes

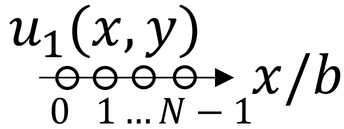
$$
\begin{aligned}
A(k_x,k_y) &= \sum_{n} w_n \mathcal{F}_{xy}\{e^{ikx_n\sin\theta_0}\delta(x-x_n)\delta(y-y_n)\} \\
&= \sum_{n} w_n e^{-i(k_x - k\sin\theta_0)x_n} e^{-ik_y y}\,,
\end{aligned}
$$

i.e.,

$$
A(k\alpha,k\beta) \mapsto A[k(\alpha-\sin\theta_0),k\beta]\,,
$$

where $U_1(k\alpha,k\beta)$ remains unaffected.

The above discussion is now generalized to rectangular arrays. Assume equal weighting $w_n = 1$ and consider first a line array, where b is the spacing between the elements of the array.



The source condition and its 2D spatial Fourier transform is

$$u_0(x, y) = \sum_{n=0}^{N-1} u_1(x - nb, y)$$
$$U_0(k_x, k_y) = U_1(k_x, k_y) A_+(k_x) \,,$$

where the array factor is

$$A_+(k_x) = \sum_{n=0}^{N-1} e^{-ink_xb} = 1 + e^{-ik_xb} + e^{-i2k_xb} + \cdots + e^{-i(N-1)k_xb} \,.$$

Since $1 + x + x^2 + \cdots + x^{N-1} = \frac{1-x^N}{1-x}$, the array factor becomes

$$A_+(k_x) = \frac{1 - e^{iNk_xb}}{1 - e^{-ik_xb}} = \frac{e^{iNk_xb/2}}{e^{-ik_xb/2}} \frac{e^{iNk_xb/2} - e^{iNk_xb/2}}{e^{iNk_xb/2} + e^{iNk_xb/2}}$$
$$= e^{-i(N-1)k_xb/2} \frac{\sin Nk_xb/2}{\sin k_xb/2} \,.$$

Now center the line array at $x = 0$:

$$u_0(x, y) = \sum_{n=0}^{N-1} u_1(x - nb + L/2, y) \,, \quad L = (N-1)b \,.$$

Then the 2D spatial Fourier transform of the source condition becomes

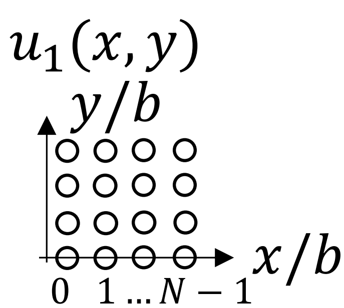
$$U_0(k_x, k_y) = U_1(k_x, k_y) A_+(k_x) e^{i(N-1)k_xb/2}$$
$$= U_1(k_x, k_y) A(k_x)$$

giving

$$A(k_x) = \frac{\sin(Nk_xb/2)}{\sin(k_xb/2)} \,,$$

which is N for $k_x = 0$ as expected.

Now consider a centered rectangular array:



In this case (presented without full derivation), the angular spectrum is

$$A(k_x, k_y) = \frac{\sin(N_x k_x b_x/2)}{\sin(k_x b_x/2)} \frac{\sin(N_y k_y b_y/2)}{\sin(k_y b_y/2)}$$

where $A(0,0) = N_x N_y$, the total number of elements in the rectangular array, again as expected.

Example: Two circular pistons of radius *a*

Suppose there are two circular pistons of radius a separated by distance b. Then

$$\begin{aligned} A(k_x) &= \frac{\sin(k_x b)}{\sin(k_x b/2)} \\ &= \frac{2\sin(k_x b/2)\cos(k_x b/2)}{\sin(k_x b/2)} = 2\cos(k_x b/2)\,. \end{aligned}$$

Thus the directivity of the two circular pistons is

$$\begin{aligned} D(\theta, \phi) &= \frac{2J_1(ka\sin\theta)}{ka\sin\theta}\cos\left[\frac{kb}{2}\sin\theta\cos\phi\right] \\ &= 1\,, \quad \theta = \phi = 0 \\ &\simeq \cos\left[\frac{kb}{2}\sin\theta\cos\phi\right], \quad ka \ll 1 \end{aligned}$$

Fresnel approximation

> *In choosing a theory, one should pay attention to simplicity in hypotheses only. Simplicity in computation can be of no weight in the balance of probabilities. Nature is not embarrassed by difficulties of analysis. She avoids complication only in means. Nature seems to have proposed to do much with little: it is a principle that the development of physics constantly supports by new evidence.*

–Augustin-Jean Fresnel

The study of focused sources motivates the Fresnel ("paraxial") approximation. The Fresnel approximation is not limited to focused sources, however, and it allows for analytical ease in the study of general diffraction phenomena. Nor is the Fresnel approximation limited in z: indeed, a far-field approximation of the Fresnel approximation can be taken.

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Focused sources

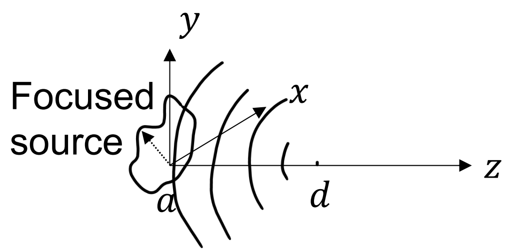
Consider a spherical wave converging at $z = d > 0$. The pressure field is then

$$\begin{aligned} p_\omega(x, y, z) &= A\frac{e^{-ikR}}{R} \\ &= A\frac{e^{ik\sqrt{x^2+y^2+(d-z)^2}}}{\sqrt{x^2+y^2+(d-z)^2}}\,, \end{aligned}$$

where the dimensions of A are pressure × distance. At $z = 0$, the pressure field is

$$p_\omega(x, y, 0) = A\frac{e^{-ik\sqrt{x^2+y^2+d^2}}}{\sqrt{x^2+y^2+d^2}}\,. \tag{1}$$

In the absence of diffraction (ray theory), any finite source at $z = 0$ with the above phasing will focus at $(x, y, z) = (0, 0, d)$:



Let us assume that $p_0(x, y)$ or $u_0(x, y) \sim 0$ for $x^2 + y^2 \geq a^2$, where a is the characteristic source radius, and that $a^2 \ll d^2$. This is to say that the so-called f-number: $N = d/2a \sim 1/2\alpha$ so $N^2 \gg 1$.

The phase in Eq. (1) can be approximated by

$$-ik\sqrt{x^2 + y^2 + d^2} = -ikd\left(1 + \frac{x^2 + y^2}{d^2}\right)^{1/2}$$
$$= -ikd + ik\frac{x^2 + y^2}{2d} + \mathcal{O}(ka^2/d^3)\,.$$

Meanwhile, the amplitude is approximated as

$$\frac{A}{\sqrt{x^2 + y^2 + d^2}} \simeq \frac{A}{d}\,,$$

which has units of pressure (pressure × distance ÷ distance = pressure). So the source condition, Eq. (1), becomes

$$p_\omega(x, y, 0) = A\frac{e^{-ikd}}{d}e^{-ik(x^2+y^2)/2d}\,.$$

Thus if

$$u_0(x, y)\,,\ p_0(x, y) = \text{ unfocused source distribution,}$$

focusing is achieved by multiplying by $e^{-ik(x^2+y^2)/2d}$, or in axisymmetric form, $e^{-ik(x^2+y^2)/2d}$. The factor Ae^{-ikd}/d can be neglected because phase is a relative quantity, and because A/d is simply a pressure amplitude that can be included in the pressure source amplitude p_0 or the velocity source amplitude $u_0 = p_0/\rho_0 c_0$. Reintroducing the time dependence renders focusing as time-advancing:

$$e^{-i\omega t}e^{-ik\rho^2/2d} = e^{-i\omega(t+\rho/2c_0 d)}\,.$$

That is to say, focusing is achieved in the time domain by the transformation

$$t \mapsto t + \frac{\rho^2}{2c_0 d}\,,$$

i.e., the further from the origin, the earlier the waveform must be launched.

Field in the focal plane of a focused source.

In the focal plane $z = d$, consider a focused velocity source

$$u(x, y, 0, t) = u_0(x, y)e^{-ik(x^2+y^2)/2d}e^{-i\omega t} .$$

The Rayleigh integral evaluated at the focal plane is

$$p_\omega(x, y, d) = -\frac{ik\rho_0 c_0}{2\pi} \iint_{-\infty}^{\infty} u_0(x_0, y_0)e^{-ik(x_0^2+y_0^2)/2d} \frac{e^{ikR}}{R} \, dx_0 \, dy_0 \, , \tag{R}$$

where

$$\begin{aligned} ikR &= ik\sqrt{(x - x_0)^2 + (y - y_0)^2 + d^2} \\ &= ikd\left(1 + \frac{x^2 + y^2}{d^2} - 2\frac{xx_0 + yy_0}{d^2} + \frac{x_0^2 + y_0^2}{d^2}\right)^{1/2} \\ &= ikd + ik\frac{x^2 + y^2}{2d} - ik\frac{xx_0 + yy_0}{d} + ik\frac{x_0^2 + y_0^2}{2d} + \mathcal{O}(ka^2/d^3) \, . \end{aligned}$$

In the amplitude, let $1/R \simeq 1/d$. Equation (R) becomes

$$p_\omega(x, y, d) = -\frac{ik\rho_0 c_0}{2\pi d} e^{ikd+ik(x^2+y^2)/2d} \iint_{-\infty}^{\infty} u_0(x_0, y_0)e^{-ikxx_0/d-ikyy_0/d} \, dx_0 dy_0 \, , \tag{2}$$

the magnitude of which is

$$\boxed{|p_\omega(x, y, d)| = \frac{k\rho_0 c_0}{2\pi d}\left|U_0(kx/d, ky/d)\right|,} \tag{3}$$

or in axisymmetric form, since $U_0(\kappa) = 2\pi U_{0H}(\kappa)$,

$$\boxed{|p_\omega(\rho, d)| = \frac{k\rho_0 c_0}{d}\left|U_{0H}(k\rho/d)\right|,} \tag{3'}$$

That is to say, *the field in the focal plane is but a spatial mapping of the far field into the focal plane: both are given in terms of the 2D spatial Fourier transform of the source condition.*

Field at the the focal *point* of a focused source.

Further insight can be gained from assessing the field at the focal *point* $(x, y, z) = (0, 0, d)$, Eq. (3) becomes

$$|p_\omega(0, 0, d)| = \frac{k\rho_0 c_0}{2\pi d} Q$$

Let $Q = \bar{u}_0 S$, where S is the surface area of the source, and where the *mean source particle velocity* is

$$\bar{u}_0 = \frac{Q}{S} = \frac{U_0(0, 0)}{S} = \frac{1}{S}\int_S u_0(x, y) dS \, .$$

Then, since $|p_\omega(0,0,d)| = \frac{k\rho_0 c_0}{2\pi d} Q$, the ratio of the pressure at the focal point to that in a plane wave with mean source particle velocity is

$$\frac{|p_\omega(0,0,d)|}{\rho_0 c_0 \bar{u}_0} = \frac{kS}{2\pi d} = \frac{S}{\lambda d}\,.$$

Define

$$\boxed{G = \frac{S}{\lambda d} = \text{ focusing gain}\,,} \tag{1}$$

which for a uniform circular piston of surface area $S = \pi a^2$ reduces to

$$G = \frac{ka^2}{2d} = \frac{z_0}{d}\,.$$

Note: the geometric focus at $(x, y, z) = (0, 0, d)$ is typically further beyond the location of the maximum axial amplitude.

The so-called "spot size" for a source of characteristic size a corresponds to the k-space "radius." From Eqs. (2) or (3'), in the focal plane, this corresponds to

$$\frac{k\rho}{d} \sim \frac{1}{a}$$

or

$$\rho \sim \frac{d}{ka} = a\frac{d}{ka^2} \sim \frac{a}{G}\,.$$

Thus beam radius is reduced by $\sim 1/G$ in the focal plane.

Focal point in the time domain.

Recall Eq. (3), copied below for convenience

$$|p_\omega(x, y, d)| = \frac{k\rho_0 c_0}{2\pi d}\Big|U_0(kx/d, ky/d)\Big|\,.$$

In the time domain, the field at the focal point is

$$\begin{aligned} |p_\omega(0,0,0,t)| &= -i\omega\frac{\rho_0 Q}{2\pi d} e^{i(kd-\omega t)} \\ &= \frac{\rho_0 \dot{Q}(t - d/c_0)}{2\pi d}\,. \end{aligned}$$

The expansion of ikR led to linear order led to the discarding of a term that was $\mathcal{O}(ka^4/d^3)$. So

$$\frac{ka^4}{8d^3} = \frac{1}{4}\frac{ka^2}{2d}\frac{a^2}{d^2} = \frac{G}{4}\left(\frac{a}{d}\right)^2.$$

For diagnostic medical ultrasound, $G = 4$, $a/d \simeq \frac{1}{8}$. So

$$\frac{G}{4}\left(\frac{a}{d}\right)^2 \sim 10^{-2} \ll 2\pi$$

In Lucas and Muir [JASA **72**, 1289 (1982)], $G \sim 40$ and $\frac{a}{d} \simeq \frac{1}{4}$, so

$$\frac{G}{4}\left(\frac{a}{d}\right) \sim 0.6 \ll 2\pi \, .$$

Example: Focused Gaussian source

The source condition for a Gaussian is

$$u_0(\rho) = u_0 e^{-\rho^2/a^2} \, ,$$

and its 2D spatial Fourier transform is

$$\begin{aligned} U_0(\kappa) &= 2\pi \mathcal{H}\{u_0(\rho)\} \\ &= 2\pi u_0 \int_0^\infty e^{-\rho^2/a^2} J_0(\kappa\rho)\, \rho d\rho \\ &= \pi a^2 u_0 e^{-\kappa^2 a^2/4} \, . \end{aligned}$$

From Eq. (3), the field in the focal plane is

$$|p_\omega(x,y,d)| = \frac{k\rho_0 c_0}{2\pi d}\left|U_0(kx/d, ky/d)\right| = \frac{ka^2\rho_0 c_0 u_0}{2d} e^{-\kappa^2 a^2 \rho^2/4d^2}$$

or, upon normalizing,

$$\frac{|p_\omega(\rho,d)|}{\rho_0 c_0 u_0} = G e^{-\rho^2/(a/G)^2} \, ,$$

where $G = ka^2/2d$. Thus the amplitude is seen to by magnified by G, and the beamwidth is shrunk by G.

This is only a sneak-peak into Gaussian beams, which are covered in more depth below.

Fresnel approximation

Begin with the Rayleigh integral

$$p_\omega(x,y,z) = -\frac{ik\rho_0 c_0}{2\pi} \iint_{-\infty}^{\infty} u_0(x_0,y_0) \frac{e^{ikR}}{R} dx_0 dy_0 \, . \tag{1}$$

Now expand R in powers of $1/z$ (rather than $1/r$, as was done in the Fraunhofer approximation):

$$
\begin{aligned}
kR &= k\sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2} \\
&= kz\left[1 + \frac{(x-x_0)^2}{z^2} + \frac{(y-y_0)^2}{z^2}\right]^{1/2} \\
&= kz + \frac{k}{2z}[(x-x_0)^2 + (y-y_0)^2] + \text{ H.O.T.}\,.
\end{aligned}
$$

Terminating the above at $\mathcal{O}(1/z)$ for the phase is less restrictive than the Fraunhofer approximation because we have retained the term

$$
\frac{k(x_0^2 + y_0^2)}{2z}\,,
$$

i.e., it is not required that $ka^2/2z \ll 1$ as in the Fraunhofer approximation. Now the restriction appears to be

$$
\frac{k}{8z^3}[(x-x_0)^2 + (y-y_0)^2]^2 \sim \frac{ka^4}{8z^3} \ll 2\pi
$$

or

$$
\frac{z}{a} \gg \frac{ka}{16\pi}
$$

or

$$
\boxed{\frac{z}{a} \gtrsim (ka)^{1/3}\,,}
$$

though this restriction can be weaker as the main contribution to the integral can come from points $(x_0, y_0) \sim (x, y)$ due to phase variations. Substituting the approximation of kR into (1) gives

$$
\boxed{p_\omega(x,y,z) = -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikz}}{z}\iint_{-\infty}^{\infty} u_0(x_0,y_0)e^{\frac{ik}{2z}[(x-x_0)^2+(y-y_0)^2]}\,dx_0\,dy_0\,.} \tag{2}
$$

For axisymmetric sources, $u_0(x,y) = u_0(\rho)$, so

$$
\begin{aligned}
(x-x_0)^2 + (y-y_0)^2 &= (\boldsymbol{\rho} - \boldsymbol{\rho}_0)\cdot(\boldsymbol{\rho} - \boldsymbol{\rho}_0) \\
&= \rho^2 + \rho_0^2 - 2\rho\rho_0\cos\phi_0
\end{aligned}
$$

for $\boldsymbol{\rho} = \rho\mathbf{e}_x$. Thus $\phi = 0$, so Eq. (2) becomes

$$
p_\omega(\rho,z) = -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikz}}{z}e^{ik\rho^2/2z}\int_0^\infty u_0(\rho_0)e^{ik\rho_0^2/2z}\rho_0\,d\rho_0\int_0^{2\pi} e^{-i(k\rho\rho_0/z)\cos\phi_0}d\phi_0\,.
$$

Taking the angular integral results in a Bessel function $2\pi J_0(k\rho\rho_0/z)$:

$$
p_\omega(\rho,z) = -ik\rho_0 c_0\frac{e^{ikz}}{z}e^{ik\rho^2/2z}\int_0^\infty u_0(\rho_0)J_0(k\rho\rho_0/z)e^{ik\rho_0^2/2z}\rho_0\,d\rho_0\,. \tag{3}
$$

In general,

$$p_\omega(x,y,z) = q_\omega(x,y,z)e^{ikz},$$

where q_ω varies slowly on the scale of a wavelength. In other words, if p_ω is a plane wave, then q_ω is a constant (the most slowly varying function).

Example: Pressure source in the Fresnel approximation

How does the paraxial approximation change for a pressure source? Recall the (exact) second Rayleigh integral:

$$p_\omega(x,y,z) = -\frac{ikz}{2\pi}\int_{-\infty}^{\infty} p_0(x_0,y_0)\left(1-\frac{1}{ikR}\right)\frac{e^{ikR}}{R^2}dx_0dy_0\,. \tag{1}$$

Since in the paraxial approximation $kR \gg 1$ and $R \sim z$, Eq. (1) becomes

$$p_\omega(x,y,z) = -\frac{ik}{2\pi}\int_{-\infty}^{\infty} p_0(x_0,y_0)\frac{e^{ikR}}{R^2}dx_0dy_0\,,$$

which is equivalent to the Rayleigh integral for a velocity source, if $p_0(x,y) = \rho_0 c_0 u(x,y)$. So, the paraxial approximation cannot tell the difference between a pressure and velocity source. That is to say, in the paraxial approximation it is consistent to use the plane wave impedance relation to convert from a velocity source to a pressure source. Further, the solutions do not distinguish between rigid and free baffles, i.e., the paraxial approximation can describe radiation from a source in free space, as well as it can describe radiation from a baffled or rigid surface.

Example: Paraxial field of unfocused Gaussian beam

Earlier, the field in the focal plane of a Gaussian beam was calculated. Now the paraxial field of an unfocused Gaussian beam is calculated. The source condition is

$$p_0(\rho) = p_0 e^{-\rho^2/a^2}.$$

The Fresnel approximation becomes

$$\begin{aligned} p_\omega(\rho,z) &= -ik\frac{e^{ikz}}{z}e^{ik\rho^2/2z}\int_0^\infty p_0(\rho_0)J_0(k\rho\rho_0/z)e^{=ik\rho_0^2/2z}\rho_0 d\rho_0 \\ &= -ik\frac{e^{ikz}}{z}e^{ik\rho^2/2z}\mathcal{H}\{e^{-(1+z_0/iz)\rho^2/a^2}\}\Big|_{\kappa=k\rho/z}, \end{aligned}$$

where $z_0 = ka^2/2$, $\mathcal{H}\{e^{-\rho^2/b^2}\} = \frac{b^2}{2}e^{-\kappa^2 b^2/4}$, and $b^2 = \frac{a^2}{1+z_0/iz}$. After taking the Hankel transform, one obtains

$$p(\rho,z) = \frac{p_0 e^{ikz}}{1+iz/z_0}\exp\left\{-\frac{\rho^2/a^2}{1+iz/z_0}\right\},$$

which equals the source condition at $z = 0$. Note that the Gaussian beam is of the form

$$p_\omega(\rho, z) = q_\omega(\rho, z) e^{ikz} .$$

Far field of Fresnel approximation

From Eq. (2), for $z \gg z_0 = \frac{1}{2}ka^2$, one obtains

$$p_\omega(x, y, z) = -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikz}}{z} e^{ik(x^2+y^2)/2z} \iint_{-\infty}^{\infty} u_0(x_0, y_0) e^{-i(kxx_0/z_0+kyy_0/z_0)} dx_0 dy_0 .$$

Thus the far field of the Fresnel approximation is

$$p_\omega(x, y, z) = -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikz}}{z} e^{ik(x^2+y^2)/2z} U_0(kx/z, ky/z) . \tag{4}$$

For an axisymmetric beam, the far-field approximation of the Fresnel approximation reads

$$p_\omega(z, \theta) = -ik\rho_0 c_0 \frac{e^{ikz}}{z} e^{i\frac{1}{2}kz\tan^2\theta} U_H(k\tan\theta) .$$

Compare Eq. (4) with the Fraunhofer approximation in spherical coordinates,

$$p_\omega(r, \theta, \phi) = -\frac{ik\rho_0 c_0}{2\pi}\frac{e^{ikr}}{r} U_0(k\alpha, k\beta) , \tag{5}$$

where

$$\alpha = x/r = \sin\theta\cos\phi ,$$
$$\beta = y/r = \sin\theta\sin\phi ,$$

where Eq. (5) is exact for all (θ, ϕ). Some important distinctions between Eqs. (4) and (5) are made:

- **Wavefront curvature.** From Eq. (5),

  $$e^{ikr} = e^{ik\sqrt{x^2+y^2+z^2}} = e^{ikz[1+(x^2+y^2)/z^2]}$$

  Only near the z axis can we expand and write

  $$e^{ikr} = e^{ikz} e^{ik(x^2+y^2)/2z} .$$

- **Angular dependence.** From Eq. (5),

  $$p_\omega(\theta, \phi) \propto |U_0(k\sin\theta\cos\phi, k\sin\theta\sin\phi)| .$$

  Only for small θ can the above equation be written as

  $$\begin{aligned} p_\omega(\theta, \phi) &\propto |U_0(k\tan\theta\cos\phi, k\tan\theta\sin\phi)| \\ &= |U_0(kx/z, ky/z)| . \end{aligned}$$

Thus in the paraxial region,

$$\sin\theta \sim \tan\theta \sim \theta\,, \quad (\theta \lesssim 20^\circ)\,.$$

- **Range of validity in z.** The Fresnel approximation is thus appropriate for sound beams for which $ka \gg 1$ and $z/a \gtrsim (ka)^{1/3}$, which is much closer to the source than in the Fraunhofer approximation.

Gaussian beams

The properties of Gaussian beams, which were introduced in the above examples, are now studied in more depth.

