Parabolic Approximation

Helmholtz eqn:

$$\nabla^2 p_\omega + k^2 p_\omega = 0 \qquad (1)$$

Let

$$p_\omega(x,y,z) = q_\omega(x,y,z)\, e^{ikz} \qquad (2)$$

Thus

$q_\omega = \text{const.}$ for plane wave

is slowly varying re λ for sound beam.

$$\frac{\partial p_\omega}{\partial z} = \left(\frac{\partial q_\omega}{\partial z} + ikq_\omega\right)e^{ikz}$$

$$\frac{\partial^2 p_\omega}{\partial z} = \left(\frac{\partial^2 q_\omega}{\partial z^2} + i2k\frac{\partial q_\omega}{\partial z} - k^2 q_\omega\right)e^{ikz}$$

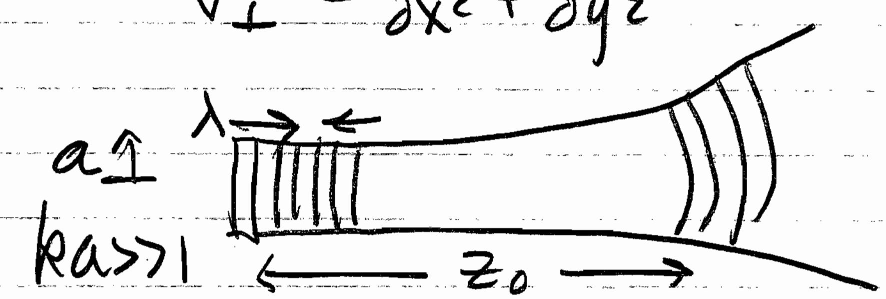
subst. in (1):

$$\frac{\partial^2 q_\omega}{\partial z^2} + i2k\frac{\partial q_\omega}{\partial z} + \nabla_\perp^2 q_\omega = 0 \qquad (1')$$

where

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

sound beam:



Thus.

$$\frac{\partial^2 q_\omega/\partial z^2}{i2k\,\partial q_\omega/\partial z} \sim \frac{q_\omega/z_0^2}{k q_\omega/z_0} \sim \frac{1}{kz_0} \sim \frac{1}{(ka)^2} \ll 1$$

so (1') well approximated by

$$\boxed{i2k\frac{\partial q_\omega}{\partial z} + \nabla_\perp^2 q_\omega = 0} \qquad (3)$$

(3) = parabolic eqn, whereas
(1) = elliptic eqn

Time domain:

$$p = p_\omega e^{-i\omega t} = q_\omega e^{i(kz-\omega t)} = q_\omega e^{-i\omega\tau}$$

where $\tau = t - z/c_0$

Then in (3): $ik = \frac{1}{c_0} i\omega \rightarrow -\frac{1}{c_0}\frac{\partial}{\partial\tau}$

(3) becomes $$\frac{\partial^2 p}{\partial z\,\partial\tau} = \frac{c_0}{2}\nabla_\perp^2 p \qquad (4)$$

where $p = p(x,y,z,\tau)$

With nonlinearity included,

$$\frac{\partial^2 p}{\partial z\,\partial\tau} = \frac{c_0}{2}\nabla_\perp^2 p + \frac{\beta}{2\rho_0 c_0^3}\frac{\partial^2 p^2}{\partial\tau^2}$$

obtain Khokhlov–Zabolotskaya (KZ) equations for sound beams (1969)

