* Ensured if (a), b or c) satisfied

Normal Mode Expansion of G

Let V be enclosed by locally reacting surfaces. Then the eigenfns satisfying

$$\nabla^2 \psi_n(\vec r) + k_n^2 \psi_n(\vec r) = 0 \qquad (1)$$

where $n = (l, m, n)$ in 3D, are orthogonal:*

$$\iiint_V \psi_n(\vec r)\psi_{n'}(\vec r)\, dV = V N_n \delta_{nn'} \qquad (2)$$

Green's fn satisfies

$$(\nabla^2 + k^2) G(\vec r|\vec r_0) = -\delta(\vec r - \vec r_0) \qquad (3)$$

Let

$$G(\vec r|\vec r_0) = \sum_n A_n \psi_n(\vec r)$$

Thus

$$\sum_n (-k_n^2 + k^2) A_n \psi_n(\vec r) = -\delta(\vec r - \vec r_0) \qquad (4)$$

Take $\iiint_V (4)\, \psi_{n'}(\vec r)\, dV$ and use (2):

$$(-k_n^2 + k^2) A_n V N_n = -\psi_n(\vec r_0)$$

so

$$A_n = \frac{\psi_n(\vec r_0)}{V N_n (k_n^2 - k^2)}$$

and

$$\boxed{G(\vec r|\vec r_0) = \frac{1}{V}\sum_n \frac{\psi_n(\vec r)\psi_n(\vec r_0)}{N_n(k_n^2 - k^2)}} = G(\vec r_0|\vec r) \qquad (5)$$

For simple source with vol. vel. Q

$$p_\omega(\vec r|\vec r_0) = -ik\rho_0 c_0 Q\, G(\vec r|\vec r_0)$$

$$= ik\rho_0 c_0 \frac{Q}{V} \sum \frac{\psi_n(\vec r)\psi_n(\vec r_0)}{N_n(k^2 - k_n^2)} \qquad (6)$$

M&I (9.5.14) for rect. room

* Include rigid rectangular duct, maybe as homework. (M&I, p. 500)

Example Rectangular Room $(L \times W \times H)$
For rigid walls

$$\psi_{lmn}(x,y,z) = \cos\frac{l\pi x}{L}\cos\frac{m\pi y}{W}\cos\frac{n\pi z}{H}$$

$$k_{lmn} = \frac{\omega_{lmn}}{c_0} = \pi\sqrt{\left(\frac{l}{L}\right)^2 + \left(\frac{m}{W}\right)^2 + \left(\frac{n}{H}\right)^2}$$

$$N_{lmn} = \frac{1}{\varepsilon_l \varepsilon_m \varepsilon_n}, \quad \varepsilon_n = 1,\ n = 0$$
$$= 2,\ n > 0$$

$(l, m, n) = (0,0,1), (0,1,0)$, etc.

Damping is important when $\omega \simeq \omega_n$.
Consider simple harmonic oscillator:

$$\ddot{x} + \frac{1}{\tau}\dot{x} + \omega_0^2 x = 0$$

Let $x = Ae^{st}$: $\quad s^2 + \frac{1}{\tau}s + \omega_0^2 = 0 \qquad (7)$

Eigenvalues:

$$s = \frac{1}{2}\left[-\frac{1}{\tau} \pm \sqrt{\frac{1}{\tau^2} - 4\omega_0^2}\,\right]$$

$$= -\frac{1}{2\tau} \pm \frac{1}{2\tau}\sqrt{1 - (2\omega_0\tau)^2}$$

$$= -\frac{1}{2\tau} \pm i\frac{1}{2\tau}\sqrt{(2\omega_0\tau)^2 - 1}, \text{ underdamped}$$
$(2\omega_0\tau > 1)$

$$\simeq -\frac{1}{2\tau} \pm i\omega_0, \text{ lightly damped}$$
$[(2\omega_0\tau)^2 \gg 1]$

So for light damping

$$x \simeq A e^{-t/2\tau} e^{\pm i\omega_0 t}$$

$$|x|^2 \simeq A e^{-t/\tau}$$

τ = exponential decay time for system <u>energy</u>

For forced response set $s=-i\omega$ on LHS of (7):

$$s^2+\frac{1}{\tau}s+\omega_0^2 = -\omega^2-\frac{i\omega}{\tau}+\omega_0^2$$
$$= -(\omega^2-\omega_0^2+i\omega/\tau)$$

Thus write

$$k^2-k_n^2 = \frac{1}{c_0^2}(\omega^2-\omega_n^2)$$

$$\rightarrow \frac{1}{c_0^2}(\omega^2-\omega_n^2+i\omega/\tau_n)$$

where

τ_n = decay time for energy in mode n

Thus for hard walls ($|z| \gg \rho_0 c_0$) with light damping [$(2\omega_n\tau_n)^2 \gg 1$]

$$p_\omega(\vec{r}|\vec{r}_0) \simeq i\omega\rho_0 c_0^2 \frac{Q}{V}\sum_n \frac{\psi_n(\vec{r})\psi_n(\vec{r}_0)}{N_n(\omega^2-\omega_n^2+i\omega/\tau_n)}$$

$$\simeq \rho_0 c_0^2 \frac{Q\tau_n}{VN_n}\psi_n(\vec{r})\psi_n(\vec{r}_0), \quad \omega=\omega_n$$

Note that quality factor of mode n is $Q_n=\omega_n\tau_n$. (Maybe use QF to distinguish from vol. vel.)