Unfocused Gaussian beam

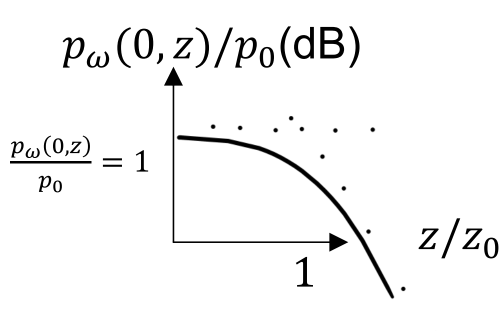
As derived above, the unfocused Gaussian beam is given in cylindrical coordinates by

$$p_\omega(\rho, z) = \frac{p_0 e^{ikz}}{1 + iz/z_0} \exp\left\{ -\frac{\rho^2/a^2}{1 + iz/z_0} \right\},$$

which is $p_0(\rho)$ for $z = 0$. Note that the solution is in the form $q_\omega(\rho, z) = p_\omega(\rho, z)e^{ikz}$, where $q_\omega(\rho, z)$ is slowly varying over the scale of a wavelength. The magnitude of the Gaussian beam is

$$\begin{aligned}
|p_\omega(\rho, z)| &= \frac{p_0 e^{ikz}}{\sqrt{1 + z/z_0}} \exp\left\{ -\frac{\rho^2/a^2}{1 + (z/z_0)^2} \right\} \\
&\simeq p_0 e^{-\rho^2/a^2}, \quad z \ll z_0 \\
&\simeq \frac{z_0}{z} p_0 e^{-(z_0/z)^2 \rho^2/a^2}, \quad z \gg z_0 \\
&= \frac{z_0}{z} p_0 e^{-(ka\tan\theta)^2/4}, \quad \tan\theta = \rho/z,
\end{aligned}$$

and the amplitude profile as a function of z is sketched below on a dB scale:



Focused Gaussian beam

Inclusion of focusing modifies the source condition to

$$
\begin{aligned}
p_0(\rho) &= p_0 e^{-\rho^2/a^2} e^{ik\rho^2/2d} \\
&= p_0 e^{-(1+ika^2/2d)\rho^2/a^2} \\
&= p_0 e^{-(1+iG)\rho^2/a^2}, \quad G = ka^2/2d \\
&= p_0 e^{-\rho^2/\tilde{a}^2}, \quad \tilde{a}^2 = \frac{a^2}{1+iG}\,.
\end{aligned}
$$

To obtain the *focused* Gaussian beam solution, simply replace a^2 by $\tilde{a}^2$ in the unfocused solution. The quantities ρ^2/a^2 and iz/z_0 become

$$
\begin{aligned}
\rho^2/a^2 &\mapsto p^2/a^2 = (1+iG)\rho^2/a^2 \\
\frac{iz}{z_0} &\mapsto \frac{izz_0}{k\tilde{a}^2} = i(1+iG)\frac{z}{z_0} = \frac{i-G}{G}\frac{z}{d} = \frac{iz}{Gd} - \frac{z}{d}\,.
\end{aligned}
$$

The pressure field is therefore

$$
p_\omega(\rho, z) = \frac{p_0 e^{ikz}}{1 - z/d + iz/Gd} \exp\left\{ -\frac{(1+iG)\rho^2/a^2}{1 - z/d + iz/Gd} \right\}. \tag{G}
$$

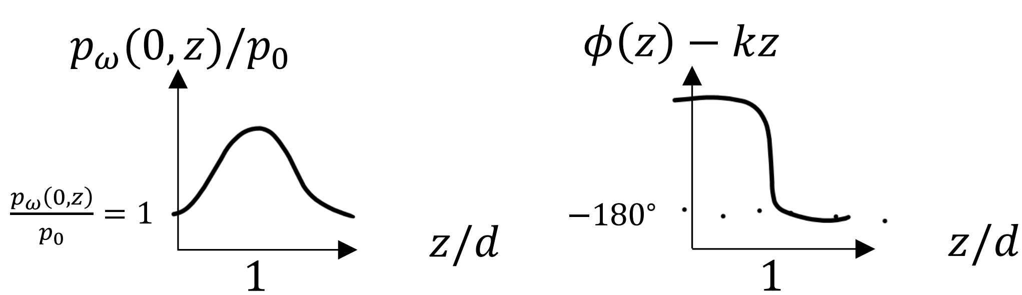
The field in the focal plane is found by setting $z = d$ in Eq. (G):

$$
p_\omega(\rho, d) = -iGp_0 e^{-G^2\rho^2/a^2} e^{i(kd+G\rho^2/a^2)} \implies \frac{|p(\rho,d)|}{p_0} = Ge^{-G^2\rho^2/a^2} = Ge^{-\rho^2/(a/G)^2}\,.
$$

The axial field is found by setting $z = 0$ in Eq. (G):

$$
\begin{aligned}
p_\omega(0, z) &= \frac{p_0 e^{i\phi(z)}}{\sqrt{(1 - z/d)^2 + (z/d)^2/G^2}}\,, \\
\text{where } \phi(z) &= kz - \arctan\frac{z/d}{G(1 - z/d)}\,.
\end{aligned}
$$

Sketches of the axial magnitude and phase of the focused Gaussian beam are shown below:



Calculation of the physical maximum in a focused Gaussian beam

The physical maximum of the field in a Gaussian beam does not correspond to geometric focus (except in the ray theory limit of $ka \to \infty$). To find where the physical maximum of the axial field occurs, take the derivative of the axial pressure magnitude with respect to z:

$$\frac{1}{p_0}\frac{d|p_\omega(0,z)|}{dz} = \frac{1 - z/d - z/G^2 d}{d[(1-z/d)^2 + (z/d)^2/G^2]^{3/2}}$$
$$= 0 \implies \frac{z}{d} = \frac{1}{1+G^{-2}}\,.$$

Some values of the above relation are tabulated below:

G	$z/d = 1/(1+G^{-2})$
10	0.99
5	0.96
2	0.80

Circular piston

The uniform circular piston is now considered. It will be seen that the discontinuous edges of the piston introduce complications in the Fresnel approximation:

The velocity source condition for the uniform circular piston is

$$u_0(\rho) = \begin{cases} u_0\,, & \rho \le a \\ 0\,, & \rho > a \end{cases}\,.$$

In this case, the Fresnel diffraction integral reads

$$p_\omega(\rho, z) = -ik\rho_0 c_0 u_0 \frac{e^{ikz}}{z} e^{ik\rho^2/2z} \int_0^\infty J_0(k\rho\rho_0/z) e^{ik\rho_0^2/2z} \rho_0\, d\rho_0\,.$$

For the axial field, $\rho = 0$, so the Bessel function $\to 1$. Meanwhile, let $x \equiv ik\rho_0^2/2z$ and $\rho_0 d\rho_0 = (z/ik)dy$. The integral becomes

$$p_\omega(0,z) = -\rho_0 c_0 u_0 e^{ikz} \int_0^{ika^2/2z} e^x dx = \rho_0 c_0 u_0 e^{ikz}(1 - e^{ika^2/2z})\,. \tag{1}$$

Compare Eq. (1) to the exact solution of the Helmholtz equation derived by Rayleigh for the (exact) axial field of a uniform circular piston:

$$p_\omega(0,z) = \rho_0 c_0 u_0 (e^{ikz} - e^{ik\sqrt{z^2+a^2}})$$
$$= \rho_0 c_0 u_0 e^{ikz}[1 - e^{ik(\sqrt{z^2+a^2}-z)}] \tag{2}$$

Now expand Eq. (2) in powers of $1/z$:

$$
\begin{aligned}
k(\sqrt{z^2+a^2}-z) &= kz\left[(1+\frac{a}{z})^{1/2}-1\right] \\
&= kz\left[\left(1+\frac{a^2}{2z^2}-\frac{a^4}{8z^4}+\dots\right)-1\right] \\
&= \frac{ka^2}{2z}-\frac{ka^4}{8z^3}+\mathcal{O}(1/z^5)\,.
\end{aligned}
$$

Thus Eq. (1) $\simeq$ (2) for

$$
\begin{aligned}
\frac{ka^4}{8z^3} &\ll 2\pi \\
(z/a)^3 &\gg ka/16\pi \\
z/a &\gtrsim (ka)^{1/3},
\end{aligned}
$$

which is the very condition of the Fresnel approximation. This suggests that the Fresnel approximation can be taken *a priori* (before any calculations have been made made), or *a posteriori* (after an exact result has been found).

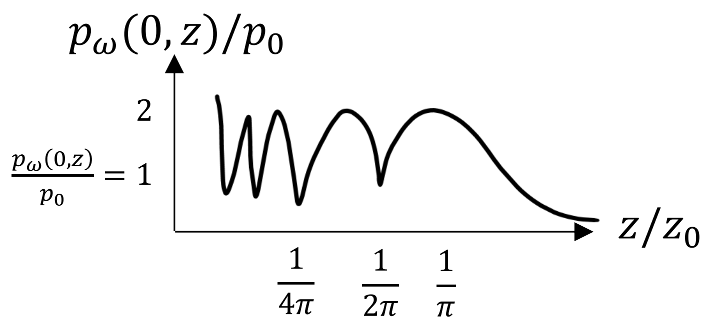
Note that Eq. (2) is singular $z = 0$, because the complex exponential as its argument approaches ∞ executes circles in the complex plane:

$$
\begin{aligned}
\frac{p_\omega(0,z)}{\rho_0 c_0 u_0} &= e^{ikz}e^{iz_0/z}(e^{-iz_0/2z}-e^{iz_0/2z})\,, \quad z_0 = ka^2/2 \\
&= -i2e^{i(kz+z_0/2z)}\sin(z_0/2z)\,.
\end{aligned}
$$

So

$$
\frac{p_\omega(0,z)}{\rho_0 c_0 u_0} = 2|\sin(z_0/2z)| = z_0/z\,, \quad z \gg z_0\,.
$$

The far field starts at $z/z_0 = 1/\pi$.



Meanwhile, the far field of the Fresnel approximation is given by

$$
p_\omega(z,\theta) = -ik\rho_0 c_0\frac{e^{ikz}}{z}e^{ik\rho^2/2z}U_H(k\tan\theta)\,.
$$

In this case, for the the uniform circular piston

$$
U_H(\kappa) = \mathcal{H}\{u_0(\rho)\} = u_0\mathcal{H}\{\,\mathrm{circ}\,(\rho/a)\} = u_0\frac{a^2}{2}\frac{2J_1(\kappa a)}{\kappa a}\,.
$$

The far field of the Fresnel approximation for a uniform circular piston is

$$\frac{p_\omega(z,\theta)}{\rho_0 c_0 u_0} = -i\frac{z_0}{z}e^{ikz}e^{i(kz/2)\tan^2\theta}\frac{2J_1(ka\tan\theta)}{ka\tan\theta}.$$

Meanwhile, for the sake of comparison, note that the Fraunhofer approximation of the uniform circular piston is given in spherical coordinates by

$$\frac{p_\omega(r,\theta)}{\rho_0 c_0 u_0} = -i\frac{z_0}{r}e^{ikr}\frac{2J_1(ka\sin\theta)}{ka\sin\theta}$$

See here for Dr. Hamilton's handwritten notes on the axial pressure of the focused circular piston. This was not covered formally in class, but is uploaded here for reference.

Rectangular piston

The field due to a rectangular piston is now calculated in the paraxial approximation.

Since the source pressure in the paraxial approximation is related to particle velocity through the plane wave impedance relation, $p = \rho_0 c_0 u$, the following source condition is considered:

$$p_0(x,y) = p_0\,, \quad x \in [-a,a]\,, \quad y \in [-b,b]\,.$$

The field is given in Cartesian coordinates by

$$p_\omega(x,y,z) = -\frac{ikp_0}{2\pi}\frac{e^{ikz}}{z}I_x I_y$$

where

$$\begin{aligned} I_x &= \int_{-a}^{a} e^{ik(x-x_0)^2/2z}dx_0\,, \\ I_y &= \int_{-b}^{b} e^{ik(y-y_0)^2/2z}dy_0\,. \end{aligned}$$

To take the integrals above, let

$$\begin{aligned} t^2 &= -\frac{ik}{2z}(x-x_0)^2 = \frac{k}{i2z}(x-x_0)^2 \\ \implies x - x_0 &= \sqrt{\frac{i2z}{k}}t\,, \quad dx_0 = -\sqrt{\frac{i2z}{k}}dt\,. \end{aligned}$$

The x integral becomes

$$I_x = -\frac{i2z}{k}\int_{\sqrt{k/i2z}(x+a)}^{\sqrt{k/i2z}(x-a)} e^{-t^2}dt\,.$$

The integral evaluates to

$$\int_\alpha^\beta e^{-t^2} dt = \left\{ \int_0^\beta - \int_0^\alpha \right\} e^{-t^2} dt$$
$$= \frac{\sqrt{\pi}}{2} (\operatorname{erf} \beta - \operatorname{erf} \alpha) ,$$

where

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \text{error function}$$
$$\to 1 , \quad z \to \infty .$$

Substituting in the appropriate limits of integration, I_x becomes

$$I_x = -\sqrt{\frac{i2z}{k}} \frac{\sqrt{\pi}}{2} \left\{ \operatorname{erf} \left[\sqrt{\frac{k}{i2z}} (x - a) \right] - \operatorname{erf} \left[\sqrt{\frac{k}{i2z}} (x + a) \right] \right\} .$$

Since the error function is odd, i.e.,

$$\operatorname{erf}(-z) = -\operatorname{erf}(z) , \tag{$\star$}$$

I_x can be written as

$$I_x = \frac{1}{2} \sqrt{\frac{i2\pi z}{k}} \left\{ \operatorname{erf} \left[\sqrt{\frac{z_a}{iz}} (1 + x/a) \right] + \operatorname{erf} \left[\sqrt{\frac{z_a}{iz}} (1 - x/a) \right] \right\} ,$$

where $z_a = ka^2/2$ and $z_b = kb^2/2$. The expression for I_y is very similar.

The pressure field is therefore

$$p_\omega(x, y, z) = \frac{1}{4} p_0 e^{ikz} \left\{ \operatorname{erf} \left[\sqrt{\frac{z_a}{iz}} (1 + x/a) \right] + \operatorname{erf} \left[\sqrt{\frac{z_a}{iz}} (1 - x/a) \right] \right\}$$
$$\times \left\{ \operatorname{erf} \left[\sqrt{\frac{z_b}{iz}} (1 + y/b) \right] + \operatorname{erf} \left[\sqrt{\frac{z_b}{iz}} (1 - y/b) \right] \right\} . \tag{1}$$

The axial field is found by setting $x = y = 0$:

$$p_\omega(0, 0, z) = p_0 e^{ikz} \operatorname{erf} \sqrt{\frac{z_a}{iz}} \operatorname{erf} \sqrt{\frac{z_b}{iz}} . \tag{2}$$

Note that

$$\sqrt{\frac{z_a}{iz}} = \pm(1 - i) \sqrt{\frac{z_a}{2z}} ,$$

but either sign gives the same result in Eqs. (1) and (2) by Eq. ($\star$).

Derivation of parabolic equation (paraxial equation)

The Fresnel approximation is now considered from the perspective of partial differential equations. Consider the Helmholtz equation:

$$\nabla^2 p_\omega + k^2 p_\omega = 0\,. \tag{1}$$

Let

$$q(x,y,z) = q_\omega(x,y,z)e^{ikz}\,. \tag{2}$$

q is slowly varying in z relative to the wavelength. Equation (2) is now inserted into Eq. (1). The appropriate derivatives are first taken:

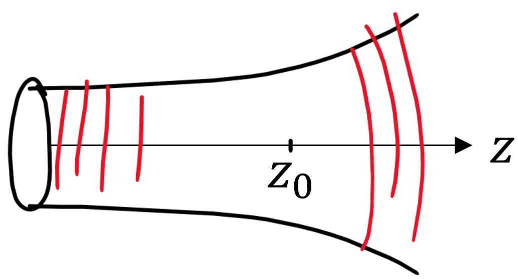
$$\begin{aligned}\frac{\partial p_\omega}{\partial z} &= \left(\frac{\partial q_\omega}{\partial z} + ikq_\omega\right)e^{ikz}\\ \frac{\partial^2 p_\omega}{\partial z^2} &= \left(\frac{\partial^2 q_\omega}{\partial z^2} + i2k\frac{\partial q_\omega}{\partial z} - k^2 q_\omega\right)e^{ikz}\,.\end{aligned}$$

Equation (1) becomes

$$\frac{\partial^2 q_\omega}{\partial z^2} + i2k\frac{\partial q_\omega}{\partial z} + \nabla_\perp^2 q_\omega = 0\,, \tag{1'}$$

where $\nabla_\perp^2$ is the Laplacian in the transverse direction, e.g., $\partial^2/\partial x^2 + \partial^2/\partial y^2$ in Cartesian coordinates. Note that

$$\frac{\partial^2 q_\omega/\partial z^2}{i2k\partial q_\omega/\partial z} \sim \frac{q_\omega/z_0^2}{kq_\omega/z_0} \sim \frac{1}{kz_0} \sim \frac{1}{(ka)^2} \ll 1\,.$$



Thus Eq. (1') is approximated by

$$\boxed{i2k\frac{\partial q_\omega}{\partial z} + \nabla_\perp^2 q_\omega = 0\,.} \tag{3}$$

Equation (3) is called the parabolic approximation of the Helmholtz equation. The parabolic equation is first order in z, reducing the elliptic equation (Helmholtz equation) to a parabolic equation.

Note that in the time domain, the parabolic equation is

$$\frac{\partial^2 p}{\partial z \partial \tau} = \frac{c_0}{2}\nabla_\perp^2 p\,,$$

where $p = p(x,y,z,\tau)$, and $\tau = t - z/c_0$.

To solve Eq. (3), one can use the standard Fourier acoustics procedure. See Dr. Hamilton's notes here in which the Fresnel diffraction integral is recovered.

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Moving media

Nature...is inexorable and immutable; she never transgresses the laws imposed upon her, or cares a whit whether her abstruse reasons and methods of operation are understandable to men.

–Galileo Galilei

The topic of moving media marks a departure from the content covered in topics 1-5. The wave equation in (nonaccelerating) motion (which can be either the motion of the observer or the motion of the background fluid) is first derived. The Doppler effect is then discussed. The remainder of the section deals with applications of the theory.

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Derivation of wave equation with flow

The wave equation in the presence of a uniform velocity flow field $\mathbf{v}_0$ is first derived by linearizing and combining the appropriate continuity, momentum, and state equations. *See Blackstock's "Fundamentals of Physical Acoustics" page 93, for a simpler version of this derivation (which is confined to a particular flow field in Cartesian coordinates).* This was not presented in class, but it was an unassigned homework problem.

The exact continuity, momentum, and state equations are

$$\frac{D\rho}{Dt} + \rho\boldsymbol{\nabla}\cdot\mathbf{v} = 0\,, \quad \rho\frac{D\mathbf{v}}{Dt} + \boldsymbol{\nabla}P = \mathbf{0}\,, \quad P = P(\rho)\,, \tag{1}$$

where $D/Dt = \partial/\partial t + \mathbf{v}\cdot\boldsymbol{\nabla}$ is the material derivative. It is to be shown that linearizing these equations, where $P = P_0 + p$, $\rho = \rho_0 + \rho'$, and $\mathbf{v} = \mathbf{v}_0 + \mathbf{u}$, leads to

$$\nabla^2 p = \frac{1}{c_0^2}\frac{\mathcal{D}^2 p}{\mathcal{D}t^2}\,, \tag{1}$$

where $\mathcal{D}/\mathcal{D}t = \partial/\partial t + \mathbf{v}_0 \cdot \boldsymbol{\nabla}$.

Begin by writing Eqs. (1) in terms of $\partial/\partial t + \mathbf{v} \cdot \boldsymbol{\nabla}$ (rather than in terms of D/Dt):

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \rho + \rho \boldsymbol{\nabla} \cdot \mathbf{v} &= 0 \\ \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} \right) + \boldsymbol{\nabla} P &= \mathbf{0} \\ P(\rho) &= P\,. \end{aligned} \tag{2}$$

Now insert $P = P_0 + p$, $\rho = \rho_0 + \rho'$, and $\mathbf{v} = \mathbf{v}_0 + \mathbf{u}$ into Eqs. (2), and note that $\partial \rho_0/\partial t = 0$, $\boldsymbol{\nabla} \rho_0 = \mathbf{0}$, $\boldsymbol{\nabla} \cdot \mathbf{v}_0 = 0$, $\partial \mathbf{v}_0/\partial t = \mathbf{0}$, $\boldsymbol{\nabla} \mathbf{v}_0 = \mathbf{0}$, and $\boldsymbol{\nabla} P_0 = \mathbf{0}$:

$$\begin{aligned} \frac{\partial \rho'}{\partial t} + (\mathbf{v}_0 + \mathbf{u}) \cdot \boldsymbol{\nabla} \rho' + (\rho_0 + \rho') \boldsymbol{\nabla} \cdot \mathbf{u} &= 0 \\ (\rho_0 + \rho') \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v}_0 + \mathbf{u}) \cdot \boldsymbol{\nabla} \mathbf{u} \right] + \boldsymbol{\nabla} p &= \mathbf{0} \\ P(\rho_0 + \rho') &= P\,. \end{aligned} \tag{3}$$

Nonlinear terms in Eqs. (3) are now neglected. For the state equation, the linearization is achieved by Taylor expanding the function $P(\rho)$ to linear order:

$$\begin{aligned} \frac{\partial \rho'}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla} \rho' + \rho_0 \boldsymbol{\nabla} \cdot \mathbf{u} &= 0 \\ \rho_0 \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla} \mathbf{u} \right) + \boldsymbol{\nabla} p &= \mathbf{0} \\ P_0 + \left(\frac{\partial P}{\partial \rho} \right)_0 (\rho - \rho_0) &= P\,. \end{aligned} \tag{4}$$

In the state equation [the third of Eqs. (4)], the quantities $p = P - P_0$, $\rho' = \rho - \rho_0$, and $(\partial P/\partial \rho)_0 = c_0^2$ are identified, resulting in the familiar state equation,

$$\rho' = p/c_0^2\,.$$

The perturbation density ρ' is therefore eliminated from Eqs. (4):

$$\frac{1}{c_0^2} \frac{\partial p}{\partial t} + \frac{1}{c_0^2} \mathbf{v}_0 \cdot \boldsymbol{\nabla} p + \rho_0 \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \tag{5}$$

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \rho_0 \mathbf{v}_0 \cdot \boldsymbol{\nabla} \mathbf{u} + \boldsymbol{\nabla} p = \mathbf{0}\,. \tag{6}$$

Taking the time derivative of Eq. (5) and the divergence of Eq. (6) results in

$$\frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} + \frac{1}{c_0^2} \mathbf{v}_0 \cdot \boldsymbol{\nabla} \left(\frac{\partial p}{\partial t} \right) + \rho_0 \boldsymbol{\nabla} \cdot \left(\frac{\partial \mathbf{u}}{\partial t} \right) = 0 \tag{7}$$

$$\rho_0 \boldsymbol{\nabla} \cdot \left(\frac{\partial \mathbf{u}}{\partial t} \right) + \rho_0 \mathbf{v}_0 \cdot \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u}) + \nabla^2 p = 0\,. \tag{8}$$

where it has been noted that $\boldsymbol{\nabla} \cdot (\mathbf{v}_0 \cdot \boldsymbol{\nabla} \mathbf{u}) = \mathbf{v}_0 \cdot \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u})$ (since $\mathbf{v}_0$ is a constant). Equation (7) is subtracting from Eq. (8), and the $\rho_0 \boldsymbol{\nabla} \cdot (\partial \mathbf{u}/\partial t)$ terms cancel, resulting in

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \rho_0 \mathbf{v}_0 \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \frac{1}{c_0^2}\mathbf{v}_0 \cdot \boldsymbol{\nabla}\left(\frac{\partial p}{\partial t}\right) = 0\,. \tag{9}$$

To obtain a wave equation in terms of single wave variable, the particle velocity needs to be eliminated from Eq. (9). This is done by solving Eq. (5) for $\rho_0 \boldsymbol{\nabla} \cdot \mathbf{u}$.

$$\rho_0 \boldsymbol{\nabla} \cdot \mathbf{u} = -\frac{1}{c_0^2}\left(\frac{\partial p}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla} p\right).$$

Insertion into Eq. (9) yields

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{c_0^2}\mathbf{v}_0 \cdot \boldsymbol{\nabla}\left(\frac{\partial p}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla} p\right) - \frac{1}{c_0^2}\mathbf{v}_0 \cdot \boldsymbol{\nabla}\left(\frac{\partial p}{\partial t}\right) = 0 \tag{10}$$

Combining terms yields

$$\nabla^2 p = \frac{1}{c_0^2}\left[\frac{\partial^2 p}{\partial t^2} + 2\mathbf{v}_0 \cdot \boldsymbol{\nabla}\left(\frac{\partial p}{\partial t}\right) + \mathbf{v}_0 \cdot \boldsymbol{\nabla}(\mathbf{v}_0 \cdot \boldsymbol{\nabla} p)\right]. \tag{11}$$

Equation (11) can be factored as

$$\nabla^2 p = \frac{1}{c_0^2}\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla}\right)^2 p\,. \tag{12}$$

Equation (12) will be reproduced by the Galilean transformation of the Helmholtz equation. Noting that $\partial p/\partial t + \mathbf{v}_0 \cdot \boldsymbol{\nabla}$ is simply the definition of $\mathcal{D}/\mathcal{D}t$, Eq. (12) can be written as

$$\nabla^2 p = \frac{1}{c_0^2}\frac{\mathcal{D}^2 p}{\mathcal{D}t^2}\,. \tag{2}$$

Galilean transformation of wave equation

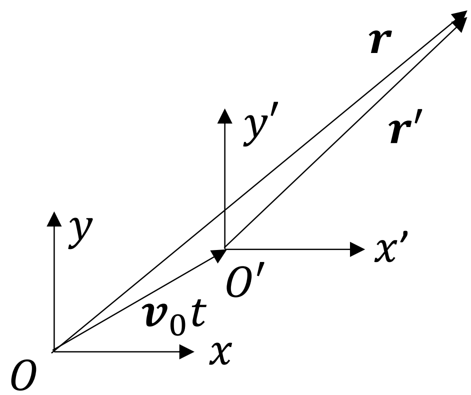
An alternate derivation of the wave equation derived above is now presented. In this derivation, the wave equation in a stationary frame is written in terms of coordinates that are moving with respect to that frame. Both this approach and the approach of the previous section lead to the same result, by the principle of relativity: motion of the background medium (standing still in a prairie with wind blowing from the north) is indistinguishable from motion of the observer in a stationary background medium (running north in a windless prairie). For more discussion, see the second and third paragraphs on page 699 in *Theoretical Acoustics* by Morse & Ingard.

Let $\mathbf{v}_0$ be the velocity at which point O moves with respect to point O', as shown in the schematic below. In this discussion, it is supposed that $\mathbf{v}_0$ is not a function of time. That is to say, O is not accelerating with respect to O'. If the reference frame containing O is called S, and that containing O' is called S', then both S and S' are regarded as inertial frames.

Let $\mathbf{r}$ be the coordinates in the frame S with origin O, and let $\mathbf{r}'$ be the cordinates in the system S' with origin O', which is moving with respect to S. Assume that sound in frame S' conforms to the wave equation

$$\nabla'^2 p = \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t'^2} , \tag{1}$$

where the $'$ symbol is used to denote the Laplacian in that frame"



Equation (1) is now written in terms of the unprimed coordinates, which are defined with respect to the primed coordinates by the Galilean transformation,

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}_0 t' \tag{2a}$$
$$t = t' . \tag{2b}$$

Taking the time derivative of Eq. (2a) results in $\dot{\mathbf{r}} = \dot{\mathbf{r}'} + \mathbf{v}_0$. Consider a one-dimensional case in which $\dot{r'} = c_0$, the speed of sound in frame S'. Then the speed in frame S is $\dot{r} = c_0 + v_0$. Therefore, the speed of sound in S and S' is different if there is any motion between S and S'. Note that this is **not** true for electromagnetic waves, i.e., the speed of light is the same in all inertial frames, motivating the linear transformation between two inertial frames that preserves this quantity (the Lorentz transformation).

In Cartesian coordinates, Eq. (2a) is represented as

$$x = x' + v_{0x} t'$$
$$y = y' + v_{0y} t'$$
$$z = z' + v_{0z} t'$$

while Eq. (2b) remains the same. Consider the derivative of a scalar-valued test function of unprimed variables $f(x, y, z, t)$ with respect to one of the spatial primed variables:

$$\frac{\partial}{\partial x'} f(x, y, z, t) = \frac{\partial f}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x'} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x'} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial x'} = \frac{\partial f}{\partial x} ,$$

i.e., they are unchanged, and thus the second spatial derivatives are also unchanged. Thus the sum of the second derivatives (the Laplacian in Cartesian coordinates) remains the same:

$$\nabla'^2 = \nabla^2 . \tag{$\star$}$$

Meanwhile, the derivative with respect to t' of a function of t is written in terms of unprimed quantities as

$$\frac{\partial f}{\partial t'} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t'} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t'} ,$$

which, upon noting that

$$\begin{aligned} \frac{\partial x}{\partial t'} &= \frac{\partial x}{\partial t} = \mathbf{e}_x \cdot \mathbf{v}_0 = v_{0x} , \\ \frac{\partial y}{\partial t'} &= \cdots = v_{0y} , \\ \frac{\partial z}{\partial t'} &= \cdots = v_{0z} , \end{aligned}$$

becomes

$$\begin{aligned} \frac{\partial f}{\partial t'} &= \frac{\partial f}{\partial t} + \left(v_{0x}\frac{\partial}{\partial x} + v_{0y}\frac{\partial}{\partial y} + v_{0z}\frac{\partial}{\partial z} \right) f \\ &= \left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla} \right) f . \end{aligned}$$

The second time derivative is therefore

$$\frac{\partial^2}{\partial t'^2} = \left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla} \right)^2 . \qquad (\star\star)$$

By Eqs. $(\star)$ and $(\star\star)$, Eq. (1) (the wave equation in frame S') becomes, in frame S,

$$\boxed{\nabla^2 p = \frac{1}{c_0^2}\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla} \right)^2 p ,} \qquad (3)$$

where $p = p(\mathbf{r}, t)$.

For $p(\mathbf{r}, t) = p_\omega(\mathbf{r}) e^{-i\omega t}$ (time-harmonic waves), Eq. (3) becomes a Helmholtz-like equation,

$$\nabla^2 p_\omega = \frac{1}{c_0^2} (-i\omega + \mathbf{v}_0 \cdot \boldsymbol{\nabla})^2 p_\omega ,$$

which can be written as

$$\boxed{\nabla^2 p_\omega + \left(\frac{\omega}{c_0} + i\mathbf{M} \cdot \boldsymbol{\nabla} \right)^2 p_\omega = 0 ,} \qquad (4)$$

where $\mathbf{M} = \mathbf{v}_0 / c_0$.

Attention is now turned to solving Eq. (3). Rather than directly solving the equation, consider Eq. (3) in the absence of motion ($\mathbf{v}_0 = 0$), whose solution is $p_{\text{NF}}(\mathbf{r}, t)$:

$$\nabla^2 p_{\mathrm{NF}} = \frac{1}{c_0^2}\frac{\partial^2 p_{\mathrm{NF}}}{\partial t^2}\,. \tag{5}$$

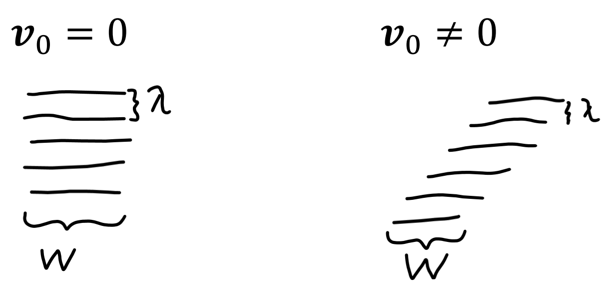
The transformation given by Eqs. (2a) (inverted) and (2b)

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t\,, \quad t' = t \tag{2'}$$

accounts for the motion:

$$p_{\mathrm{NF}}(\mathbf{r}', t') = p_{\mathrm{NF}}(\mathbf{r} - \mathbf{v}_0 t, t)\,. \tag{6}$$

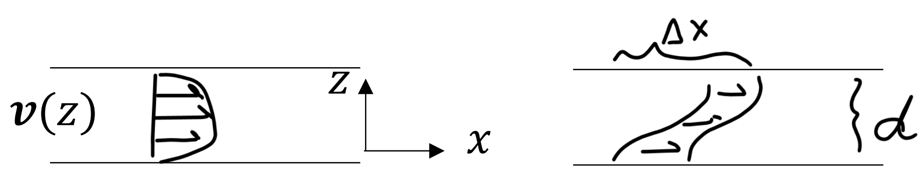
That is to say, Eq. (6) is the pressure field in which the effects of motion have been accounted. Motion therefore generally "shears" the field, as illustrated below:



The following example illustrates how the simple substitution in Eq. (6) gives the field in the moving medium.

Example: Laminar cross-flow

In this example, sound in a laminar cross-flow between two parallel plates is considered. The flow profile is given by the function $\mathbf{v}(z)$, as shown below:



Since the sound field is translated (sheared) by the differential amount

$$dx = v\,dt = v\frac{dz}{c_0} = M(z)dz\,,$$

the total shift is

$$\Delta x = \int_0^d M(z)dz\,.$$

The laminar flow between parallel plates is (from fluid mechanics)

$$M(z) = M_0(1 - 4z^2/d^2)\,, \quad -\frac{d}{2} \leq z \leq \frac{z}{2}\,,$$

so the shift is

$$\Delta x = 2M_0 \int_0^{d/2} \left(1 - 4\frac{z^2}{d^2}\right) dz = 2M_0 \left(z - \frac{4}{3}\frac{z^3}{d^2}\right)\Bigg|_0^{d/2}$$
$$= \frac{2}{3} M_0 d \,.$$

The solution of the wave equation is therefore given by the solution in the absense of flow, only displaced by the amount $2M_0 d/3$, i.e., $p_{\mathrm{NF}}(x - 2M_0 d/3, z, t)$.

Now suppose the flow profile is constant, i.e., $M = M_0 =$ constant. Thus $\Delta x = M_0 d$. So, the solution to the wave equation in that case is $p_{\mathrm{NF}}(x - M_0 d, z, t)$.

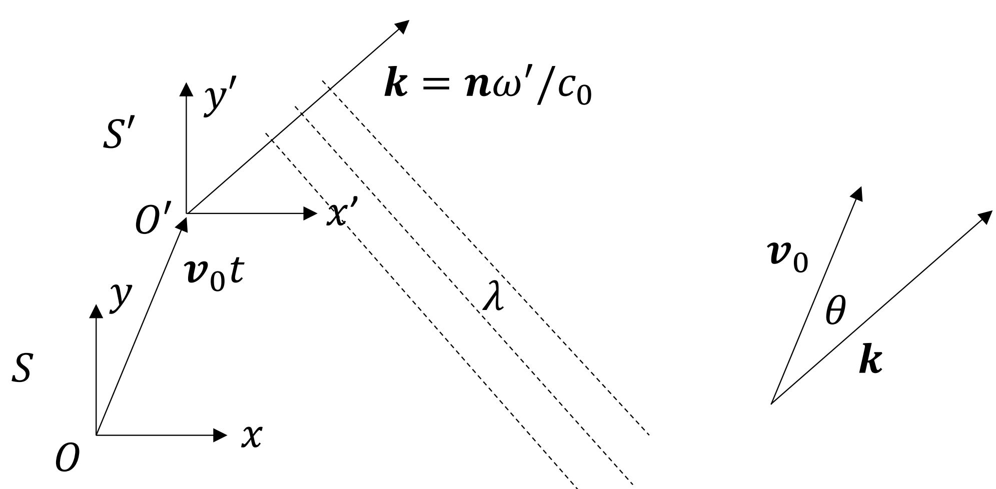
Doppler effect

The Doppler effect is simply a consequence of the motion between two inertial frames of reference.

Consider a plane wave propagating in direction $\mathbf{k}$ in a frame S' that is moving with velocity $\mathbf{v}_0$ with respect to S, where θ is the angle between the direction of the motion [i.e., $\theta = \arccos(\mathbf{k} \cdot \mathbf{v}_0 / k v_0)$],

$$p(\mathbf{r}', t) = A e^{-i\omega(t - \mathbf{n}\cdot\mathbf{r}'/c_0)}$$
$$= A e^{i(\mathbf{k}\cdot\mathbf{r}' - \omega' t)} \,, \tag{7}$$

where $\mathbf{n}$ is the normal to the wavefront, $\mathbf{k} = \omega' \mathbf{n}/c_0$ is the wavenumber, and $\lambda = 2\pi/|\mathbf{k}| = 2\pi c_0/\omega'$ is the wavelength, as illustrated below. Note that λ is the same in both S and S', but the speed of sound is $c_0 + v_0$ in the moving frame [by Eq. (2a)], and the frequency measured in frame S will be different from that measured in frame S'. The motion between frames causes the number of crests and troughs encountered *per unit time* by the observer in frame S and S' to be different. Since the number of events per unit time is by definition frequency, it is the frequency that varies between frame S and S'.



Substituting the plane wave given by Eq. (7) into the Helmholtz equation in frame S [Eq. (4)] with

$$p_\omega = Ae^{i\mathbf{k}\cdot\mathbf{r}'}\,, \quad \mathbf{M} = \frac{\mathbf{v_0}}{c_0}\,, \quad \mathbf{k} = \frac{\omega'}{c_0}\mathbf{n}\,,$$

requires the following quantities to be calculated:

$$\nabla^2 p_\omega = \nabla'^2 p_\omega = -k^2 p_\omega = -(\omega'/c_0)^2 p_\omega$$
$$i\mathbf{M}\cdot\boldsymbol{\nabla} = i\mathbf{M}\cdot\boldsymbol{\nabla}' = i\mathbf{M}\cdot(i\omega'\mathbf{n}/c_0) = -M(\omega'/c_0)\cos\theta\,.$$

Upon making these substitutions, Eq. (4) becomes

$$-(\omega'/c_0)^2 + [\omega/c_0 - M(\omega'/c_0)\cos\theta]^2 = 0\,.$$

Solving for ω and tossing the negative solution yields $\omega - M\omega'\cos\theta = \omega'$ or

$$\boxed{\omega = (1 + M\cos\theta)\omega'\,.}$$

Thus it has been shown that a Galilean transformation of the wave equation results in the frequency between two frames to be related through $\omega = (1 + M\cos\theta)\omega'$.

Alternatively, the Doppler shift can be derived by applying the Galilean transformation *to the plane-wave solution* of the wave equation (See Morse & Ingard, page 700). Start with the plane wave in the moving frame S',

$$p = Ae^{i\mathbf{k}\cdot\mathbf{r}' - i\omega' t}\,,$$

where $\mathbf{k} = \omega'\mathbf{n}/c_0$. Substituting in $\mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t$ into the equation for the plane wave gives

$$p = Ae^{i\mathbf{k}\cdot\mathbf{r} - i(\omega' + \mathbf{k}\cdot\mathbf{v}_0)t} \equiv Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\,,$$

where it has been identified that

$$\begin{aligned}\omega &\equiv \omega' + \mathbf{k}\cdot\mathbf{v}_0 \\ &= \omega' + \frac{\omega'}{c_0}\mathbf{n}\cdot\mathbf{v}_0 \\ &= (1 + M\cos\theta)\omega'\,.\end{aligned}$$

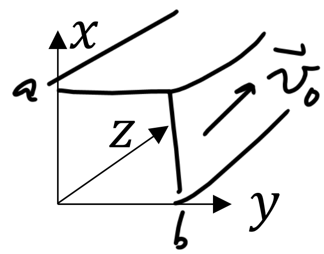
As mentioned in the beginning of this section, the Doppler shift is a natural consequence of the motion between two inertial frames of reference. The theory is blind to the nature of the motion i.e., whether it is due to the medium flowing past a stationary observer, or due to the observer moving through a stationary medium. Those two scenarios are kinematically equivalent (See the fourth paragraph on page 699 in Morse).

This checks with everyday experience. An example of the Doppler effect arising due to the medium flowing past a stationary observer is the sound of wind "howling." Imagine you are standing in a prairie and the wind begins to blow, creating a whistling/howling sound (Such tones are generated by turbulent flow of air as the wind hits your ear, which is beside the point). The stronger the wind, the higher the frequency of the howl, and when the wind dies down, the howl frequency decrease. The wind is in the frame of reference S', and the governing wave equation is that frame is Eq. (1). You are in the frame of reference S, and the governing wave equation in your frame is Eq. (3).

Now suppose there is no wind in the prairie, but that you are now running through the prairie. If you run at the same speed at which the wind was previously blowing, you will hear *exactly the same sound* as what was described in the previous paragraph. You are still in the frame of reference S, and the world around you is still in the frame S'.

Sound in ducts with uniform flow

A practical problem involving waves in moving media is that of sound in ducts (e.g., air conditioning ducts, pipelines, etc.), for which the bulk fluid motion is in one direction, and in which modal solutions exist in the transverse directions. This topic is covered in Morse and Ingard, pages 714-715, or Ingard, page 317-319.



For fluid motion in the z direction, as depicted in the figure below, the Helmholtz equation describing sound in frame S becomes

$$\begin{aligned}\nabla^2 p_\omega &= -\left(\frac{\omega}{c_0} + i\mathbf{M}\cdot\boldsymbol{\nabla}\right)^2 p_\omega \\ &= -\left(\frac{\omega}{c_0} + iM\frac{\partial}{\partial z}\right)^2 p_\omega\,. \end{aligned} \quad (1)$$

A dispersion relation $k_z(\omega)$ is sought by assuming Eq. (1) has solutions of the form

$$p_\omega = A\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{ik_z z}\,.$$

Substitution into Eq. (1) gives, on the left-hand side,

$$\left[-\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 - k_z^2\right]p_\omega = -\left(\frac{\omega_{mn}^2}{c_0^2} + k_z^2\right)p_\omega\,,$$

where

$$\omega_{mn} = \sqrt{\left(\frac{m\pi c_0}{a}\right)^2 + \left(\frac{n\pi c_0}{b}\right)^2}\,,$$

which is the cutoff frequency for $v_0 = 0$. Meanwhile, the right-hand side of Eq. (1) becomes,

$$-(\omega/c_0 + iM\partial/\partial z)^2 p_\omega = -(\omega/c_0 - Mk_z)^2 p_\omega\,.$$

Thus one obtains from the Helmholtz equation a dispersion relation:

$$\frac{\omega_{mn}^2}{c_0^2} + k_z^2 = \left(\frac{\omega}{c_0} - Mk_z\right)^2$$
$$= \frac{\omega^2}{c_0^2} - 2M\frac{\omega}{c_0}k_z + M^2k_z^2 .$$

Rearranging the above yields

$$(1 - M^2)k_z^2 + 2M\frac{\omega}{c_0}k_z - \left(\frac{\omega^2}{c_0^2} - \frac{\omega_{mn}^2}{c_0^2}\right),$$

which is solved for k_z by the quadratic formula:

$$\boxed{k_z = \frac{\omega/c_0}{1 - M^2}\left(\sqrt{1 - \Omega_{mn}^2/\omega^2} - M\right),} \tag{2}$$

where $\Omega_{mn} = \sqrt{1 - M^2}\ \omega_{mn}$, which is the cutoff frequency for $v_0 \neq 0$, and where the $\pm$ sign corresponds to the propagation direction. As a sanity check, note that for $M = 0$ (no flow), one obtains

$$k_z = \pm\frac{\omega}{c_0}\sqrt{1 - \omega_{mn}^2/\omega^2} ,$$

which is the familiar projection of the wavenumber in the z direction in a waveguide (from Acoustics I/II).

The phase speed and group speeds are

$$c_{\text{ph}} = \omega/k_z = \frac{(1 - M^2)c_0}{\pm\gamma - M} , \quad \gamma = \sqrt{1 - \Omega_{mn}^2/\omega^2} .$$
$$c_{\text{gr}} = \frac{d\omega}{dk_z} = \frac{1}{dk_z/d\omega} = \frac{\gamma(1 - M^2)c_0}{\pm 1 - \gamma M} .$$

For this discussion, consider only + direction propagation. Then, as another sanity check, for $\omega \to \infty$, $\gamma \to 1$, for which the sound propagation is non-dispersive, i.e.,

$$c_{\text{ph}} = c_{\text{gr}} = (1 + M)\, c_0 = c_0 + v ,$$

which checks with intuition, i.e., the energy travels at the speed that is the sum of the wave speed and the ambient speed. Meanwhile, the cutoff frequency is the frequency at which the group velocity vanishes, which corresponds to setting $\gamma = 0$, for which

$$\omega_c = \Omega_{mn} = \sqrt{1 - M^2}\, \omega_{mn} .$$

Then, from Eq. (2), for $\omega \leq \omega_c = \Omega_{mn}$,

$$k_z = \frac{\omega/c_0}{1 - M^2}\left[i\sqrt{(\Omega_{mn}/\omega)^2 - 1} - M\right].$$

The above equation is an interesting result, since it shows that evanescent waves "sail" along with the motion of the fluid in the z direction at the speed v_0. In this sense, the evanescent waves propagate, because their wavenumber has a real component.

Refraction with flow

Now the refraction of sound is discussed in the context of a medium with flow. This topic is covered in Morse and Ingard, pages 708-710.

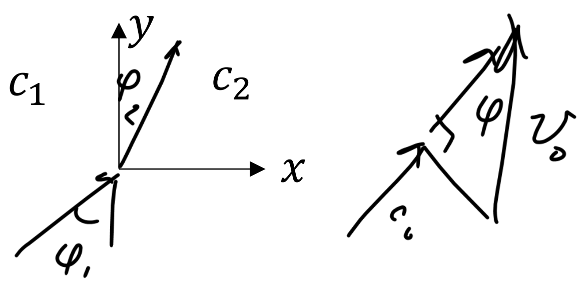
Consider the incident and transmitted plane waves,

$$p_i = A_i e^{i\mathbf{k}_1 \cdot \mathbf{r}}$$
$$p_t = A_t e^{i\mathbf{k}_2 \cdot \mathbf{r}} ,$$

where for $n = 1, 2$,

$$\mathbf{k}_n = k_n (\sin \phi_n \mathbf{e}_x + \cos \phi_n \mathbf{e}_y) ,$$

as illustrated below:



The Helmholtz equation in frame S,

$$\nabla^2 p_\omega = -\left(\frac{\omega}{c_0} + i\mathbf{M} \cdot \boldsymbol{\nabla} \right)^2 p_\omega ,$$

when assuming spatially harmonic solutions ($\boldsymbol{\nabla} \mapsto i\mathbf{k}$) and denoting the Mach number as $\mathbf{M} = \mathbf{v}_0 / c_0$, becomes

$$-k^2 p_\omega = -\left(\frac{\omega}{c_0} - \mathbf{M} \cdot \mathbf{k} \right)^2 p_\omega = -\left(\frac{\omega}{c_0} - Mk \cos \phi \right)^2 p_\omega ,$$

where it has been noted that the angle between the direction of the wave propagation and the motion is $\phi = \arccos (\mathbf{M} \cdot \mathbf{k} / Mk)$. Thus

$$k = \frac{\omega}{c_0} - Mk \cos \phi ,$$

which, solving for k, yields

$$k = \frac{\omega / c_0}{1 + M \cos\phi} = \frac{\omega}{c_0 + v_0 \cos\phi} . \tag{a}$$

The original equations for p_i and p_t are now evaluated at $x = 0$, and the continuity of the pressure field is enforced:

$$A_i e^{ik_1 y \cos\phi_1} = A_t e^{ik_2 y \cos\phi_2} .$$

To satisfy this equation, the phases must match

$$k_1 \cos\phi_1 = k_2 \cos\phi_2 . \tag{b}$$

Combining Eqs. (a) and (b) yields

$$\frac{\omega \cos\phi_1}{c_1 + v_1 \cos\phi_1} = \frac{\omega \cos\phi_2}{c_2 + v_2 \cos\phi_2} .$$

which, solving for $\cos\phi_2$ (requires about five lines of algebra), reads

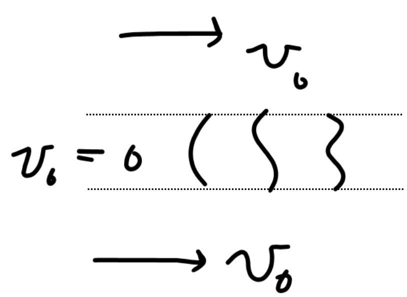
$$\cos\phi_2 = \frac{c_2 \cos\phi_1}{c_1 - \Delta v \cos\phi_1}$$

where $\Delta v = v_2 - v_1$. The critical angle, which is the ϕ_{1c} that sets $\phi_2 = 0$, can therefore be found:

$$c_2 \cos\phi_{1c} = c_1 - \Delta v \cos\phi_{1c} \implies \cos\phi_{1c} = \frac{c_1}{c_2 + \Delta v} .$$

Example: Wind channels creating a waveguide for sound

The equation for the critical angle, $\cos\phi_{1c} = c_1/(c_2 + \Delta v)$, reduces to $1/(1 + M)$ if $c_1 = c_2 = c_0$, $v_1 = 0$, $v_2 = v_0$. These are the conditions that describe wind in an otherwise homogeneous medium. As long as sound is incident at angles less than the critical angle given by $\phi_{1c} = \arccos[(1 + M)^{-1}]$, no sound can enter the moving medium. Consider two such interfaces, as shown below:



Assuming the sound in the middle layer in which $v_0 = 0$ is confined to angles smaller than the critical angle, the interfaces with the moving media serve as idealized boundaries. Thus, modes can develop in the middle layer, and the moving medium engulfing the stationary channel act as walls of a waveguide for sound.

Interestingly, the flow at the boundary of two media results in *nonreciprocity* (also known as "broken symmetry"), which refers to the exchange of source and observer having an effect on the measured signal. This is illustrated by example below.

Example: Nonreciprocity due to reflection at a boundary with flow

Suppose $c_1 = c_2 = c_0$, i.e., there is only one medium, but that part of the medium is moving with respect to another part of the medium. Thus $\cos\phi_2 = \frac{c_2 \cos\phi_1}{c_1 - \Delta v \cos\phi_1}$ becomes

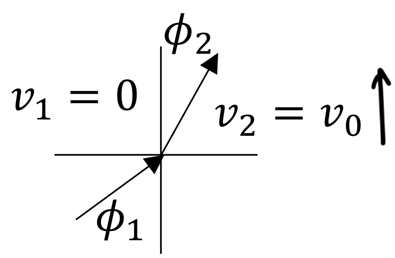
$$\cos\phi_2 = \frac{c_0 \cos\phi_1}{c_0 - (v_2 - v_1)\cos\phi_1} = \frac{1}{\sec\phi_1 - (M_2 - M_1)}\,.$$

Two cases are now considered, the first corresponding to wave motion from medium 1 to medium 2, and the second corresponding to motion from medium 2 to medium 1:

Case I.

For motion from medium 1 to medium 2, let $M_1 = 0, M_2 = M = v_0/c_0$:

$$\cos\phi_2 = \frac{1}{\sec\phi_1 - M} \implies \sec\phi_2 = \sec\phi_1 - M\,. \qquad (\star)$$

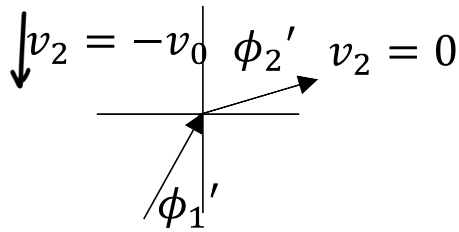


Case II.

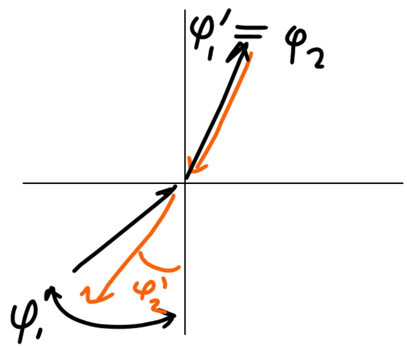
Now consider propagation in the reverse direction by setting $\phi_1' = \phi_2, M_1 = -M, M_2 = 0$. These assignments can be obtained by rotating the figure for Case I above by 180° and noting that the wave originating from the second medium would be traveling against the direction of the fluid motion. Then, the equation for $\cos\phi_2'$ becomes [using Eq. $(\star)$ in the third equality in the first line below]

$$\begin{aligned}\cos\phi_2' &= \frac{1}{\sec\phi_1' - M} = \frac{1}{\sec\phi_2 - M} = \frac{1}{\sec\phi_1 - M - M}\\ &= \frac{1}{\sec\phi_1 - 2M} = \frac{\cos\phi_1}{1 - 2M\cos\phi_1}\\ &> \cos\phi_1\end{aligned}$$

The inequality above holds because $1 - 2M\cos\phi_1$ is less than 1. Since $\cos\phi_2' > \cos\phi_1$, it is concluded that ϕ_2' is less than ϕ_1. The main takeaway is that $\phi_2' \neq \phi_1$. Thus the exchanging source and receiver changes the wave's behaviour, and reciprocity is broken.



The full situation is written below, following Fig. 11.5 in Morse and Ingard (but beware of the typo in their figure: ϕ_2 on the LHS should be primed).



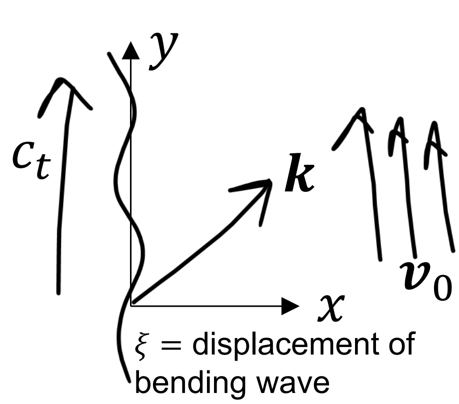
Bending wave in flow

Radiation (and lack thereof) from bending waves in flow is now discussed. See pages 705-708 of Morse and Ingard for more on this topic.

The displacement of the bending wave is given by

$$\begin{aligned} \xi &= \xi_0 e^{-i\omega(t-y/c_t)} \\ &= \xi_0 e^{i(k_t y-\omega t)}\,, \quad k_t = \omega/c_t\,, \end{aligned} \tag{1}$$

as depicted below:



The pressure wave radiated into the fluid is a plane wave (since the plate is considered to be infinite),

$$p = Ae^{i\mathbf{k}\cdot\mathbf{r}-\omega t}$$

which in the Helmholtz domain is

$$p_\omega = Ae^{ik(x\sin\phi+y\cos\phi)}\,. \tag{2}$$

p_ω satisfies the Helmholtz equation with $\mathbf{v}_0 = v_0\mathbf{e}_y$:

$$\nabla^2 p_\omega = -(\omega/c_0 + i\mathbf{M}\cdot\boldsymbol{\nabla})^2 p_\omega\,.$$

Thus

$$k = \frac{\omega/c_0}{1+M\cos\phi} = \frac{\omega}{c_0+v_0\cos\phi}\,. \tag{3a}$$

At the boundary $x = 0$, the field must be continuous, requiring that

$$k_t = k\cos\phi \implies \frac{\omega}{c_t} = \frac{\omega\cos\phi}{c_0+v_0\cos\phi}\,,$$

which, upon rearranging, gives

$$\cos\phi = \frac{c_0}{c_t - v_0}\,. \tag{3b}$$

Thus, for sound to radiate from the plate,

$$\boxed{c_t > c_0 + v_0\,.}$$

Otherwise, $\sin\phi = \sqrt{1-\cos^2\phi}$, which is imaginary and corresponds to evanescent waves.

Now the amplitudes A and ξ_0 are related to each other. Following Morse and Ingard's approach, the (exact) momentum equation is appealed to:

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\frac{D^2\xi}{Dt^2} = -\boldsymbol{\nabla}P\,,$$

where $D/Dt = \partial/\partial t + (\mathbf{v}_0 + \mathbf{u})\cdot\boldsymbol{\nabla}$ is the material derivative. Linearization results in

$$\rho_0\left(\frac{\partial}{\partial t} + \mathbf{v}_0\cdot\boldsymbol{\nabla}\right)^2\xi = -\boldsymbol{\nabla}p\,.$$

Since ξ and $\mathbf{v}_0$ are perpendicular, i.e., $\xi = \xi\mathbf{e}_x$ and $\mathbf{v} = v_0\mathbf{e}_y$, the momentum equation reduces to

$$\begin{aligned}\frac{\partial p}{\partial x} &= -\rho_0(i\omega + v_0\partial/\partial y)^2\\ &= -\rho_0(-i\omega + ik_z v_0)^2\xi\\ &= \frac{\rho_0\omega^2}{c_t^2}(c_t - v_0)^2\xi\end{aligned} \tag{4}$$

From Eq. (2),

$$
\begin{aligned}
\frac{\partial p}{\partial x} &= ik\sin(\phi)p\,, \quad k = k_t/\cos\phi \\
&= i\frac{\omega\sin\phi}{c_t\cos\phi}p \\
&= i\frac{\omega}{c_t}\frac{\sqrt{1-\cos^2\phi}}{\cos\phi} \\
&= i\frac{\omega}{c_t}\left[\frac{1}{\cos\phi}-1\right]^{1/2} \\
&= i\frac{\omega}{c_t c_0}\left[(c_t-v_0)^2-c_0^2\right]^{1/2}p\,.
\end{aligned} \tag{5}
$$

Now, Eq. (4) is equated to Eq. (5), resulting, at $x = 0$, in

$$
i\frac{\omega}{c_t c_0}\sqrt{(c_t-v_0)^2-c_0^2}\,A = \frac{\rho_0\omega^2}{c_t^2}(c_t-v_0)^2\xi_0\,.
$$

Solving for A results in

$$
\boxed{A = \rho_0 c_0 u_0\frac{(c_t-v_0)^2}{c_t\sqrt{(c_t-v_0)^2-c_0^2}}\,,}
$$

where $u_0 = -i\omega\xi_0$, which is the plate's particle velocity.

As a sanity check, eliminate the flow, i.e., $v_0 = 0$:

$$
A = \frac{\rho_0 c_0 u_0}{\sqrt{1-c_0^2/c_t^2}}\,, \quad \cos\phi = c_0/c_t\,.
$$

This matches previous results for a bending wave in a stationary medium.

Now consider the case in which $c_t = 0$, which corresponds to steady flow across a stationary corrugated surface. Since $k_t = \omega/c_t$ (which would go to ∞ in the limit that $c_t = 0$), define

$$
k_t = \omega/c_t = 2\pi/L \implies \omega = \frac{2\pi c_t}{L}\,,
$$

where L is the spatial period of the corrugation. Then,

$$
u_0 = -i\omega\xi_0 = -i\frac{2\pi c_t}{L}\xi_0\,,
$$

so

$$
\begin{aligned}
A &= -i2\pi\rho_0 c_0 \xi_0 \frac{(c_t - c_0)^2}{L\sqrt{(c_t - v_0)^2 - c_0^2}}\bigg|_{c_t \to 0} \\
&= -i2\pi\rho_0 c_0 \xi_0 \frac{v_0}{L\sqrt{v_0^2 - c_0^2}} \\
&= 2\pi\rho_0 c_0^2 \frac{\xi_0}{L} \frac{M^2}{\sqrt{1 - M^2}}\,, \quad M = v_0/c_0\,. \\
&\to 0\,, \quad M \to 0
\end{aligned}
$$

Note that

$$
\begin{aligned}
k_y &= k\cos\phi = k_t = 2\pi/L \\
k_x &= k\sin\phi = \sqrt{k^2 - (2\pi/L)^2}\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
k &= \frac{\omega}{c_0 + v_0\cos\phi}\,, \quad \cos\phi = \frac{1}{c_t - v_0} \\
&= \frac{\omega(c_t - v_0)}{c_0(c_t - v_0) + v_0 c_)} \\
&= \frac{\omega}{c_t}\frac{c_t - v_0}{c_0} \\
&\to \frac{2\pi}{L}(-v_0/c_0)\,, \quad c_t \to 0
\end{aligned}
$$

After some algebra it is found that

$$
\begin{aligned}
k_x &= \frac{2\pi}{L}\sqrt{M^2 - 1} \\
&= i\frac{2\pi}{L}\sqrt{1 - M^2}\,, 1 > M \text{ (subsonic)}.
\end{aligned}
$$

Thus the radiated sound field,

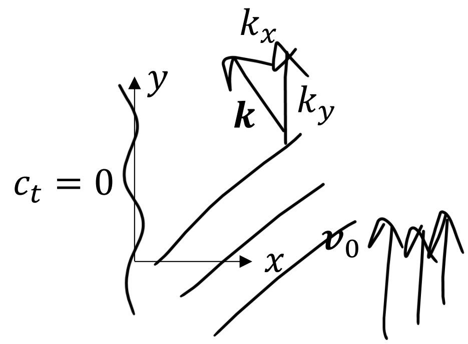
$$
\boxed{p(x, y, t) = 2\pi\rho_0 c_0^2 \frac{\xi_0}{L}\frac{M^2}{\sqrt{1 - M^2}} e^{-(1-M^2)^{1/2} 2\pi x/L} e^{i2\pi y/L}\,, \quad 1 > M}
$$

Note that the wave is stationary in space (e.g., in the frame *S*). It is sinusoidal in *y* and exponential decay in *x*. Thus, no power is radiated by the plate. The penetration depth in the *x* direction is *d* $\sim L/2\pi\sqrt{1 - M^2}$, which is ∞ for $M = 1$ (the condition at which radiation begins).

Meanwhile, for $M > 1$,

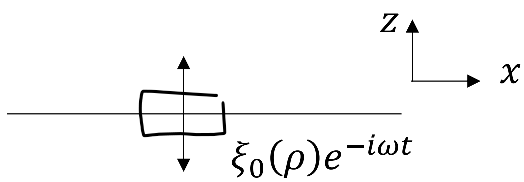
$$
p(x, y, t) \propto e^{i\frac{2\pi}{L}(-\sqrt{M^2-1}\,x + y)}\,,
$$

with the fluid field depicted below:



Piston radiation into crossflow

Now the radiation due to a piston in a crossflow is studied. Determining the boundary condition for the piston is essential to this problem, as Ingard showed. See pages 715-716 in Morse and Ingard (and Ingard, pages 405-409).



Ingard pointed out that the fundamental boundary condition is on the displacement field, not the velocity. Note that the velocity field is given in terms of the displacement field by

$$\mathbf{u} = \frac{D\xi}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla}\right)\xi\,, \tag{1}$$

and let the z component of the displacement be

$$\xi_z\Big|_{z=0} = \xi_0(\rho)e^{-i\omega t}\,.$$

Consider the source condition

$$\xi_0(\rho) = \xi_0 \operatorname{circ}(\rho/a) = \xi_0 H(a-\rho)$$

and suppose the crossflow is oriented purely in the x direction, i.e., $\mathbf{v}_0 = v_0\mathbf{e}_x$. Then, from Eq. (1), the velocity in the z direction is

$$\begin{aligned} u_0(\rho,\phi) &= (-i\omega + v_0\partial/\partial x)\xi_0 H(a-\rho) \\ &= [1 + i(M/k)\partial/\partial x]u_0 H(a-\rho)\,, \end{aligned} \tag{2}$$

where $M = v_0/c_0$ and $u_0 = -i\omega\xi_0$. The derivative of the step function is taken, noting that a delta function arises at $\rho = a$, and that $x = \rho\cos\phi$:

$$\frac{\partial}{\partial x} H(a-\rho) = \delta(a-\rho)(-\partial\rho/\partial x), \quad \rho = (x^2+y^2)^{1/2}$$
$$= \delta(a-\rho)(-2x/2\rho)$$
$$= -\delta(\rho-a)\cos\phi .$$

Combining the above result with Eq. (2) shows that the velocity field at the boundary is

$$u_0(\rho,\phi) = u_0[H(a-\rho) - i(M/k)\delta(\rho-a)\cos\phi] . \tag{3}$$

where ϕ is the polar angle in the x-y plane. Thus the velocity at the boundary imparts a dipolar radiation pattern.

To calculate the radiated field, first note that the angular spectrum of the source condition given by Eq. (3) is

$$U_0(k_x,k_y) = [1 + i(M/k)ik_x]u_0\,\mathcal{F}_{xy}\{H(a-\rho)\} , \tag{3}$$

where the factor of ik_x appears in the above by the sifting property of the delta function when the spatial Fourier transform is taken. Noting that

$$\mathcal{F}_{xy}\{H(a-\rho)\} = \pi a^2 \frac{2J_1(\kappa a)}{\kappa a}, \quad \kappa = \sqrt{k_x^2+k_y^2} ,$$

the angular spectrum becomes

$$U_0(k_x,k_y) = \pi a^2 u_0\,(1 - Mk_x/k)\frac{2J_1\left[(k_x^2+k_y^2)^{1/2}a\right]}{(k_x^2+k_y^2)^{1/2}a}$$

The spectrum width is of order $1/a$ in k-space, so k_x is also of order $1/a$. Thus

$$\frac{\text{edge effect}}{\text{surface effect}} = \mathcal{O}(M/ka) .$$

That is to say, the edge effect becomes important for $ka \ll 1$.

The pressure field is thus

$$p_\omega(x,y,z) = \rho_0 c_0 \mathcal{F}_{xy}^{-1}\left\{\frac{k_0 - Mk_x}{k_z} U_0(k_x,k_y)e^{ik_z z}\right\}$$
$$k_z = \sqrt{(k_0 - Mk_x)^2 - k_x^2 - k_y^2}, \quad k_0 = \omega/c_0 .$$

From the 2D spatial Fourier transform of momentum equation,

$$\mathbf{U} = \frac{\mathbf{k}P}{\rho_0 c_0(k_0 - \mathbf{M}\cdot\mathbf{k})}$$
$$k_z = \sqrt{(k_0 - \mathbf{M}_\perp\cdot\boldsymbol{\kappa})^2 - \kappa^2}$$

where $\mathbf{M}_\perp = M_x\mathbf{e}_x + M_y\mathbf{e}_y$

Periodic media

While I am certainly not asking you to close your eyes to the experiences of earlier generations, I want to advise you not to conform too soon and to resist the pressure of practical necessity. Free imagination is the inestimable prerogative of youth and it must be cherished and guarded as a treasure.

–Felix Bloch

This section on periodic media consists of only two sections. The first discusses the mass-spring transmission line, through which wave propagation is dispersive. Three special cases of the dispersion relation are studied. The second section discusses waves in layered media. This topic involves lots of algebra, much of which is skipped over. Invoking Bloch-Floquet theory, a dispersion relation is derived, and two limiting cases are considered.

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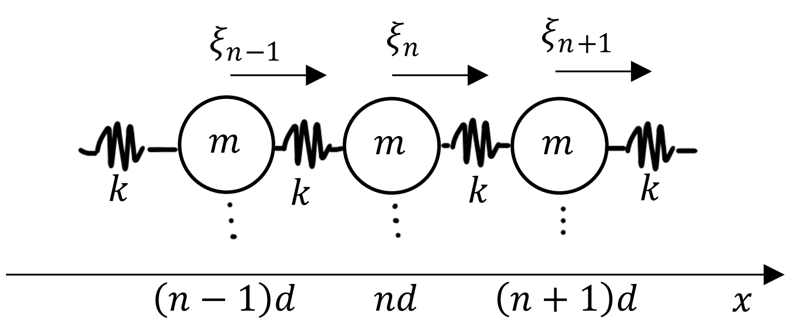
Mass-spring transmission line

Consider a mass-spring transmission line:



The position of the nth mass is given by

$$x_n = nd + \xi_n$$

where d = equilibrium separation distance

ξ_n = displacement

The dynamical equation $\mathbf{F} = m\mathbf{a}$ for the nth mass is one-dimensional:

$$m\frac{\partial^2 \xi_n}{\partial t^2} = k(\xi_{n-1} - \xi_n) - k(\xi_n - \xi_{n+1})$$
$$= k(\xi_{n-1} - 2\xi_n + \xi_{n+1})\,. \tag{1}$$

The signs can be determined by considering the case if the middle mass is held still (ξ_n = 0), but the mass on the left is moved to the right ($\xi_{n-1} > 0$). Then the middle mass experiences a positive force. Thus the first term on the RHS of Eq. (1) is positive. Meanwhile, if the middle mass is held still (ξ_n = 0) while the mass on the right is moved to the right ($\xi_{n+1} > 0$), Then the middle mass experiences a negative force, explaining why the second term on the RHS of Eq. (1) is negative.

A solution of Eq. (1) is sought that describes wave motion. Thus a traveling wave solution is assumed,

$$\xi_n = Ae^{i(\beta nd - \omega t)}$$
$$\text{where } \beta = \text{ propagation constant}\,.$$

Note that β and α are flipped in Morse and Ingard. Let us denote

$$\lambda = \text{ spatial period}$$
$$x = nd = \text{spatial coordinate}\,.$$

Then the spatial part of the complex exponential in the assumed form of solution can be written as

$$e^{i\beta nd} = e^{i\beta x} = e^{i\beta(x+\lambda)}\,, \tag{2}$$

which is satisfied by

$$\beta\lambda = 2\pi \implies \lambda = 2\pi/\beta.$$

This condition ensures that the motion is periodic in λ. Substituting Eq. (2) into (1) results in

$$-m\omega^2\xi_n = k(e^{-i\beta d} - 2 + e^{i\beta d})\xi_n$$
$$= k(e^{i\beta d/2} - e^{i\beta d/2})^2\xi_n$$
$$= -4k\sin^2(\beta d/2)\xi_n\,.$$

Nontrivial solutions ($\xi_n \neq 0$) arise for

$$\omega = 2\sqrt{k/m}\,\sin(\beta d/2)\,,$$

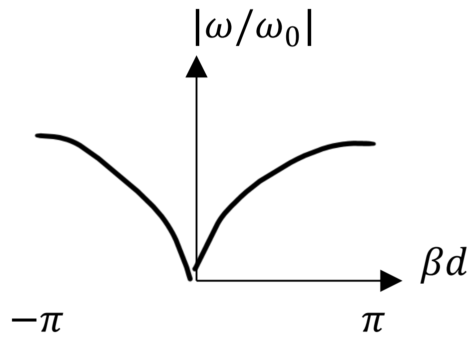
or, identifying $\omega_0 = 2\sqrt{k/m}$,

$$\boxed{\omega = \omega_0 \sin(\beta d/2)\,.}$$

Three few cases of this dispersion relation are studied:

Case I

First consider frequencies $\omega < \omega_0$. In this frequency range, the dispersion relation satisfies $\omega/\omega_0 = \sin(\beta d/2) < 1$, which implies that $-\pi/2 < \beta d/2 < \pi/2$, or simply $-\pi < \beta d < \pi$. The magnitude of the dispersion relation, $|\omega/\omega_0| = |\sin(\beta d/2)|$, is plotted below in this domain over βd:



Since $\lambda = 2\pi/\beta$, at $\omega = \omega_0$,

$$\lambda \equiv \lambda_0 = \left.\frac{2\pi}{\beta}\right|_{\beta=\pi/d} = 2d\,,$$

which is to say that the spatial period is $2d$.

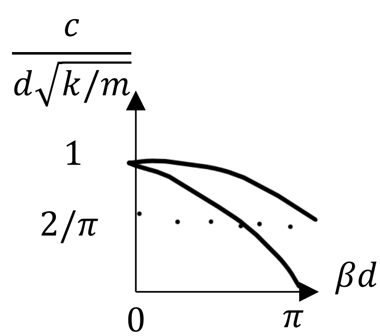
The corresponding phase and group velocities are

$$\begin{aligned} c_{\text{ph}} &= \frac{\omega}{\beta} = \frac{\omega_0}{\beta}\sin\frac{\beta d}{d} = \frac{1}{2}\omega_0 d\frac{\sin(\beta d/2)}{\beta d/2}\,. \\ c_{\text{gr}} &= \frac{d\omega}{d\beta} = \frac{1}{2}\omega_0 d\cos\frac{\beta d}{2}\,. \end{aligned}$$

In the limit that $|\beta d| \ll 1$ (which corresponds to $\omega \ll \omega_0$), the phase and group velocities become

$$c_{\text{ph}} \simeq c_{\text{gr}} \simeq \frac{1}{2}\omega_0 d = d\sqrt{k/m}\,.$$

This asymptote is used to normalize the plot below of the phase and group speeds:



Example: Recovery of continuum in the low frequency limit

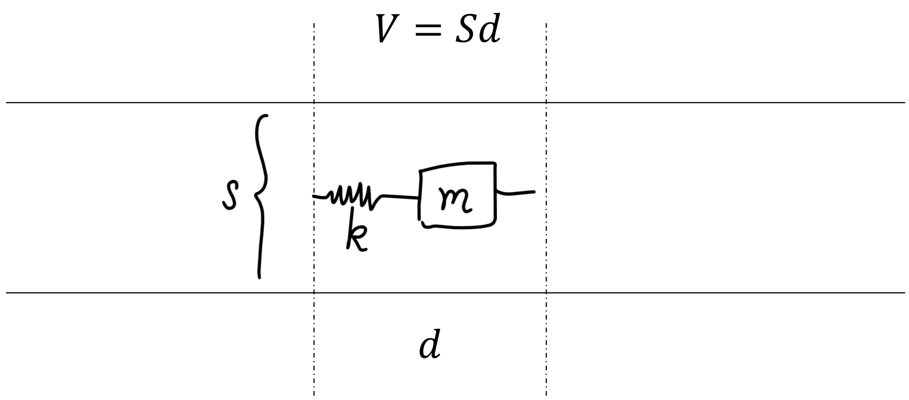
The propagation of sound in a fluid can be thought of the limiting case of the mass-spring chain. Specifically, it can be thought of lots of small mass-spring oscillators whose length scale is much smaller than a wavelength, i.e., $d \ll \lambda$, which is equivalent to the limit $|\beta d| \ll 1$ (because $\lambda/d = 2\pi/\beta d \gg 1$). In this limit, as noted above,

$$c_{\text{ph}} = c_{\text{gr}} = d\sqrt{k/m} = c_0\,.$$

The mass and stiffness of a fluid are related to the quantities

$$\rho_0 = \text{ density } = m/V$$
$$B = \text{ bulk modulus } = -V\frac{\partial P}{\partial V} = -d\frac{\partial P}{\partial x},$$

while the pressure can be described by the force divided by the surface area, $P = kx/S$, as illustrated below:



Thus the bulk modulus becomes

$$B = -d\frac{d}{dx}(-kx/S) = kd/S \implies k = SB/d\,.$$

Inserting $k = sB/d$ and $m = \rho_0 Sd$ into $c_0 = d\sqrt{k/m}$, the sound speed is found to be

$$c_0 = d\sqrt{\frac{SB/d}{\rho_0 Sd}} = \sqrt{B/\rho_0}$$

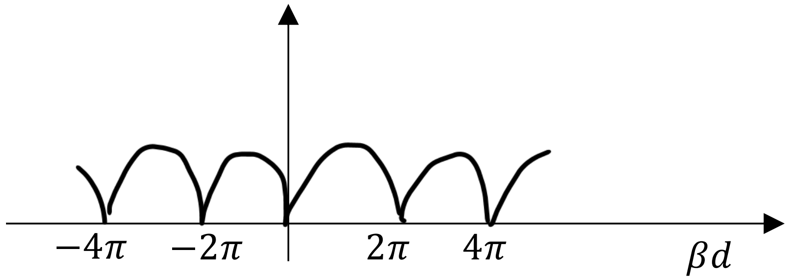
which is familiar sound speed of a fluid. *For more discussion, see page 88 of Morse & Ingard.*

Example: Wavelength ambiguity

"Wavelength ambiguity" is now discussed as an example of the case $\omega < \omega_0 = 2\sqrt{k/m}$, for which

$$\frac{\omega}{\omega_0} = \sin(\beta d/2)\,,\, \beta \in \Re \tag{1}$$

which is valid for **all** βd, as shown below:



That is to say, from the displacements

$$\xi_n = Ae^{i(n\beta d - \omega t)},$$

are unaltered if βd is replaced by

$$(\beta d)' = \beta d \pm 2\pi m, \quad m = \text{ integer}.$$

In other words, all the points separated by $2\pi m$ on the dispersion curve give the same solution, which is ambiguous. That is why the domain is restricted to

$$-\pi \le \beta d \le \pi$$

or

$$2d \le \lambda \le \infty.$$

Wavelengths λ less than $2d$ are called aliased (or "undersampled").

Case II

Now consider $\omega > \omega_0$, which is the case that the drive frequency is greater than $\omega_0 = 2\sqrt{k/m}$. Then, the dispersion relation must satisfy $\omega/\omega_0 = \sin(\beta d/2) > 1$, which requires that β must be complex (since $d/2$ is real). Thus define

$$\beta d/2 = \gamma + i\alpha,$$

insert this identification into the dispersion relation, and use the sine double-angle identity:

$$\begin{aligned}\frac{\omega}{\omega_0} &= \sin(\gamma + i\alpha)\\ &= \sin\gamma\cos i\alpha + \cos\gamma\sin i\alpha\\ &= \sin\gamma\cosh\alpha + i\cos\gamma\sinh\alpha.\end{aligned}$$

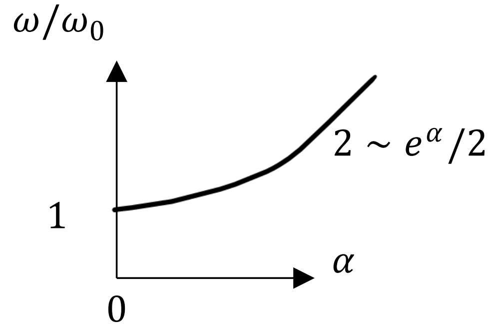
The drive frequency ω must be real, which requires $\gamma = \pi/2$ in the above relation. So

$$\omega/\omega_0 = \cosh\alpha \ge 1.$$

Then the motion of the nth mass is therefore

$$\begin{aligned}\xi_n &= Ae^{i(n\beta d-\omega t)}, \quad \beta d = \pi + i2\alpha \\ &= Ae^{i(n\pi+i2n\alpha-\omega t)} \\ &= (-1)^n Ae^{-2n\alpha}e^{-i\omega t} \implies |\xi| = Ae^{-2n\alpha},\end{aligned}$$

That is to say, the motion of each mass is exponential decay, and the direction in which the masses move alternates from mass to mass. The dispersion relation $\omega/\omega_0 = \cosh\alpha$ is plotted below:



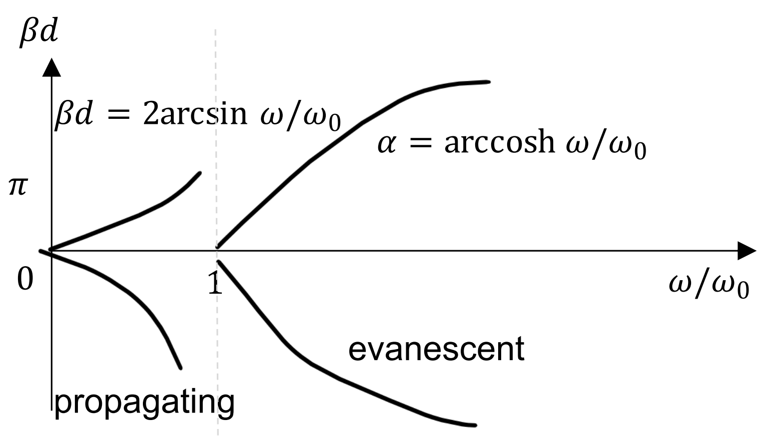
Case III

Finally, the case $\omega = \omega_0$ is considered. Then the dispersion relation reads $1 = \sin\beta d/2$, whose solution is $\beta d = \pi$ (and thus $\lambda = 2\pi/\beta = 2d$). The motion of the nth mass is therefore given by $\xi_n = Ae^{in\pi-i\omega}$, which gives

$$\xi_n = (-1)^n Ae^{i\omega t}.$$

Thus for the case $\omega = \omega_0$, the motion of all adjacent masses are equal and opposite to any given mass, and each moves as a simple harmonic oscillator.

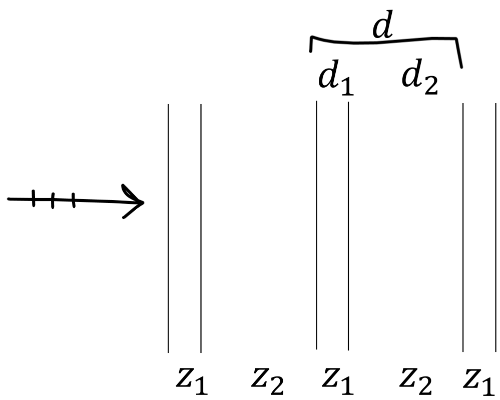
In summary, all three cases are shown in the dispersion curve below:



Propagation occurs only drive frequencies less than ω_0, i.e., the stopband is seen to extend for $\omega/\omega_0 > 1$.

Waves in layered media

Consider a layered medium with alternating impedances z_1 and z_2, where the z_1 medium has thickness d_1 while the thickness of the z_2 medium has thickness d_2:



The impedances are denoted $z_j = \rho_j c_j$, where $j = 1, 2$. The wave equations are

$$\frac{\partial^2 p_j}{\partial x^2} = \frac{1}{c_j^2}\frac{\partial^2 p_j}{\partial t^2}.$$

Assume the form of the pressure (and thus the particle velocity, by the momentum equation) is

$$p_j = (A_j e^{ik_j x} + B_j e^{-ik_j x})e^{-i\omega x} \tag{1}$$

$$u_j = \frac{1}{z_j}(A_j e^{ik_j x} - B_j e^{-ik_j x})e^{-i\omega x}, \tag{2}$$

where $k_j = \omega / c_j$. There are four unknowns in the unit cell, which is d wide, from $-d_1 \leq x \leq d_2$:

$$A_1, B_1, A_2, B_2 .$$

The pressure and particle velocity must match at $x = 0$:

$$p_1(0) = p_2(0)$$
$$u_1(0) = u_2(0) .$$

So from Eqs. (1) and (2) (resectively),

$$A_1 + B_1 = A_2 + B_2 \tag{3}$$

$$Z_2(A_1 - B_1) = Z_1(A_2 - B_2) . \tag{4}$$

To close the system algebraically, two more conditions are needed. For this, the concept of periodicity is invoked through

$$q = \text{ Bloch wave number}$$
$$P, U = \text{ Bloch wave functions},$$

such that

$$p_j = P_j(x)e^{i(qx-\omega t)} \tag{5}$$

$$u_j = U_j(x)e^{i(qx-\omega t)}, \tag{6}$$

where, by equating Eqs. (1) and (2) to Eqs. (5) and (6), respectively, P_j and U_j are identified,

$$P_j(x) = A_j e^{ik_j^- x} + B_j e^{-ik_j^+ x} \tag{7}$$
$$U_j(x) = \frac{1}{Z_j}\left(A_j e^{ik_j^- x} - B_j e^{-ik_j^+ x}\right), \tag{8}$$

where

$$k_j^{\pm} = k_j \pm q\,.$$

According to Floquet theory, because p and u are continuous across interfaces, P and U must be periodic:

$$P_1(-d_1) = P_2(d_2)$$
$$U_1(-d_1) = U_2(d_2)\,.$$

Note that p and u are not necessarily periodic in d. (Recall, for example, the mass-spring lattice, for which $\lambda \in [2d, \infty)$, e.g., the wavelength of p and u can be much larger than d). Enforcing Floquet's result on Eqs. (7) and (8) results in

$$A_1 e^{-ik_1^- d_1} + B_1 e^{ik_1^+ d_1} = A_2 e^{ik_2^- d_2} + B_2 e^{-ik_2^+ d_2} \tag{9}$$
$$Z_2(A_1 e^{-ik_1^- d_1} - B_1 e^{ik_1^+ d_1}) = Z_1(A_2 e^{ik_2^- d_2} - B_2 e^{-ik_2^+ d_2})\,. \tag{10}$$

Now Eqs. (3), (4), (9), and (10) are combined:

$$[A]\cdot\begin{bmatrix} A_1 \\ B_1 \\ -A_2 \\ -B_2 \end{bmatrix} = [0]\,,$$

where

$$[A] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ Z_2 & -Z_2 & Z_1 & -Z_1 \\ e^{-ik_1^- d_1} & e^{ik_1^+ d_1} & e^{ik_2^- d_2} & e^{-ik_2^+ d_2} \\ Z_2 e^{-ik_1^- d_1} & -Z_2 e^{ik_1^+ d_1} & Z_1 e^{ik_2^- d_2} & -Z_1 e^{-ik_2^+ d_2} \end{bmatrix}.$$

Nontrivial solutions correspond to $\det[A] = 0$, which result in the characteristic equation

$$\boxed{\cos qd = \cos k_1 d_1 \, \cos k_2 d_2 - \frac{1}{2}\left(\frac{z_1}{z_2} + \frac{z_2}{z_1}\right)\sin k_1 d_1 \sin k_2 d_2\,.} \tag{11}$$

Equation (11) is wirtten in the form

$$\cos qd = f(\omega)\,.$$

Then [Bradley, JASA **96**, 1844 (1994)],

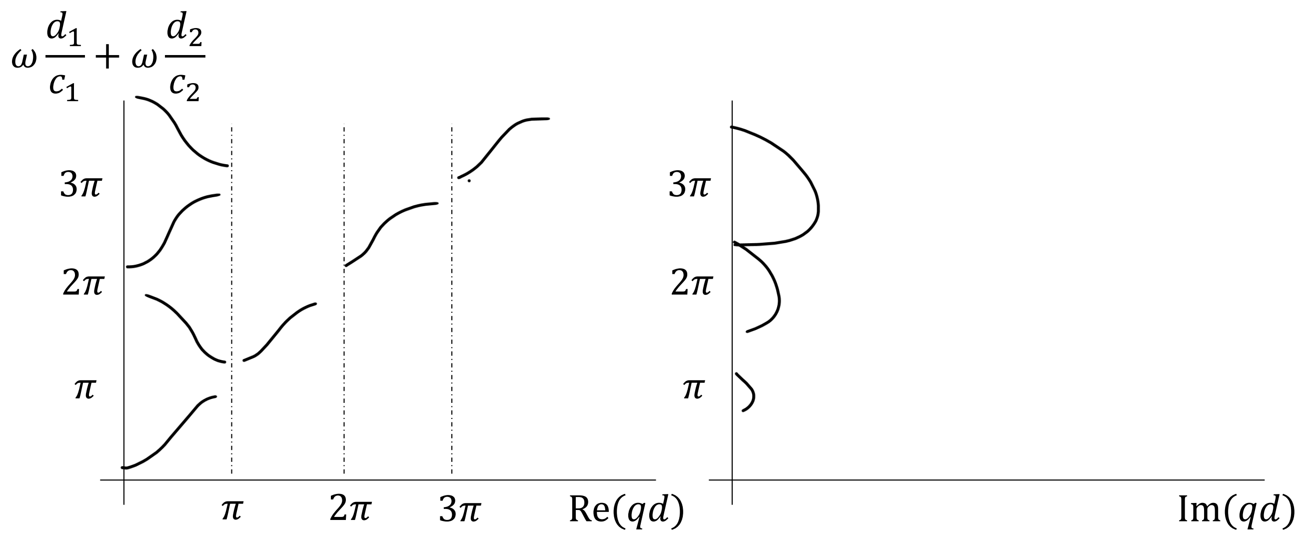
$$qd = \begin{cases} \arccos(f) = \Re\,, & -1 \le f \le 1 \\ n\pi + i\text{arccosh}^{-1} f\,, n \text{ even} & , f > 1 \\ n\pi + i\text{arccosh}^{-1} |f|\,, n \text{ odd} & , f \text{ less than } 1 \end{cases}$$

Thus the Bloch wavenumber is

$$
\begin{aligned}
q &= q_R + i q_I \\
q_I &= 0\,, \quad \text{passband} \\
q_I &\neq 0\,, \quad \text{stopband}
\end{aligned}
$$

and the Bloch wave phase speed is

$$c_B = \omega / q_R\,.$$



Two limiting cases of Eq. (11) are now considered.

Limiting case I: $z_1 = z_2$

The dispersion relation becomes

$$
\begin{aligned}
\cos qd &= \cos k_1 d_1 \cos k_2 d_2 - \sin k_1 d_1 \sin k_2 d_2 \\
&= \cos(k_1 d_1 + k_2 d_2)\,.
\end{aligned}
$$

Thus there are no stopbands, e.g., q is always real and equal to ω / c_B, where c_B is the Bloch wave speed defined above. From this limiting form, the relation

$$\frac{d}{c_B} = \frac{d_1}{c_1} + \frac{d_2}{c_2}\,,$$

which results in an analytical expression for the Bloch wave speed:

$$c_B = \frac{d}{d_1/c_1 + d_2/c_2}\,.$$

Limiting case II: $\lambda \gg d$ ($\omega \to 0$, the effective medium limit)

The terms of dispersion relation in this low-frequency limit become

$$\cos qd \sim 1 - \frac{1}{2}(qd)^2$$
$$\sin kd \sim kd\,.$$

Through $\mathcal{O}(q^2, k^2)$, the dispersion relation reads

$$1 - \frac{1}{2}(qd)^2 = 1 - \frac{1}{2}(k_1 d_1)^2 - \frac{1}{2}(k_2 d_2)^2 - \frac{1}{2}\left(\frac{z_1}{z_2} + \frac{z_2}{z_1}\right) k_1 d_1 k_2 d_2\,.$$

Rearranging this relation results in

$$\frac{d^2}{c_B^2} = \frac{d_1^2}{c_1^2} + \left(\frac{z_1}{z_2} + \frac{z_2}{z_1}\right)\frac{d_1 d_2}{c_1 c_2} + \frac{d_2^2}{c_2^2}\,,$$

which is a constant (i.e., no dispersion). Interestingly, if $c_1 = c_2 \equiv c_0$ and yet $\rho_1 \neq \rho_2$, then $c_B < c_0$, which is not necessarily an obvious result.

Now attention is turned back to the eigenvectors corresponding to q. The matrix equation is re-labeled: (Where does this come from?)

$$\begin{pmatrix} A_1 \\ B_1 \\ -A_2 \\ -B_2 \end{pmatrix} = A \begin{pmatrix} C_{11} \\ C_{12} \\ C_{13} \\ C_{14} \end{pmatrix}\,,$$

where

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

is the cofactor of A_{ij}, and where A is an arbitrary constant. Then, define from Eq. (7)

$$P_0(x) = A[C_{11} e^{i(k_1 - q)x} + C_{12} e^{-i(k_1 + q)x}]\,, \quad -d_1 \leq x \leq 0\,. \tag{12}$$
$$= -A[C_{13} e^{i(k_2 - q)x} + C_{14} e^{-i(k_2 + q)x}]\,, \quad 0 \leq x \leq d_2\,. \tag{13}$$
$$= 0 \text{ elsewhere}\,. \tag{2}$$

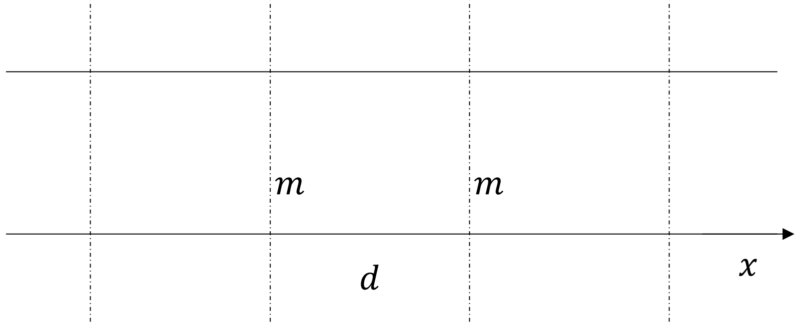
Then $P(x) = P_0(x)$ in the reference unit cell, labeled $n = 0$. Since P_0 is repeated in other cells, then

$$p(x,t) = [P_0(x) * \sum_{n=-\infty}^{\infty} \delta(x - nd)] e^{i(qx - \omega t)} \tag{14}$$

where $P_0(x) * \sum_{n=-\infty}^{\infty} \delta(x - nd) = \Phi(x)$ is the so-called Bloch function.

Example: Thin periodic barriers

Consider the limits $k_2 d_2 \ll 1$ and $z_2/z_1 \gg 1$, e.g., the mass law. Layer 2 is thus a thin mass, e.g., $d_1 \simeq d$. Call $\rho_1 c_1 = \rho_0 c_0$:



The mass per unit area is $m = \rho_2 d_2$. Then the dispersion relation becomes

$$\cos qd = \cos(\omega d/c_0) - \frac{1}{2}\frac{z_2}{z_1} k_2 d_2 \sin(\omega d/c_0)\,.$$

Noting that the factor $\frac{z_2}{z_1} k_2 d_2$ can be written

$$\frac{z_2}{z_1} k_2 d_2 = \frac{\rho_2 c_2}{\rho_0 c_0}\frac{\omega d_2}{c_2} = \frac{\rho_2 d_2 \omega}{\rho_0 c_0} = \frac{m\omega}{\rho_0 c_0}\,,$$

the dispersion relation becomes

$$\cos qd = \cos(\omega d/c_0) - \frac{m\omega}{2\rho_0 c_0}\sin(\omega d/c_0).$$

This relation can be written in a dimensionless form by defining $Q \equiv qd$, $\Omega = \omega d/c_0$, and $M = m/2\rho_0 d$:

$$\cos Q = \cos\Omega - M\Omega\sin\Omega\,. \tag{1}$$

For increasing $M\Omega$, one obtains increasingly narrower passbands centered at $\Omega \sim n\pi$.

In the Bloch function, set $d_1 = d$ and $k_2 d_2 = 0$ in $[A]$:

$$C_{11} = \det\begin{bmatrix} -z_2 & z_1 & -z_1 \\ e^{ik^+d} & 1 & 1 \\ -Z_2 e^{ik^+d} & z_1 & -z_1 \end{bmatrix} = 2z_1 z_2[1 - e^{i(k+q)d}]$$

$$C_{12} = -\det\begin{bmatrix} z_2 & z_1 & -z_1 \\ e^{ik^-d} & 1 & 1 \\ Z_2 e^{ik^-d} & z_1 & -z_1 \end{bmatrix} = 2z_1 z_2[1 - e^{-i(k-q)d}]$$

Thus from $-d \le x \le 0$,

$$\boxed{P_0(x) = [1 - e^{i(k+q)d}]e^{i(k-q)x} + [1 - e^{-i(k-q)d}]e^{-i(k+q)x}\,.}$$

and $P_0(x) = 0$ elsewhere. By Eq. (14), the solution in the entire medium is

$$\boxed{p(x,t) = \underbrace{P_0\left[\sum_{n=-\infty}^{\infty} P_0(x - nd)\right]}_{\text{periodic in } d}\underbrace{e^{i(qx-\omega t)}}_{\text{periodic in } 2\pi/\Re q}\,.}$$

As a sanity check, note that for $M = 0$, $q = \omega/c_0 = k$, and thus

$$P_0(x) = A(1 - e^{i2kd}) \rightarrow B = \text{ const.}$$

Thus

$$p(x,t) = Be^{i(kx-\omega t)} \,.$$

Note that although $d_2 \rightarrow 0$, $P_0(0) \neq P_1(d_2) = P_0(-d_1)$ because of pressure jumps across the mass.

Scattering theory

> *The scientist's urge to investigate, like the faith of the devout or the inspiration of the artist, is an expression of mankind's longing for something fixed, something at rest in the universal whirl: God, Beauty, Truth. Truth is what the scientist aims at. He finds nothing at rest, nothing enduring, in the universe. Not everything is knowable, still less predictable. But the mind of man is capable of grasping and understanding at least a part of Creation; amid the flight of phenomena stands the immutable pole of law.*

–Max Born

The scattering of waves is described by adding an inhomogeneous term (a δ function) to the Helmholtz equation, which is solved by the Green's function. Green's functions in 1D, 2D, and 3D are developed in the first two sections. Reciprocity's constraints on the Green's function for situations in which there exist boundaries are then discussed. See here for Dr. Hamilton's notes on how to expand G using normal mode expansions (not covered in the lecture). The Helmholtz-Kirchhoff integral is derived. It is then shown that the multipole expansion can be expressed entirely in terms of Green's functions and their derivatives. The exact theory of scattering is then derived. Three useful approximations of of this exact theory are then derived: far-field, Born, and Rayleigh. The remainder of the course applies these approximations to the scattering of sound by bubbles.

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General definition of Green's function

Consider a linear differential operator $\mathcal{L}$ (e.g., $\nabla^2 + k^2$). A Green's function g satisfies

$$\mathcal{L}\{g(\mathbf{r}|\mathbf{r}_0)\} = -\delta(\mathbf{r} - \mathbf{r}_0)\,. \tag{1}$$

Now consider the general inhomogeneous differential equation

$$\mathcal{L}\{\phi(\mathbf{r})\} = -f(\mathbf{r})\,. \tag{2}$$

Let the right-hand side of Eq. (2) be written in terms of an integral over a delta function, by the sifting property:

$$f(\mathbf{r}) = \int f(\mathbf{r}_0)\delta(\mathbf{r} - \mathbf{r_0})dV_0 \,.$$

Then Eq. (2) becomes

$$\mathcal{L}\{\phi(\mathbf{r})\} = -\int f(\mathbf{r}_0)\delta(\mathbf{r} - \mathbf{r_0})dV_0.$$

Replacing $-\delta(\mathbf{r} - \mathbf{r}_0)$ with the right-hand side of Eq. (1), Eq. (2) becomes

$$\begin{aligned}\mathcal{L}\{\phi(\mathbf{r})\} &= \int f(\mathbf{r}_0)\mathcal{L}\{g(\mathbf{r}|\mathbf{r}_0)\}dV_0 \\ &= \mathcal{L}\left\{\int f(\mathbf{r}_0)g(\mathbf{r}|\mathbf{r}_0)dV_0\right\} \,,\end{aligned}$$

where $\mathcal{L}$ is removed from the integral by the linearity of the integral operation. The particular solution of Eq. (2) is therefore

$$\phi(\mathbf{r}) = \int f(\mathbf{r_0})g(\mathbf{r}|\mathbf{r}_0)dV_0$$

where $g(\mathbf{r}|\mathbf{r}_0)$ is the so-called free space Green's function is defined by Eq. (1). g satisfies $\mathcal{L}\{g(\mathbf{r}|\mathbf{r}_0)\}$ = 0 everywhere except for at $\mathbf{r} = \mathbf{r}_0$.

The "−" sign in Eq. (1) is motivated by electrostatics. Gauss's law states that $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, where $\mathbf{E}$ is the electric field, ρ is the charge density, and ϵ_0 is the permittivity of free space. Meanwhile, the definition of the electric field is given in terms of the electric potential as $\mathbf{E} = -\nabla V$. Combining these equations leads to Poisson's equation, which has a negative right-hand side: $\nabla^2 V = -\rho/\epsilon_0$. Thus it proves convenient in electrostatics to let $g(\mathbf{r}|\mathbf{r}_0)$ solve the inhomogeneous Poisson equation with negative right-hand side.

Green's functions of inhomogeneous 1D, 2D, 3D Helmholtz equations

1D and 3D cases are discussed first. Attention is then turned to the more challenging 2D case, which is handled by modifying the 3D result.

1D Green's function of the Helmholtz equation

Consider the inhomogeneous Helmholtz equation in 1D, for which $\mathcal{L} = \partial^2/\partial x^2 + k^2$:

$$\frac{\partial^2 g}{\partial x^2} + k^2 g = -\delta(x - x_0) \,. \tag{1}$$

The solution $g(x|x_0)$ must have the form

$$g(x|x_0) = Ae^{ik(x-x_0)}\,, \quad x > x_0$$
$$= Ae^{-ik(x-x_0)}\,, \quad x < x_0\,,$$

which can be written as

$$g(x|x_0) = Ae^{ik|x-x_0|}, \quad x \neq x_0\,. \tag{2}$$

To calculate the constant A, Eq. (1) is integrated:

$$\left.\frac{\partial g}{\partial x}\right|_{x_0-\epsilon}^{x_0+\epsilon} + k^2 \int_{x_0-\epsilon}^{x_0+\epsilon} g dx = -1\,. \tag{3}$$

The first term in Eq. (3) is

$$\left.\frac{\partial g}{\partial x}\right|_{x_0-\epsilon}^{x_0+\epsilon} = ikAe^{ik\epsilon} - (-ik)Ae^{ik\epsilon}$$
$$= 2ikA\,, \quad \epsilon \to 0\,.$$

In the second term of Eq. (3), the integral is

$$\int_{x_0-\epsilon}^{x_0+\epsilon} g dx = \left(\int_{x_0-\epsilon}^{x_0} + \int_{x_0}^{x_0+\epsilon}\right) g dx$$
$$= \frac{A}{ik}(1 - e^{ik\epsilon}) - \frac{A}{ik}(e^{-ik\epsilon} - 1)$$
$$\to 0\,, \quad \epsilon \to 0\,.$$

Therefore the constant A is found to be

$$A = -\frac{1}{2ik}\,,$$

so the Green's function of the 1D Helmholtz equation is

$$\boxed{g(x|x_0) = \frac{i}{2k} e^{ik|x-x_0|}\,.} \tag{1}$$

3D Green's function of the Helmholtz equation

The inhomogeneous Helmholtz equation is now

$$\nabla^2 g + k^2 g = -\delta(\mathbf{r} - \mathbf{r}_0)\,. \tag{1}$$

The solution is supposed to have the form

$$g(\mathbf{r}|\mathbf{r}_0) = A\frac{e^{ikR}}{R}\,,$$

where

$$R = |\mathbf{r} - \mathbf{r}_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}\,.$$

To determine the constant A, Eq. (1) is integrated over the volume of a sphere of radius ϵ centered at $\mathbf{r} = \mathbf{r}_0$. Noting that the surface area of this integral is $S = 4\pi\epsilon^2$, its volume is $V = \frac{4}{3}\pi\epsilon^3$, and $dV = 4\pi R^2 dR$, the integral is taken,

$$\int \nabla^2 g dV + k^2 \int g dV = -1 .$$

The second integral is easy to take:

$$\begin{aligned}
\int g dV &= \int_0^\epsilon A \frac{e^{ikR}}{R} 4\pi R^2 dR \\
&= 4\pi A \int_0^\epsilon e^{ikR} R dR \\
&= 4\pi A \frac{e^{ikR}}{R} (R - 1/ik) \Big|_0^\epsilon \to 0 , \quad \epsilon \to 0 .
\end{aligned}$$

Meanwhile, for the first integral, Gauss's theorem is invoked:

$$\begin{aligned}
\int \nabla^2 g dV &= \int \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} g) dV \\
&= \oint (\boldsymbol{\nabla} g) \cdot d\mathbf{S} \\
&= \oint \frac{\partial g}{\partial R} dS \\
&= \oint \left(-\frac{1}{R} + ik \right) A \frac{e^{ikR}}{R} dS \\
&= \left(-\frac{1}{\epsilon} + ik \right) A \frac{e^{ik\epsilon}}{\epsilon} 4\pi\epsilon^2 \\
&= -4\pi A , \quad \epsilon \to 0 .
\end{aligned}$$

Thus A is determined to be $1/4\pi$. The Green's function in 3D is therefore

$$\boxed{g(\mathbf{r}|\mathbf{r}_0) \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{4\pi|\mathbf{r}-\mathbf{r}_0|} = \frac{e^{ikR}}{4\pi R} .}$$

2D Green's function of the Helmholtz equation

The approach presented here uses the result derived above for the 3D Green's function of the Helmholtz equation. However, it can more straightforwardly be derived by applying the same approach taken in the 3D case, only using Green's theorem in this form, instead of the 3D divergence theorem. This approach is assigned as a homework problem.

In 2D, the inhomogeneous Helmholtz equation in Cartesian coordinates is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) g + k^2 = -\delta(x - x_0)\delta(y - y_0) .$$

Equivalently, the inhomogeneous Helmholtz equation can be written in 3D as

$$\nabla^2 g + k^2 g = -f(x, y) , \tag{1}$$

where the Laplacian is the 3D operator, and where

$$f(x,y) = \delta(x-x_0)\delta(y-y_0)\,.$$

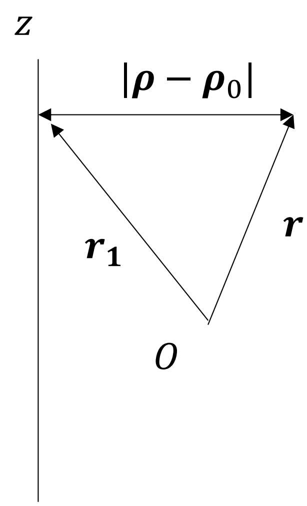
The 3D solution of Eq. (1) was found to be

$$g(\boldsymbol{\rho}|\boldsymbol{\rho}_0) = \int f(x_0,y_0)g(\mathbf{r}|\mathbf{r_1})dV_1\,,\quad g(\mathbf{r}|\mathbf{r}_1) = \frac{e^{ik|\mathbf{r}-\mathbf{r}_1|}}{4\pi|\mathbf{r}-\mathbf{r}_1|}\,.$$

Thus in 2D Cartesian coordinates, the Green's function is

$$\begin{aligned} g(\boldsymbol{\rho}|\boldsymbol{\rho}_0) &= \int \delta(x_1-x_0)\delta(y_1-y_0)\frac{e^{ik\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}}{4\pi\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}dV_1\,,\\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{e^{ik\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_1)^2}}}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_1)^2}}dz_1\,,\end{aligned}$$

where $dV_1 = dx_1dy_1dz_1$. Note that $(x-x_0)^2+(y-y_0)^2 = |\boldsymbol{\rho}-\boldsymbol{\rho}_0|^2$, as illustrated below:



Employing the change-of-variables

$$k(z-z_1) = -t\,,\quad dz_1 = \frac{dt}{k}\,,$$

the integral above becomes

$$g(\boldsymbol{\rho}|\boldsymbol{\rho}_0) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{e^{ik\sqrt{k^2|\boldsymbol{\rho}-\boldsymbol{\rho}_0|^2+t^2}}}{4\pi\sqrt{k^2|\boldsymbol{\rho}-\boldsymbol{\rho}_0|^2+t^2}}dt\,.$$

Now note that an integral representation of the 0th order Hankel function of the first kind is

$$H_0^{(1)}(x) = \frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{e^{i\sqrt{x^2+t^2}}}{\sqrt{z^2+t^2}}dt\,.$$

The 2D Green's function is therefore

$$\boxed{g(\boldsymbol{\rho}|\boldsymbol{\rho}_0) = \frac{i}{4} H_0^{(1)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}_0|)\,.}$$

Note that since

$$H_0^{(1)}(x) \simeq \sqrt{\frac{2}{\pi z}} e^{i(z-\pi/4)}\,, \quad |z| \gg 1\,,$$

the 2D Green's function in the far field is

$$g(\boldsymbol{\rho}|\boldsymbol{\rho}_0) \simeq \frac{e^{i\pi/4}}{\sqrt{8\pi}} \frac{e^{ik|\boldsymbol{\rho}-\boldsymbol{\rho}_0|}}{\sqrt{k|\boldsymbol{\rho}-\boldsymbol{\rho}_0|}}\,, \quad k|\boldsymbol{\rho}-\boldsymbol{\rho}_0| \gg 1\,.$$

The above approximation of the 2D Green's function of the Helmholtz equation is often a starting-point for the ocean acousticians, who always seem to work in the far field.

Reciprocity

Reciprocity is the invariance under the exchange of source and receiver, which corresponds to the exchange of $\mathbf{r}$ and $\mathbf{r}_0$ in a Green's function. The free space Green's function $g(\mathbf{r}|\mathbf{r}_0)$ of the the Helmholtz equation in 1D, 2D, and 3D is always reciprocal because $\mathbf{r}$ and $\mathbf{r}_0$ always appear in the combination $|\mathbf{r} - \mathbf{r}_0|$ which is equal to $|\mathbf{r}_0 - \mathbf{r}|$. However, the issue of reciprocity becomes more subtle when considering the presence of boundaries. Thus this discussion starts with some remarks about boundaries when solving an inhomogeneous Helmholtz equation.

Consider a Green's function that satisfies

$$\nabla^2 G + k^2 G = -\delta(\mathbf{r} - \mathbf{r}_0)\,, \tag{1}$$

where (following Morse and Ingard's notation, Eq. 7.1.15)

$$G(\mathbf{r}|\mathbf{r}_0) = g(\mathbf{r}|\mathbf{r}_0) + h(\mathbf{r})$$

where

$$g(\mathbf{r}|\mathbf{r}_0) = \frac{e^{ikR}}{4\pi R}\,, R = |\mathbf{r} - \mathbf{r}_0|\,,$$

and where $h(\mathbf{r})$ is the homogeneous solution, i.e.,

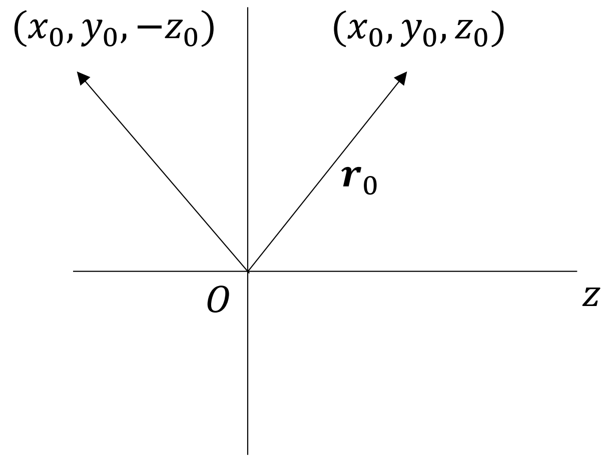
$$\nabla^2 h + k^2 h = 0\,. \tag{2}$$

That is to say, h is added to g to make the Green's function G satisfy not only 1 but also any boundary condition(s) involved in a physical situation.

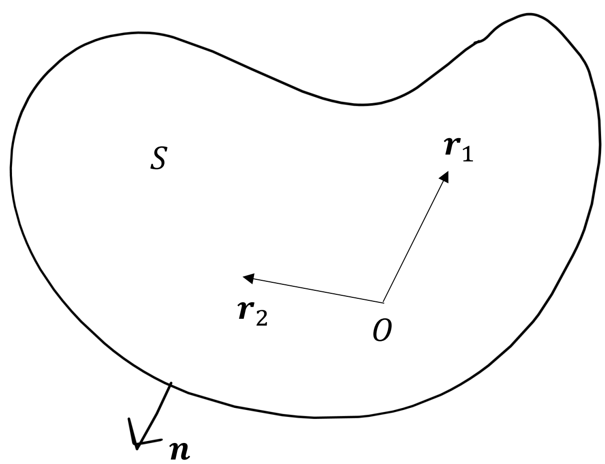
Example: Rigid wall

Let $h(\mathbf{r}) = g(\mathbf{r}|x_0, y_0, -z_0)$. Since the image $h(\mathbf{r})$ is in the half-space $z < 0$, Eq. (2) is satisfied in the half-space $z > 0$ containing the sound field, and thus G is satisfied by both Eq. (1) and the boundary condition

$$\left.\frac{\partial G}{\partial z}\right|_{z=0} = 0\,.$$



To introduce the concept of reciprocity, consider a sound field is enclosed by a surface S with two source points at positions $\mathbf{r}_1$ and $\mathbf{r}_2$, as illustrated below:



At these positions,

$$(\nabla^2 + k^2)G(\mathbf{r}|\mathbf{r}_1) = -\delta(\mathbf{r} - \mathbf{r}_1) \tag{a}$$
$$(\nabla^2 + k^2)G(\mathbf{r}|\mathbf{r}_2) = -\delta(\mathbf{r} - \mathbf{r}_2)\,. \tag{b}$$

Now the following volume integral is calculated:

$$\int_V \{G(\mathbf{r}|\mathbf{r}_2)[\text{Eq. } (a)] - G(\mathbf{r}|\mathbf{r}_1)[\text{Eq. } (b)]\}dV\,.$$

The k^2 terms cancel, leaving

$$\int_V \{G(\mathbf{r}|\mathbf{r}_2)\nabla^2 G(\mathbf{r}|\mathbf{r}_1) - G(\mathbf{r}|\mathbf{r}_1)\nabla^2 G(\mathbf{r}|\mathbf{r}_r)dV = -G(\mathbf{r}_1|\mathbf{r}_2) + G(\mathbf{r}_2|\mathbf{r}_1)\,.$$

By the divergence theorem,

$$\oint \left[G(\mathbf{r}|\mathbf{r}_2) \frac{\partial G(\mathbf{r}|\mathbf{r}_1)}{\partial n} - G(\mathbf{r}|\mathbf{r}_1) \frac{\partial G(\mathbf{r}|\mathbf{r}_2)}{\partial n} \right] dS = -G(\mathbf{r}_1|\mathbf{r}_2) + G(\mathbf{r}_2|\mathbf{r}_1) \,. \tag{3}$$

The statement of reciprocity is that $G(\mathbf{r}_1|\mathbf{r}_2) = G(\mathbf{r}_2|\mathbf{r}_1)$, which corresponds to the the left-hand side of Eq. (3) vanishing. This is achieved on S for the three following conditions:

$$\begin{aligned} G &= 0\,, \quad \text{Dirichlet, free surface} \\ \frac{\partial G}{\partial n} &= 0\,, \quad \text{Neumann, rigid surface} \\ \frac{G}{\partial G/\partial n} &= \text{const.}\,, \quad \text{can be complex} \end{aligned}$$

The third condition includes the first two conditions and corresponds to

$$\frac{p_\omega}{\mathbf{u}_\omega \cdot \mathbf{n}} = Z = \text{ acoustic impedance}\,,$$

which is said to correspond to a "locally reacting" medium, which does not support wave motion. For more insight, note that from the space- and time-harmonic momentum equation,

$$-ik\rho_0 c_0 \mathbf{u}_\omega \cdot \mathbf{n} + \frac{\partial p_\omega}{\partial n} = 0\,.$$

The condition $\frac{G}{\partial G/\partial n}$ therefore becomes

$$\frac{G}{\partial G/\partial n} = \frac{Z}{ik\rho_0 c_0}.$$

Thus, if the surfaces are locally reacting (or at $r = \infty$) and G satisfies any of the above conditions, then

$$\boxed{G(\mathbf{r}_1|\mathbf{r}_2) = G(\mathbf{r}_2|\mathbf{r}_1)},$$

which implies that the source and receiver can be reversed in space without changing the measured sound.

Helmholtz-Kirchhoff integral

The Helmholtz-Kirchhoff integral is a starting-point for the study of both scattering and diffraction. It calculates the field radiated by an arbitrary volume and/or surface distribution of scattering or radiating elements. It is shown in the example below that the Helmholtz-Kirchhoff integral can be used to derive the first Rayleigh integral, which was previously derived in this course using Fourier acoustics and the convolution theorem. The application of the Helmholtz-Kirchhoff integral to scattering will be seen in the following section.

Begin by considering two inhomogeneous Helmholtz equations:

$$(\nabla^2 + k^2) p_\omega(\mathbf{r}) = -f(\mathbf{r}) \tag{1}$$

$$(\nabla^2 + k^2) G(\mathbf{r}|\mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r_0}) \tag{2}$$

Take G times Eq. (1) minus p_ω times Eq. (2):

$$G(\mathbf{r}|\mathbf{r}_0)\nabla^2 p_\omega(\mathbf{r}) - p_\omega(\mathbf{r})\nabla^2 G(\mathbf{r}|\mathbf{r}_0) = -G(\mathbf{r}|\mathbf{r}_0) f(\mathbf{r}) + p_\omega(r)\delta(\mathbf{r} - \mathbf{r}_0)\,. \tag{3}$$

Now rename $\mathbf{r}$ with $\mathbf{r}_0$ and visa versa, and require G to be reciprocal whether or not any boundary conditions are satisfied, e.g., can let $G = g = e^{ikR}/4\pi R$:

$$G(\mathbf{r}|\mathbf{r}_0) = G(\mathbf{r}_0|\mathbf{r})\,.$$

Now take $\int_V [\text{Eq. (3)}] dV_0$:

$$\int [G(\mathbf{r}|\mathbf{r}_0)\nabla_0^2 p_\omega(\mathbf{r}_0) - p_\omega(\mathbf{r}_0)\nabla_0^2 G(\mathbf{r}|\mathbf{r}_0) dV_0]$$
$$= -\iiint G(\mathbf{r}|\mathbf{r}_0) f(\mathbf{r}_0) dV_0 + \int p_\omega(\mathbf{r}_0)\delta(\mathbf{r} - \mathbf{r}_0) dV_0\,,$$

Now use Gauss's theorem to solve (implicitly) for $p_\omega(\mathbf{r})$:

$$\boxed{p_\omega(\mathbf{r}) = \int f(\mathbf{r}) G(\mathbf{r}|\mathbf{r}_0) dV_0 + \oint \left[G(\mathbf{r}|\mathbf{r}_0)\frac{\partial p_\omega(\mathbf{r}_0)}{\partial n_0} - p_\omega(\mathbf{r_0})\frac{\partial G(\mathbf{r}|\mathbf{r}_0)}{\partial n_0}\right] dS_0} \tag{5}$$

where $\mathbf{n}$ is the outward unit normal. If

$$\frac{p_\omega}{\partial p_\omega/\partial n_0} = \frac{G}{\partial G/\partial n_0} = \frac{Z}{ik\rho_0 c_0}\,,$$

then the surface integral vanishes, and the complete solution is

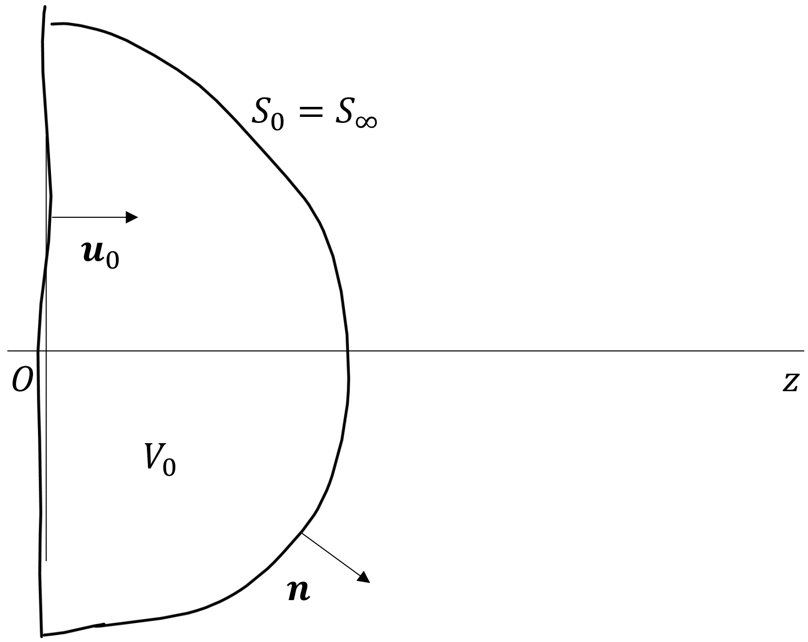
$$p_\omega(\mathbf{r}) = \int_V f(\mathbf{r}_0) G(\mathbf{r}|\mathbf{r}_0) dV_0\,.$$

Example: Radiation into a half-space ($z > 0$) by velocity source

Consider the velocity source ($z = 0$)

$$u(x, y, t) = u_0(x, y)\, e^{-i\omega t}\,.$$

Then the derivative $\partial/\partial n_0$ is simply $-\partial/\partial z_0$ on S_0. Also assume there are no source inside the volume. Then $f(\mathbf{r}) = 0$. Further, suppose the field vanishes at infinity.



Thus the field is, from Eq. (5),

$$p_\omega(\mathbf{r}) = -\iint_{-\infty}^{\infty} \left[G(\mathbf{r}|\mathbf{r}_0) \frac{\partial p_\omega(\mathbf{r}_0)}{\partial z_0} - p_\omega(\mathbf{r_0}) \frac{\partial G(\mathbf{r}|\mathbf{r}_0)}{\partial z_0} \right] dx_0 dy_0 \,. \tag{1}$$

Since an equation that maps particle velocity to pressure is desired, the momentum equation is invoked:

$$\left.\frac{\partial p_\omega}{\partial z_0}\right|_{z_0=0} = ik\rho_0 c_0 u_0(x_0, y_0) \,. \tag{2}$$

Meanwhile, it is desired to eliminate $p_\omega(\mathbf{r}_0)$ from the integral. As such, G is chosen such that

$$\left.\frac{\partial G}{\partial z_0}\right|_{z_0=0} = 0 \,, \tag{3}$$

which is satisfied for the choice

$$G(\mathbf{r}|\mathbf{r}_0) = g(\mathbf{r}|x_0, y_0, z_0) + g(\mathbf{r}|x_0, y_0, -z_0) \,,$$

where $g(\mathbf{r}|\mathbf{r}_0) = e^{ikR}/4\pi R$ is the free-space Green's function. This was shown in Acoustics II; see problem 6 on my quals site for the math. Thus in the source plane $z_0 = 0$,

$$G(\mathbf{r}|x_0, y_0, 0) = 2g(\mathbf{r}|x_0, y_0, 0) = \frac{e^{ikR}}{2\pi R} \,, \tag{4}$$

where

$$R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2},$$

Substituting Eqs. (2)-(4) in Eq. (1) results in the first Rayleigh integral:

$$p_\omega(x,y,z) = \frac{-ik\rho_0 c_0}{2\pi} \iint_{-\infty}^{\infty} u_0(x_0,y_0) \frac{e^{ikR}}{R} dx_0 dy_0\,.$$

Multipole expansion

It proves expedient to introduce the multipole expansion, since monopoles and dipoles arise in the theory of scattering to follow. The multipole expansion was covered in Acoustics II in the context of radiation from spherical and cylindrical sources [e.g., a pulsating sphere, shaking sphere, bipolar pulsating sphere, pulsating cylinder, shaking cylinder (string), etc.], while the emphasis here is on how monopolar and dipolar fields arise due to the scattering of sound.

Monopole

Let $u_\omega(a) = u_0$ be a time-harmonic velocity source of radius a at location $\mathbf{r}_0 = \mathbf{0}$. The pressure field due to this source is

$$p_\omega(r) = A\frac{e^{ikr}}{r}\,, \quad \mathbf{r}_0 = \mathbf{0}\,.$$

The constant A is determined by the 1D radial momentum equation $-i\omega\rho_0 u_\omega + \partial p_\omega/\partial r = 0$:

$$p_\omega(a) = A\frac{e^{ika}}{a} = \frac{\rho_0 c_0 u_\omega(a)}{1 - 1/ika} = -\frac{ika\rho_0 c_0 u_0}{1 - ika} \implies A = -\frac{ika^2\rho_0 c_0 u_0}{1 - ika}e^{-ika}\,.$$

Thus the pressure field is

$$p_\omega(\mathbf{r}) = -\frac{ika^2\rho_0 c_0 u_0}{1 - ika}\frac{e^{ik(r-a)}}{r}\,.$$

To obtain the field due to a monopole (referred to a "simple source"), take the limit of $ka \ll 1$:

$$p_\omega(\mathbf{r}) = -ika^2\rho_0 c_0 u_0 \frac{e^{ikr}}{r} = -ik\rho_0 c_0 Q\frac{e^{ikr}}{4\pi r}\,. \tag{1}$$

where the volume velocity (surface area times particle velocity) is

$$Q = 4\pi a^2 u_0\,.$$

Recall that the free-space Green's function in 3D is $g(\mathbf{r}|\mathbf{r}_0) = e^{ikR}/4\pi R$. If the source location is shifted from $\mathbf{r}_0 = \mathbf{0}$ to an arbitrary location $\mathbf{r}_0 \neq \mathbf{0}$, then r in Eq. (1) becomes $R = |\mathbf{r} - \mathbf{r}_0|$, allowing for the pressure field due to a monopole at $\mathbf{r} = \mathbf{r}_0$ to be expressed as

$$\boxed{p_\omega(\mathbf{r}|\mathbf{r}_0) = -ik\rho_0 c_0 Q g(\mathbf{r}|\mathbf{r}_0)\,,}$$

where $g(\mathbf{r}|\mathbf{r_0}) = e^{ikR}/4\pi R$, and where $R = |\mathbf{r} - \mathbf{r}_0|$.

Dipole

Now the dipole is considered by considering two out-of-phase monopoles separated by distance d. The field is simply

$$\begin{aligned} p_\omega(\mathbf{r}) &= -ik\rho_0 c_0 Q[g(\mathbf{r}|z_0 = d/2) - g(\mathbf{r}|z_0 = -d/2)] \\ &\to -ik\rho_0 c_0 Qd[\partial g/\partial z_0]_{r0=0}\,, \quad d \to 0\,, \end{aligned}$$

where $g(\mathbf{r}|\mathbf{r_0}) = e^{ikR}/4\pi R$ from above. Generally, the dipole pressure is given by the *gradient* of the Green's function, where the dipole moment is defined as $\mathbf{D} = Q\mathbf{d}$, where $\mathbf{d}$ is the vector extending from the negative to positive monopoles defining the dipole:

$$\boxed{p_\omega(\mathbf{r}|\mathbf{r}_0) = -ik\rho_0 c_0 \mathbf{D}\cdot\nabla g(\mathbf{r}|\mathbf{r}_0)\,.}$$

To derive an expression for the dipole pressure in the far field, the derivative is evaluated in Cartesian coordinates:

$$\frac{\partial g}{\partial z_0} = \frac{\partial g}{\partial R}\frac{\partial R}{\partial z_0} = (ik - 1/R)g\frac{\partial R}{\partial z_0}\,,$$

where

$$\frac{\partial R}{\partial z_0} = \frac{\partial}{\partial z_0}\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = -\frac{z-z_0}{R}\,.$$

Noting that $R|_{r_0=0} = r$ and $\partial R/\partial z_0|_{r_0=0} = -z/r = \cos\theta$, it is found that

$$p_\omega(r,\theta) = -k^2\rho_0 c_0 Q_d(1 - 1/ikr)g(r|0)\cos\theta\,,$$

where $Q_d = Qd$ is the dipole strength (magnitude of its moment), and where

$$g(r|0) = [g(\mathbf{r}|\mathbf{r}_0)]|_{r_0=0} = \frac{e^{ikr}}{4\pi r}.$$

In the far field, then, the pressure field is

$$p_\omega(r,\theta) = -k^2\rho_0 c_0 Q_d g(r|0)\cos\theta\,.$$

Quadrupole

The exercise above for dipoles is repeated for quadrupoles in the homework assignment. It is found that lateral and longitudinal quadrupoles can be expressed in terms of the *second* derivatives of the free-space Green's function.

Scattering from inhomogeneities

The above development of Green's functions, the Helmholtz-Kirchhoff integral, and multipoles is finally used to describe the scattering of sound.

First, a wave equation must be developed that includes the presence of inhomogeneities. The development of this wave equation was not covered in class, but it is essential to my research project, so I follow the derivation leading up to Morse and Ingard's Eq. (8.1.10), which corresponds to Dr. Hamilton's Eq. (1) below (except I neglect the time dependence of the inhomogeneities, a la Pierce).

Equation of state

Begin by letting $\rho(\mathbf{r}) = \rho_0 + \delta_I(\mathbf{r})$ be the (spatially dependent) ambient density, while ρ' is the acoustic field quantity. The subscript "I" stands for "inhomogeneity." Similarly, let $\kappa(\mathbf{r})$ be the ambient bulk modulus. *Note that Morse and Ingard use $\kappa(\mathbf{r}) = 1/B(\mathbf{r})$, which is the compressibility. Compressibility will be used until returning to the content of Dr. Hamilton's lecture.* Pierce provides the appropriate equation of state [see first footnote on p. 15 of *Acoustics: An Introduction to Its Physical Principles and Applications* (1989)]:

> If the ambient state is inhomogeneous, $p = p(\rho, s_0)$ cannot be used and one falls back on $p = p(\rho, s)$, $Ds/Dt = 0$ as a starting point. If $p_0(\mathbf{x})$ and $\rho_0(\mathbf{x})$ are independent of t, these lead to
>
> $$\frac{\partial p'}{\partial t} + \mathbf{v}' \cdot \nabla p_0 = c^2 \left(\frac{\partial \rho'}{\partial t} + \mathbf{v}' \cdot \nabla \rho_0 \right)$$
>
> as the linear equation that replaces $[p' = c^2 \rho', c^2 = (\partial p/\partial \rho)_0]$.

Using the notation in the present work, and noting that $\nabla p_0 = 0$ here, Pierce's state equation reads

$$\frac{\partial p}{\partial t} = c^2 \left[\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho(\mathbf{r}) \right] .$$

Solving this relation for $\partial \rho'/\partial t$ yields

$$\frac{\partial \rho'}{\partial t} = \frac{1}{c^2} \frac{\partial p}{\partial t} - \mathbf{u} \cdot \nabla \rho(\mathbf{r}) . \tag{S}$$

This relation will be used to eliminate the perturbation density from the systems of partial differential equations that follow.

Conservation of mass

The exact conservation of mass equation is

$$\frac{\partial}{\partial t} [\rho(\mathbf{r}) + \rho'(\mathbf{r}, t)] + \nabla \cdot \{[\rho(\mathbf{r}) + \rho'(\mathbf{r}, t)]\mathbf{u}\} = 0 .$$

Neglecting the nonlinear term $\rho'(\mathbf{r}, t)\mathbf{u}$ on the right-hand side of the above and expanding the divergence of the product $\rho(\mathbf{r})\mathbf{u}$ results in

$$\frac{\partial \rho'(\mathbf{r}, t)}{\partial t} + \rho(\mathbf{r}) \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho(\mathbf{r}) = 0 .$$

Inserting the linearized state equation given by Eq. (S) results in

$$\frac{1}{c^2}\frac{\partial p}{\partial t} - \mathbf{u}\cdot\nabla\rho(\mathbf{r}) + \rho(\mathbf{r})\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\rho(\mathbf{r}) = 0\,.$$

The $\mathbf{u}\cdot\nabla\rho(\mathbf{r})$ terms cancel, resulting in

$$\nabla\cdot\mathbf{u} = -\kappa(\mathbf{r},t)\frac{\partial p}{\partial t}\,. \tag{$\star\star$}$$

Conservation of momentum

Meanwhile, conservation of momentum requires that

$$\nabla p + \frac{\partial}{\partial t}\{[\rho(\mathbf{r}) + \rho'(\mathbf{r},t)]\mathbf{u}\} = 0\,.$$

Neglecting the nonlinear term $\rho' u$ yields

$$\nabla p + \frac{\partial}{\partial t}[\rho(\mathbf{r})\mathbf{u}] = 0\,.$$

Since ρ is not a function of time, it can be removed from the time derivative:

$$\nabla p + \rho(\mathbf{r})\frac{\partial\mathbf{u}}{\partial t} = 0\,.$$

Solving for $\partial\mathbf{u}/\partial t$ gives

$$\frac{\partial\mathbf{u}}{\partial t} = -\frac{1}{\rho(\mathbf{r})}\nabla p\,. \tag{$\star\star\star$}$$

Taking the time derivative of Eq. ($\star\star$), taking the divergence of Eq. ($\star\star\star$), and subtracting the resulting equations results in

$$\frac{\partial}{\partial t}(\nabla\cdot\mathbf{u}) - \nabla\cdot\left[\frac{\partial\mathbf{u}}{\partial t}\right] = -\kappa(\mathbf{r})\frac{\partial^2 p}{\partial t^2} + \nabla\cdot\left[\frac{1}{\rho(\mathbf{r})}\nabla p\right],$$

The left-hand side of the above equation can be seen to be 0, resulting in

$$\nabla\cdot\left[\frac{1}{\rho(\mathbf{r})}\nabla p\right] - \kappa(\mathbf{r})\frac{\partial^2 p}{\partial t^2} = 0\,.$$

Using the bulk modulus $B = 1/\kappa$ results in Dr. Hamilton's wave equation (1):

$$\nabla\cdot\left[\frac{\nabla p}{\rho(\mathbf{r})}\right] = \frac{1}{B(\mathbf{r})}\frac{\partial^2 p}{\partial t^2}\,. \tag{1}$$

Defining the reference density, bulk modulus, and sound speed (which are constants)

$$\rho_0\,,\quad B_0\,,\quad c_0 = \sqrt{B_0/\rho_0}\,,$$

Eq. (1) can be written as

$$\nabla\cdot\left[\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)\nabla p+\frac{\nabla p}{\rho_0}\right]=\left(\frac{1}{B}-\frac{1}{B_0}\right)\frac{\partial^2 p}{\partial t^2}+\frac{1}{B_0}\frac{\partial^2 p}{\partial t^2}\,.$$

Expanding the above equation yields

$$\frac{1}{\rho_0}\nabla^2 p-\frac{1}{B_0}\frac{\partial^2 p_\omega}{\partial t^2}=\left(\frac{1}{B}-\frac{1}{B_0}\right)\frac{\partial^2 p}{\partial t^2}+\nabla\cdot\left[\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)\nabla p\right]\,.$$

This equation is rearranged further in preparation to identify dimensionless spatially dependent contrast factors:

$$\nabla^2 p-\frac{\rho_0}{B_0}\frac{\partial^2 p}{\partial t^2}=\frac{\rho_0}{B_0}\left(\frac{B_0}{B}-1\right)\frac{\partial^2 p}{\partial t^2}+\nabla\cdot\left[\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)\nabla p\right]\,.$$

This equation can be written as

$$\nabla^2-\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}=\frac{\gamma_B(\mathbf{r})}{c_0^2}\frac{\partial^2 p}{\partial t^2}+\nabla\cdot\left[\gamma_\rho(\mathbf{r})\nabla p\right]\,, \tag{2}$$

where

$$\gamma_B(\mathbf{r})=\frac{B_0}{B(\mathbf{r})}-1\,,\quad \gamma_\rho(\mathbf{r})=1-\frac{\rho_0}{\rho(\mathbf{r})}\,.$$

In the frequency domain, this equation reads

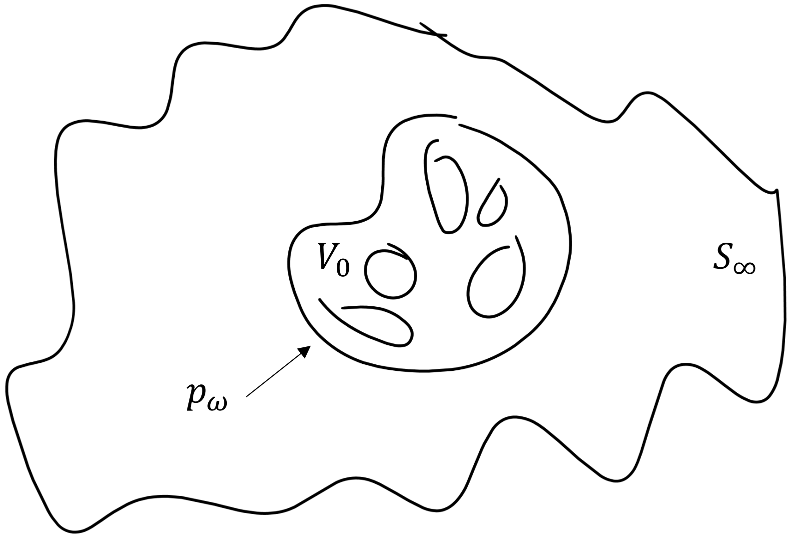
$$\nabla^2 p_\omega+k^2 p_\omega=-k^2\gamma_B p_\omega+\nabla\cdot(\gamma_\rho\nabla p_\omega)\,, \tag{3}$$

where $k=\omega/c_0$. *Note that there is a sign error in Morse and Ingard's corresponding Eq. (8.1.12).* Equation (3) can be recasted as

$$\nabla^2 p_\omega+k^2 p_\omega=-f(\mathbf{r})\,,$$

where $f(\mathbf{r})=k^2\gamma_B p_\omega-\nabla\cdot(\gamma_\rho\nabla p_\omega)$, which allows for implicit solution via the Helmholtz-Kirchhoff integral [Eq. (5)].

To prepare invoking the Helmholtz-Kirchhoff integral, let the inhomogeneities be confined to the volume V, as illustrated schematically below:



Denote the incident wave as $p_i(\mathbf{r})$ (which is not necessarily planar). Further, consider an unbounded medium, which allows for the use of the free-space Green's function, $g(\mathbf{r}|\mathbf{r}_0) = e^{ik|\mathbf{r}-\mathbf{r}_0|}/4\pi|\mathbf{r}-\mathbf{r}_0|$. Then, the solution of the inhomogeneous Helmholtz equation, with $f(\mathbf{r})$ as defined above, is

$$p_\omega(\mathbf{r}) = \int_V f(\mathbf{r}_0)g(\mathbf{r}|\mathbf{r}_0)dV_0 + I_\infty(\mathbf{r}),$$

where

$$I_\infty(\mathbf{r}) = \oint\left[g(\mathbf{r}|\mathbf{r}_0)\frac{\partial p_\omega(\mathbf{r}_0)}{\partial n_0} + p_\omega(\mathbf{r_0})\frac{\partial g(\mathbf{r}|\mathbf{r}_0)}{\partial n_0}\right]dS.$$

Since the surface is $S_0 = S_\infty$, and since V_0 is finite, then

$$I_\infty(\mathbf{r}) = p_i(\mathbf{r}),$$

which is to say that the pressure field infinitely far away from the scatterer is simply the field due to the incident wave, not the scattered wave. Therefore, the total field becomes

$$p_\omega(\mathbf{r}) = p_i(\mathbf{r}) + \int_{V_0}\left\{k^2\gamma_B(\mathbf{r}_0)p_\omega(\mathbf{r}_0) - \boldsymbol{\nabla}_0\cdot\left[\gamma_\rho(\mathbf{r}_0)\boldsymbol{\nabla}_0 p_\omega(\mathbf{r}_0)\right]\right\}g(\mathbf{r}|\mathbf{r}_0)dV_0 \tag{4}$$

By the vector calculus identity $\boldsymbol{\nabla}\cdot(a\mathbf{b}) = a\boldsymbol{\nabla}\cdot\mathbf{b} + \mathbf{b}\cdot\boldsymbol{\nabla}a$, the second term in Eq. (4) can be written as (identifying $\mathbf{b} \equiv \gamma_\rho\boldsymbol{\nabla}p$ and $a \equiv g$ in the identity),

$$\boldsymbol{\nabla}\cdot(\gamma_\rho\boldsymbol{\nabla}p)g = \boldsymbol{\nabla}\cdot[\gamma_\rho(\boldsymbol{\nabla}p)g] - \gamma_\rho\boldsymbol{\nabla}p\cdot\boldsymbol{\nabla}g.$$

Applying the divergence theorem to the volume integral of the term $\boldsymbol{\nabla}\cdot[\gamma_\rho(\boldsymbol{\nabla}p)g]$ results in

$$\int_{V_0}\boldsymbol{\nabla}\cdot\{\gamma_\rho(\mathbf{r}_0)[\boldsymbol{\nabla}p_\omega(\mathbf{r}_0)]g(\mathbf{r}|\mathbf{r}_0)\}dV_0 = \oint\gamma_\rho(\mathbf{r}_0)\frac{\partial p_\omega}{\partial n_0}g(\mathbf{r}|\mathbf{r}_0)dS_0,$$

which vanishes for S_0 at ∞ and finite V_0, because $\gamma_\rho = 0$ at ∞. After making these considerations, Eq. (4) becomes

$$\boxed{p_\omega(\mathbf{r}) = p_i(\mathbf{r}) + \int_{V_0}\left[k^2\gamma_B(\mathbf{r}_0)p_\omega(\mathbf{r}_0)g(\mathbf{r}|\mathbf{r}_0) + \gamma_\rho(\mathbf{r}_0)\boldsymbol{\nabla}_0 p_\omega(\mathbf{r}_0)\cdot\boldsymbol{\nabla}_0 g(\mathbf{r}|\mathbf{r}_0)\right]dV_0.} \tag{5}$$

Equation (5) can be thought of as the contribution of monopoles and dipoles,

$$p_\omega = -ik\rho_0 c_0 Q g(\mathbf{r}|\mathbf{r}_0) = \text{ monopole}$$
$$p_\omega = -ik\rho_0 c_0 \mathbf{D}\cdot\nabla g(\mathbf{r}|\mathbf{r}_0) = \text{ dipole}\,,$$

where $D = |\mathbf{D}| = Qd$. Thus set

$$k^2\gamma_B p_\omega \Delta V = -ik\rho_0 c_0 \Delta Q$$
$$\gamma_\rho \nabla p_\omega \Delta V = -ik\rho_0 c_0 \Delta\mathbf{D}\,,$$

which leads to

$$q = \frac{\Delta Q}{\Delta V} = \frac{\text{volume velocity}}{\text{unit volume}} = \frac{ik}{\rho_0 c_0}\gamma_B(\mathbf{r})p_\omega(\mathbf{r})$$
$$\mathbf{d} = \frac{\Delta\mathbf{D}}{\Delta V} = \frac{\text{dipole strength}}{\text{unit volume}} = -\frac{1}{ik\rho_0 c_0}\gamma_\rho(\mathbf{r})\nabla p_\omega(\mathbf{r})\,.$$

Then Eq. (5) can be written in terms of monopole strengths and dipole moments:

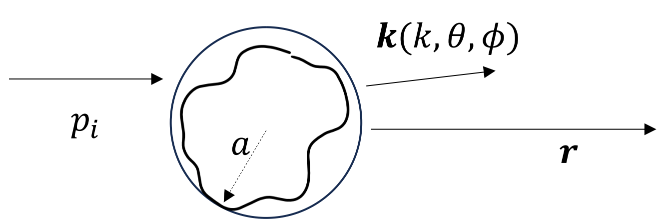
$$\boxed{p_\omega(\mathbf{r}) = p_i(\mathbf{r}) - ik\rho_0 c_0 \int_{V_0} [q(\mathbf{r}_0)g(\mathbf{r}|\mathbf{r}_0) + \mathbf{d}(\mathbf{r}_0)\cdot\nabla_0 g(\mathbf{r}|\mathbf{r}_0)]dV_0\,.}$$

Far-field, Born, and Rayleigh scattering

Using the results of the previous section, several scattering approximations are now made. These approximations are increasingly liberal. The far-field approximation is implicit and simply considers in far-field limit of the exact theory. The Born approximation builds on the far-field approximation and further assumes that the scattered field is much weaker than the incident field, making the scattering theory implicit. Rayleigh scattering builds on the Born approximation (though not historically) and further assumes the scatterer is subwavelength.

Far-field approximation, $r \gg a$

The far field of the scattered wave is now considered, as illustrated schematically below:



To describe the scattered wave in the far field, consider the free-space green's function

$$g(\mathbf{r}|\mathbf{r}_0) = \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{4\pi|\mathbf{r}-\mathbf{r}_0|}\,,$$

where

$$|\mathbf{r}-\mathbf{r}_0| = \sqrt{(\mathbf{r}-\mathbf{r}_0)\cdot(\mathbf{r}-\mathbf{r}_0)} = \sqrt{r^2 - 2\mathbf{r}\cdot\mathbf{r}_0 + r_0^2} = r\left(1 - 2\frac{\mathbf{r}\cdot\mathbf{r}_0}{r^2} + \frac{r_0^2}{r^2}\right)^{1/2} .$$

For $r \gg a$, where a is the characteristic radius of the scatterer, binomial expansion and retention to linear order of the above expression leads to

$$|\mathbf{r}-\mathbf{r}_0| \simeq r - \mathbf{e}_r \cdot \mathbf{r}_0 , \quad \mathbf{e}_r = \mathbf{r}/r ,$$

so

$$k|\mathbf{r}-\mathbf{r}_0| \simeq kr - \mathbf{k}_s \cdot \mathbf{r}_0 ,$$

where $\mathbf{k}_s = k\mathbf{e}_r = \mathbf{k}(k, \theta, \phi)$ is the wave vector in the direction of the scattering. So the Green's function and its gradient become

$$g(\mathbf{r}|\mathbf{r}_0) \simeq \frac{e^{ikr}}{4\pi r} e^{-i\mathbf{k}_s\cdot\mathbf{r}_0} , \quad \nabla_0 g = -i\mathbf{k}_s g .$$

The total field is then

$$p_\omega(\mathbf{r}) = p_i(\mathbf{r}) + p_s(\mathbf{r}), \tag{1}$$

where now the scattered field is

$$p_s(\mathbf{r}) = k^2 \frac{e^{ikr}}{4\pi r} \int \left[\gamma_B(\mathbf{r}_0)p_\omega(\mathbf{r}_0) - \gamma_\rho(\mathbf{r}_0)\frac{i\mathbf{k}_s}{k^2}\cdot\nabla_0 p_\omega(r_0)\right] e^{-i\mathbf{k}_s\cdot\mathbf{r}_0} dV_0 .$$

The above result can be written as a *3D* Fourier transform:

$$p_s(\mathbf{r}) = k^2 \frac{e^{ikr}}{4\pi r} \int q_s(\mathbf{r}_0) e^{-i\mathbf{k}_s\cdot\mathbf{r}_0} dV_0 , \tag{2}$$

where

$$q_s(\mathbf{r}) \equiv \gamma_B(\mathbf{r})p_\omega(\mathbf{r}) - \gamma_\rho(\mathbf{r})\frac{i\mathbf{k}_s}{k^2}\cdot\nabla p_\omega(\mathbf{r}) . \tag{3}$$

As such, Eqs. (2) can be written using the old notation of Fourier transforms:

$$p_s(\mathbf{r}) = k^2 \frac{e^{ikr}}{4\pi r} \mathcal{F}_{\mathrm{3D}}\{q_s(\mathbf{r})\}_{\mathbf{k}=\mathbf{k}_s} .$$

Note that Eqs. (2) and (3) are still *implicit*. That is to say, the scattered field is given in terms of the scattered field.

Born approximation

If in Eq. (1) the scattered field is much weaker than the incident field, i.e., $|p_s| \ll |p_i|$, then p_s is a small correction to p_i, so let $p_\omega = p_i$ in Eq. (3):

$$q_s(\mathbf{r}) = \gamma_B(\mathbf{r})p_i(\mathbf{r}) - \gamma_\rho(\mathbf{r})\frac{i\mathbf{k}_s}{k^2}\cdot\nabla p_i(\mathbf{r}) . \tag{5}$$

Thus the solution given by Eq. (2) is now explicit.

Rayleigh scattering

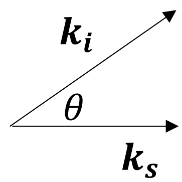
In this approximation, it is assumed that the scattering is in the far field, satisfies the Born approximation, and is additionally *subwavelength*, i.e., that $ka \ll 1$. This allows for $|\mathbf{k}_s \cdot \mathbf{r}_0| \ll 1$ in Eq. (2), resulting in

$$p_s(\mathbf{r}) = k^2 \frac{e^{ikr}}{4\pi r} \int q_s(\mathbf{r}_0) dV_0 \,. \tag{6}$$

Assuming the incident field is a plane wave $p_i(\mathbf{r}) = p_0 e^{i\mathbf{k}_i \cdot \mathbf{r}}$, Eq. (5) becomes

$$q_s(\mathbf{r}) = \left[\gamma_B(\mathbf{r}) + \gamma_\rho(\mathbf{r}) \frac{\mathbf{k}_s \cdot \mathbf{k}_i}{k^2} \right] p_i(\mathbf{r}) \,.$$

The relationship between $\mathbf{k}_s$ and $\mathbf{k}_i$ is shown below:



$$\cos\theta = \boldsymbol{k}_s \cdot \boldsymbol{k}_i / k^2$$

Writing q_s in terms of the angle θ subtending the scattered and incident wave vectors results in

$$q_s(\mathbf{r}) = \left[\gamma_B(\mathbf{r}) + \gamma_\rho(\mathbf{r}) \cos\theta \right] p_i(\mathbf{r}) \,. \tag{7}$$

Notating the spatial average of a function f as $\langle f(\mathbf{r}) \rangle = \frac{1}{V} \int f(\mathbf{r})\, dV$, and noting that for $ka \ll 1$ the incident wave is $p_i(\mathbf{r}_0) \simeq p_0$ for $ka \ll 1$, Eqs. (6) and (7) are combined:

$$\boxed{p_s(r,\theta) = p_0 k^2 V_0 \frac{e^{ikr}}{4\pi r} \left[\langle \gamma_B \rangle + \langle \gamma_\rho \rangle \cos\theta \right] .} \tag{8}$$

The scattered intensity is therefore

$$I_s(r,\theta) = \frac{|p_s|^2}{2\rho_0 c_0} \propto k^4 = \frac{1}{\lambda^4} \,,$$

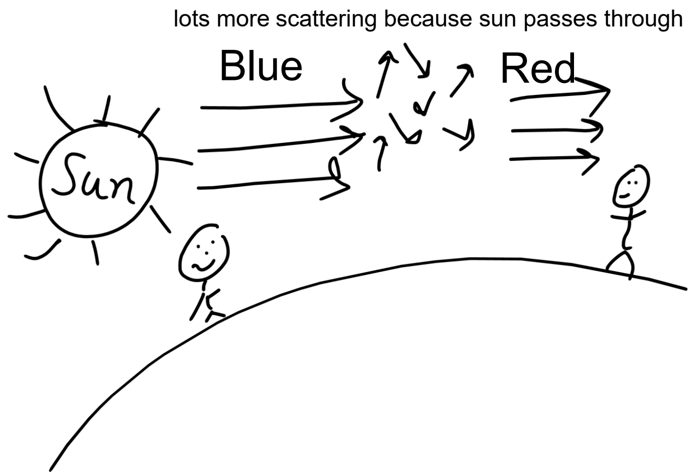
which is a hallmark of Rayleigh scattering.

Example: Blue sky

Rayleigh scattering explains why the sky is blue. Red light has a longer wavelength than blue light:

$$\left(\frac{\lambda_{\text{red}}}{\lambda_{\text{blue}}} \right)^4 = \left(\frac{700 \text{ nm}}{400 \text{ nm}} \right) \sim 10 \,.$$

Therefore, the intensity of red light scattered by the molecules in the atmosphere is 10 times weaker than that of the scattered blue light.



The sky appears yellow, orange, and red during sunset and sunrise because the sunlight passes through more atmosphere when it is near the horizon. As the sunlight passes through this atmosphere, is blue and green light is scattered away, leaving yellow, orange, and red light behind.

The same effect explains why the moon appears red during lunar eclipses.

Born scattering from bubbles below resonance

In this section and the following, the theory and approximations derived above are applied to calculating the scattered field due to bubbles in different contexts. This section focuses on scattering from bubbles *below* resonance.

The resonance frequency of a bubble can be calculated using a lumped-element approximation, as was presented in Acoustics I. The result is

$$f_0 = \frac{1}{2\pi a}\sqrt{3\kappa P_0/\rho_0},$$

where a is the bubble radius, P_0 is the atmospheric pressure, and κ is the polytropic index, which is a continuous function having values between 1 and 1.4 (unlike γ), such that $p_{\text{gas}} = \rho_{\text{gas}}^{\kappa}$. The subscripted 0s in these equations refer to the properties of the liquid surrounding the bubble.

For bubbles, the scattering parameters introduced in the "Scattering from inhomogeneities" section have the values

$$\gamma_B = \frac{B_0}{B_{\text{gas}}} - 1 \simeq \frac{B_0}{B_{\text{gas}}} = \frac{\rho_0 c_0^2}{\kappa P_0} = 1.54 \times 10^4 \text{, air bubble in water, } \kappa = 1.4$$
$$\gamma_\rho = 1 - \frac{\rho_0}{\rho_{\text{gas}}} \simeq -\frac{\rho_0}{\rho_{\text{gas}}} = -825 \text{, air in water}$$

Since

$$|\gamma_B/\gamma_\rho| = 19 \gg 1,$$

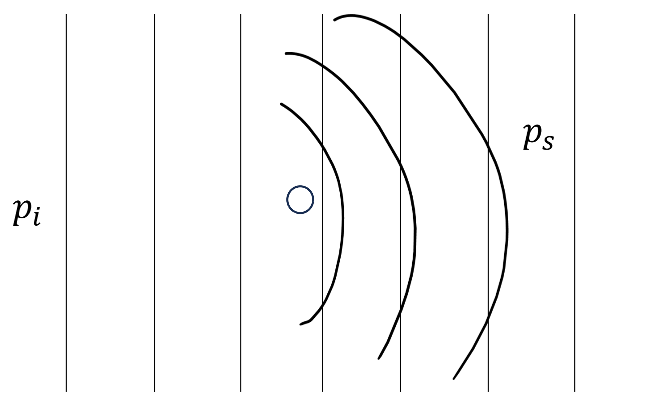
the density contrast factor γ_ρ can be ignored. Thus in the Born approximation, the scattered field is given in terms of the bulk modulus contrast factor:

$$\begin{aligned} p_s(\mathbf{r}) &= k^2 \int \gamma_B(\mathbf{r}_0) p_i(\mathbf{r}_0) g(\mathbf{r}|\mathbf{r}_0) dV_0 \\ &= k^2 \frac{\rho_0 c_0^2}{\kappa P_0} \int \phi_B(\mathbf{r}_0) p_i(\mathbf{r}_0) g(\mathbf{r}|\mathbf{r}_0) dV_0 , \end{aligned} \tag{1}$$

where ϕ = volume of gas/unit volume, regarded as the "volume fraction" (or sometimes "void fraction").

Scattering from single bubble at r = 0

The scattered field due to a single bubble, illustrated schematically below, is first considered.



At the resonance frequency f_0 of a bubble, $ka = 0.014 \ll 1$. That is to say, the bubble is deeply subwavelength and can therefore be described by a δ function in space:

$$\phi(\mathbf{r}) = \frac{4}{3}\pi a^3 \delta(\mathbf{r}) = V_0 \delta(\mathbf{r}) .$$

Therefore, the scattered field given by Eq. (1) is trivial to evaluate by the sifting property:

$$p_s(r) = k^2 \frac{\rho_0 c_0^2}{\kappa P_0} V_0 p_i(0) g(r|0) .$$

Since $p_i(0) = p_0$ and $g(r|0) = e^{ikr}/4\pi r$, the scattered field is

$$p_s(r) = \frac{\rho_0 \omega^2 V_0}{4\pi \kappa P_0} p_0 \frac{e^{ikr}}{r} ,$$

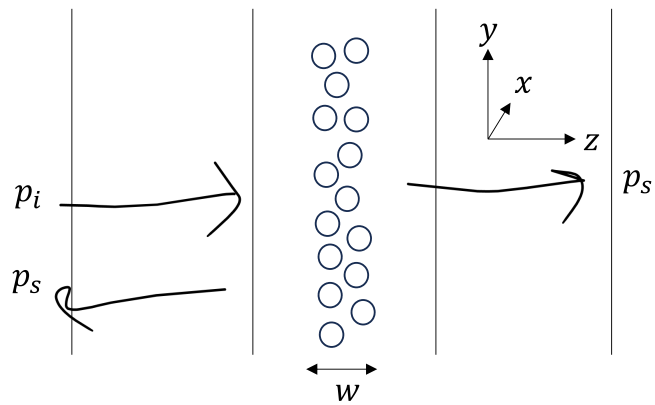
or

$$|p_s(r)/p_0| = \frac{\rho_0 \omega^2 V_0}{4\pi \kappa P_0 r} .$$

It is noted that the scattered field is proportional to ω^2 and inversely proportional to r.

Scattering from thin bubble screen of width w

Now the scattered field due to a thin screen of bubbles is considered, illustrated below, where the horizo



In this case, the volume fraction has the form

$$\phi(\mathbf{r}) = \phi(x, y) , \quad kw \ll 1 .$$

Suppose the time-harmonic incident field has the form $p_i(\mathbf{r}) = p_0 e^{ikz}$, and that the scatterer coordinates are

$$\mathbf{r}_0 = (x_0, y_0, z_0) = (x_0, y_0, 0).$$

Thus the incident field at the scatterer (the thin screen of bubbles) is $p_i(\mathbf{r}_0) = p_0$. Meanwhile, consider the field coordinate

$$\mathbf{r} = (x, y, z) = (0, 0, z),$$

for which the 3D free-space Green's function becomes

$$g(\mathbf{r}|\mathbf{r}_0) = \frac{e^{ik\sqrt{x_0+y_0+z^2}}}{4\pi \sqrt{x_0^2 + y_0^2 + z}} .$$

Thus Eq. (1) becomes

$$p_s(z) = k^2 \frac{\rho_0 c_0^2}{\kappa P_0} \frac{p_0 w}{4\pi} \iint \phi(x_0, y_0) \frac{e^{ik\sqrt{x_0+y_0+z^2}}}{4\pi \sqrt{x_0^2 + y_0^2 + z}} dx_0\, dy_0 .$$

To facilitate the integration, suppose the volume fraction is a constant, ϕ_0. That is to say, the thin screen of bubbles is approximated to be a homogeneous medium with volume fraction ϕ_0. The integral becomes

$$p_s(z) = k^2 \frac{\rho_0 c_0^2}{\kappa P_0} \frac{p_0 w \phi_0}{4\pi} I(z)\,, \quad I(z) = \int_0^\infty \frac{e^{ik\sqrt{\rho^2+z^2}}}{\sqrt{\rho_0^2 + z^2}} 2\pi \rho_0 d\rho_0\,,$$

where $\rho_0 = \sqrt{x_0^2 + y_0^2}$, and where $dx_0 dy_0 = 2\pi \rho_0 d\rho_0$. The integral $I(z)$ is taken by letting $u = \sqrt{\rho_0^2 + z^2}$, and thus $udu = \rho_0 d\rho_0$, and $k \mapsto k + i\epsilon$:

$$\begin{aligned} I &= 2\pi \int_{|z|}^\infty e^{i(k+i\epsilon)u} du \\ &= 2\pi \frac{e^{i(k+i\epsilon)u}}{i(k+i\epsilon)}\Bigg|_{|z|}^\infty \\ &= 2\pi i \frac{e^{ik|z|}}{z}\,, \quad \epsilon \to 0\,. \end{aligned}$$

So the scattered field is

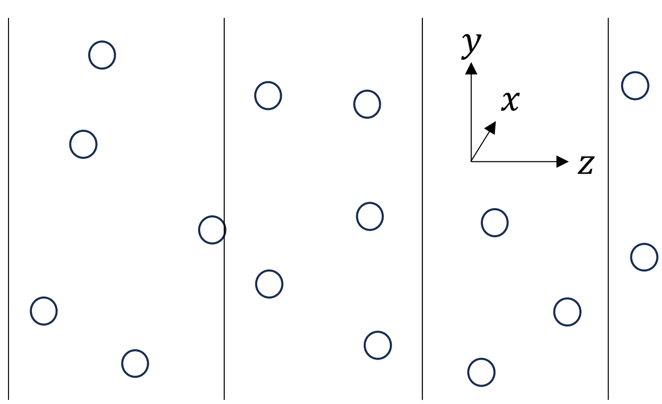
$$p_s(z) = \frac{ik\rho_0 c_0^2}{2\kappa P_0} p_0 w \rho_0 e^{ik|z|}\,.$$

The magnitude of the scattering coefficient is

$$\begin{aligned} |p_s/p_0| &= \frac{\omega \rho_0 c_0}{2\kappa P_0} w \rho_0 \\ &= \omega \qquad \text{vs. } \omega^2 \text{ for a single bubble} \\ &\equiv |R|\,, \qquad \text{reflection coefficient}\,. \end{aligned}$$

Propagation in bubbly liquid for $f \ll f_0$

The previous two cases have considered the scattering of sound due to bubbles fairly localized in space. Now the scatterers are considered to be well-distributed throughout the liquid, as illustrated below:



From Eq. (2) from the "Scattering from inhomogeneities" section, the wave equation becomes, upon neglecting γ_ρ,

$$\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = \frac{\gamma_B(\mathbf{r})}{c_0^2}\frac{\partial^2 p}{\partial t^2}\,. \qquad (\star)$$

To describe the well-distributed collection of bubbles in the medium, the ratio in bulk moduli factor γ_B is set to

$$\gamma_B \simeq (\rho_0 c_0^2/\kappa P_0)\phi(\mathbf{r}) = (B_l/B_g)\phi\,,$$

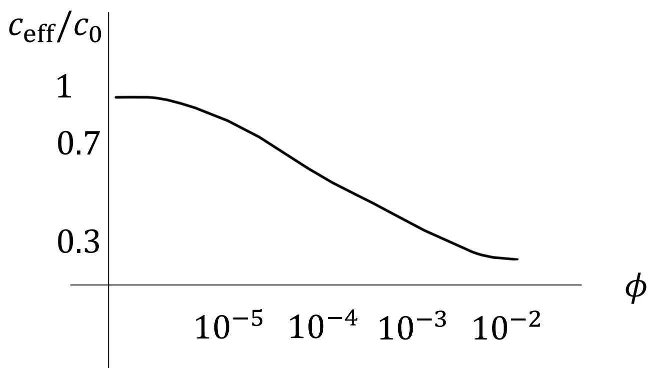
where ϕ is constant (equaling the total volume of air divided by the total volume of water). Equation $(\star)$ then becomes

$$\nabla^2 p = \left(1 + \frac{B_l}{B_g}\phi\right)\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} \equiv \frac{1}{c_{\text{eff}}^2}\frac{\partial^2 p}{\partial t^2}\,,$$

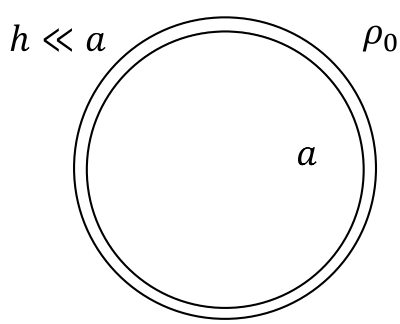
where

$$c_{\text{eff}} = \frac{c_0}{\sqrt{1 + \phi B_l/B_g}}\,.$$

This effective sound speed is the so-called Wood model for $\phi \ll 1$. (The model falls apart for any appreciable ϕ, e.g., $\phi \gtrsim 0.01$, since the underlying physics changes. See the homework problem that investigates this further.) The effective sound speed's dependence on ϕ is sketched below:



In practical settings in which contrast agents are used (like in biomedical applications), bubbles may have thin shells, as illustrated below:



In that case, the bulk modulus becomes (The subscript "CA" stands for "contrast agents")

$$B_g \mapsto B_{\text{CA}} = \kappa P_0 + 4(h/a)\mu_{\text{sh}}$$

where μ_{sh} is the shear modulus of the the shell, where $\mu_{\text{sh}} \ll B_{\text{sh}}$. So the resonance frequency becomes

$$f_0 = \frac{1}{2\pi a}\sqrt{3B_{\text{CA}}/\rho_0} = \frac{1}{2\pi a}\sqrt{\frac{3}{\rho_0}\left(\kappa P_0 + 4\frac{h}{a}\mu_{\text{sh}}\right)}$$

Scattering from bubbles at arbitrary frequencies

Now a theory for the scattering from bubbles at arbitrary frequencies (not just at frequencies below the bubble resonance frequency) is developed.

Begin with the inhomogeneous wave equation for volume sources [See Blackstock's *Fundamentals of Physical Acoustics*, Eq. (10-D-10)]:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -\rho_0 \frac{\partial q}{\partial t} \, ,$$

where q is the ratio of the volume velocity to the unit volume. Let the pressure be

$$\begin{aligned} P &= P_0 + p_\omega e^{-i\omega t} \\ V &= V_0 + v_\omega e^{-i\omega t} \\ Q &= \frac{dV}{dt} = Q_\omega e^{-i\omega t} = \text{ volume velocity} \, , \\ n_0 &= \frac{\text{number of bubbles}}{\text{unit volume}} . \end{aligned}$$

Then

$$q_\omega = n_0 Q_\omega \, , \quad Q_\omega = -i\omega v_\omega \, ,$$

and

$$\frac{\partial q}{\partial t} = -i\omega q_\omega = -i\omega n_0 Q_\omega = -n_0 \omega^2 v_\omega \, .$$

Thus a Helmholtz equation is obtained from Blackstock's Eq. (10-D-10):

$$(\nabla^2 + k^2) p_\omega = n_0 \rho_0 \omega^2 v_\omega \, . \qquad (1)$$

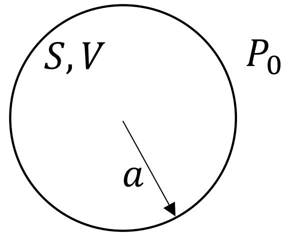
In addition to Eq. (1), an additional equation is needed to relate p_ω and v_ω. To obtain this equation, the bubble dynamics are needed.

Bubble dynamics

The acoustic impedance at the bubble wall of a bubble having volume $V_0 = 4\pi a^3/3$, surface area $S_0 = 4\pi a^2 0$, and radius a is

$$Z_{ac} = \frac{p_\omega}{-Q_\omega} = \frac{p_\omega}{i\omega v_\omega},$$

where $Q(t) = \dot{V}(t)$.



The minus sign is used in front of Q_ω because pressure increases with reduced volume. To relate p_ω to v_ω, note that for a spherical wave

$$Z_{ac} = \frac{\text{sp. acoustic impedance}}{\text{area}} = \frac{1}{4\pi a^2}\frac{\rho_0 c_0}{1 - 1/ika} + \frac{1}{-i\omega C_{ac}},$$

where the first two terms are the radiation impedance, and where the third term is is the bubble compliance $C_{ac} = V_0/\gamma P_0 \to V_0/\kappa P_0$. Noting that the term

$$\frac{1}{1 - 1/ika} = \frac{-ika}{1 - ika} = -ika(1 - ika)^{-1} = -ika + (ka)^2 + \mathcal{O}[(ka)^3].$$

The acoustic impedance becomes

$$Z_{ac} = \frac{\rho_0 c_0}{4\pi a^2}\left(-i\omega\frac{a}{c_0} + \omega^2\frac{a^2}{c_0^2}\right) - \frac{\kappa P_0}{i\omega V_0}.$$

Equating the above to the expression $Z_{ac} = \frac{p_\omega}{i\omega v_\omega}$ and solving for p_ω/v_ω results in

$$\frac{p_\omega}{v_\omega} = -\frac{\kappa P_0}{V_0}F(\omega), \tag{2}$$

where

$$F(\omega) = 1 - \frac{\omega^2}{\omega_0^2} - ik_0 a\frac{\omega^3}{\omega_0^3} \tag{3}$$

$$\omega_0^2 = \frac{3\kappa P_0}{\rho_0 a^2}, \quad k_0 = \omega_0/c_0, \quad \frac{\rho_0\omega_0^2}{4\pi a} = \frac{\kappa P_0}{V_0}.$$

Now write

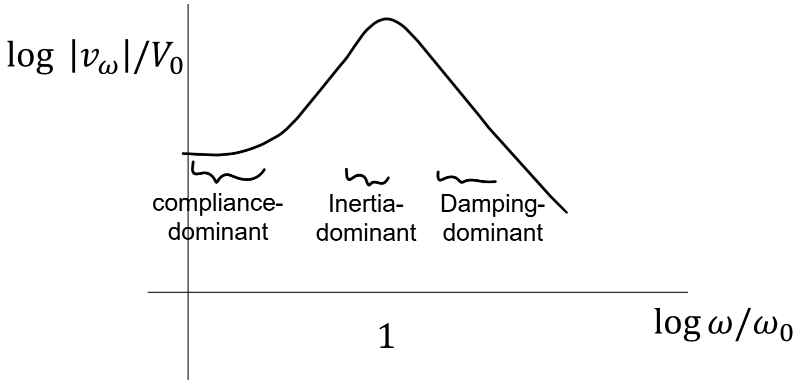
$$F(\omega) = 1 - \frac{\omega^2}{\omega_0^2} - i\delta(\omega)\frac{\omega}{\omega_0}, \quad \delta(\omega) = k_0 a\frac{\omega^2}{\omega_0^2}$$

Then call $\delta_0 = \delta(\omega_0) = k_0 a$, which is the radiation impedance at resonance.

From Eq. (2), with $p_\omega = p_0 =$ constant, then

$$\frac{v_\omega}{V_0} = -\frac{p_0/\kappa P_0}{1 - \omega^2/\omega_0^2 - i\delta\omega/\omega_0} \,. \tag{4}$$

Equation (4) is plotted below:



For air and water, at resonance

$$\frac{1}{\delta} = \frac{1}{k_0 a} = \frac{1}{0.014} \sim 72.$$

However, this result only accounts for loss of energy due to radiation. Generally, however,

$$\delta = \delta_{\text{rad}} + \delta_{\text{vis}} + \delta_{\text{th}} \,,$$

where

$$\delta_{\text{rad}} = k_0 a \frac{\omega^2}{\omega_0^2}$$
$$\delta_{\text{visc}} = \frac{4\mu}{\rho_0 \omega_0 a^2}$$
$$\delta_{\text{th}} = 3(\gamma - 1) l_{\text{th}}/a \,, \quad a \gg l_{\text{th}} \,,$$

where $\gamma = C_p/C_v$, $l_{\text{th}} = \sqrt{2\chi/\omega}$ is the boundary-layer thickness, and $\partial T/\partial t = \chi \nabla^2 T$.

Now use Eq. (4) to eliminate v_ω in Eq. (1). This can be rewritten as

$$(\nabla^2 + \tilde{k}^2) p_\omega = 0 \,,$$

where $\tilde{k}^2 \equiv \omega^2/\tilde{c}^2(\omega)$, where

$$\frac{1}{\tilde{c}^2(\omega)} = \frac{1}{c_0^2} + \frac{\phi \rho_0}{\kappa P_0 F(\omega)} \,, \quad \phi = r_0 V_0 = \text{volume fraction} \,.$$

Rearranging this relation and inserting the equation for $F(\omega)$ yields

$$\frac{c_0^2}{\tilde{c}^2(\omega)} = 1 + \frac{\phi b}{1 - \omega^2/\omega_0^2 - i\delta\omega/\omega_0} \,, \tag{5}$$

where $b = \rho_0 c_0^2/\kappa P_0 = B_{\text{liq}}/B_{\text{gas}} \sim 10^4$ for air/water. Thus the bubbly liquid has been converted to an effective medium.

Now the phase speed is found by inserting the form of solution

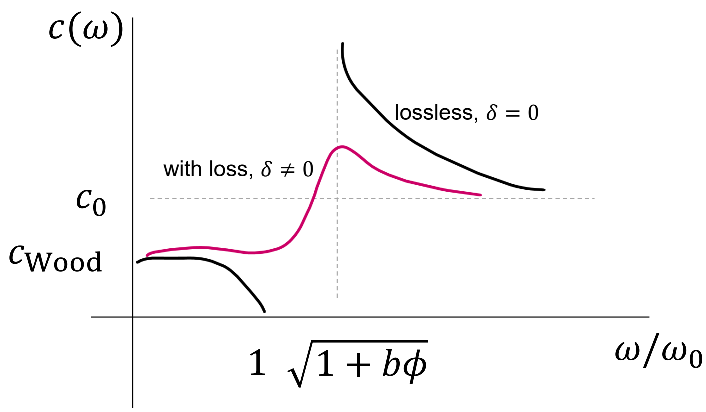
$$p_\omega = p_0 e^{i\tilde{x}} \equiv p_0 e^{ikx} e^{-\alpha x} .$$

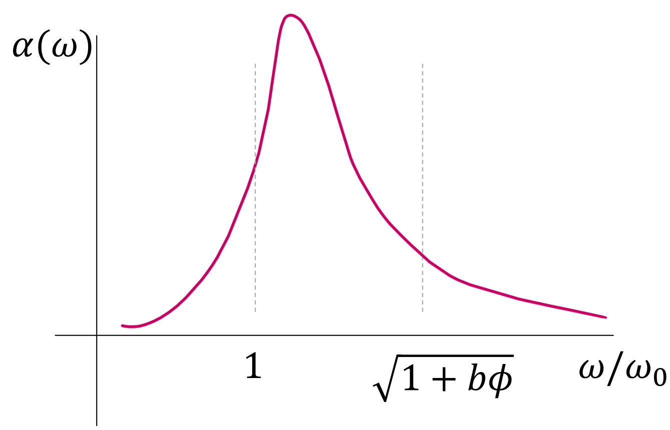
Also define

$$\tilde{k} = \frac{\omega}{\tilde{c}(\omega)} \equiv k + i\alpha = \frac{\omega}{c(\omega)} + i\alpha(\omega) ,$$

where $\alpha(\omega) = \Im\tilde{k}$ is the attenuation coefficient, and where $c(\omega) = \omega/k = \omega/\Re\tilde{k}$ is the phase speed. Ignoring losses, e.g., $\delta = 0$, Eq. (5) becomes

$$\begin{aligned}
\frac{\tilde{c}(\omega)}{c_0} &= \left(1 + \frac{\phi b}{1 - \omega^2/\omega_0^2}\right)^{-1/2} = \left(\frac{1 - \omega^2/\omega_0^2 + \phi b}{1 - \omega^2/\omega_0^2}\right)^{-1/2} \\
&= \sqrt{\frac{1 - \omega^2/\omega_0^2}{1 - \omega^2/\omega_0^2 + \phi b}} \\
&\simeq \frac{1}{\sqrt{1 + \phi b}} , \quad \omega^2 \ll \omega_0^2 , \quad \text{Wood's relation, pure compliance} \\
&\simeq 1 \quad \omega^2 \gg \omega_0^2 , \quad \text{hard bubble} \\
&= -i\sqrt{\frac{\omega^2/\omega_0^2 - 1}{1 + \phi b - \omega^2/\omega_0^2}} , \quad 1 < \omega/\omega_0 < \sqrt{1 + b\phi} .
\end{aligned}$$





Fruit bat photographed by Stephen P. Blackstock at Centennial Park in Sydney, Australia, December 2023.

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