Integral Formulation

$$ik\,\frac{\partial q_\omega}{\partial z} + \nabla_\perp^2 q_\omega = 0$$

Let

$$Q(k_x, k_y, z) = \mathcal{F}\{q_\omega(x, y, z)\}$$

so

$$ik\,\frac{dQ}{dz} - (k_x^2 + k_y^2)\,Q = 0$$

or

$$\frac{dQ}{dz} + \frac{i(k_x^2 + k_y^2)}{2k}\,Q = 0$$

$$Q(k_x, k_y, z) = Q_0(k_x, k_y)\,e^{-i(k_x^2 + k_y^2)z/2k}$$

where $Q_0(k_x, k_y) = Q(k_x, k_y, 0)$

$$q_\omega(x, y, z) = \mathcal{F}^{-1}\{Q_0(k_x, k_y)\,e^{-i(k_x^2 + k_y^2)z/2k}\} \quad (5)$$

$$= q_0(x, y) ** \mathcal{F}^{-1}\{e^{-i(k_x^2 + k_y^2)z/2k}\}$$

$$\mathcal{F}^{-1}\{\cdot\} = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} e^{-\frac{iz}{2k}(k_x^2 + k_y^2)} \cdot e^{i(k_x x + k_y y)}\,dk_x\,dk_y$$

$$\int_{-\infty}^{\infty} e^{-\alpha t^2 \pm \beta t}\,dt = \sqrt{\frac{\pi}{\alpha}}\,e^{\beta^2/4\alpha}$$

where $\alpha = \frac{iz}{2k}$, $\beta = ix$ or iy

$$\mathcal{F}^{-1}\{\cdot\} = \frac{1}{4\pi^2}\,\frac{\pi}{iz/2k}\,\exp\left\{-\frac{x^2+y^2}{2iz/k}\right\}$$

$$= -\frac{ik}{2\pi z}\,e^{\frac{ik}{2z}(x^2+y^2)}$$

so

$$q_\omega(x,y,z) = -\frac{ik}{2\pi z}\, q_0(x,y) ** e^{\frac{ik}{2z}(x^2+y^2)}$$

$$= -\frac{ik}{2\pi z}\iint_{-\infty}^{\infty} q_0(x_0,y_0) \cdot e^{\frac{ik}{2z}[(x-x_0)^2+(y-y_0)^2]}\,dx_0\,dy_0 \qquad (6)$$

Since $p_\omega = q_\omega e^{ikz}$, (6) is identical to Fresnel approx.

Compare (5) with exact sol'n:

$$p_\omega(x,y,z) = \mathcal{F}^{-1}\{P_0(k_x,k_y)\,e^{i(k^2-k_x^2-k_y^2)^{1/2}z}\}$$

(5) is obtained with

$$\sqrt{k^2-k_x^2-k_y^2} = k\left(1-\frac{k_x^2+k_y^2}{k^2}\right)^{1/2}$$

$$\simeq k - \frac{k_x^2+k_y^2}{2k} \qquad (7)$$

- Valid for narrow angular spectrum ($ka \gg 1$)
- Does not account for evanescent waves because (7) is always real

Numerical soln

$$\frac{\partial q_\omega}{\partial z} = \frac{i}{2k}\nabla_\perp^2 q_\omega \qquad (*)$$

$$q_\omega(x,y,z+\Delta z) \simeq q_\omega(x,y,z) + \frac{i\Delta z}{2k}\nabla_\perp^2 q_\omega(x,y,z)$$

Simple marching scheme.

Easily augmented to include on RHS of (*):

- absorption: $-\alpha_\omega q_\omega$
- inhomogeneity: $i\left[\frac{c_0^2}{c^2(\vec{r})} - 1\right]\frac{k_0}{2} q_\omega$
- nonlinearity: $-\frac{i n\omega\beta}{4\rho_0 c_0^3}\sum_m q_m q_{n-m}$
  $(q_\omega \to q_n)$

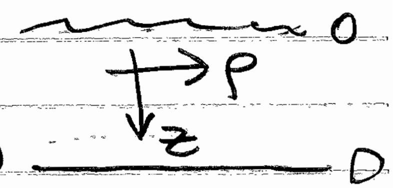
etc.

* Comp. Ocean Acs. (2011): "the PE method has now become the most popular wave-theory technique for solving range-dependent propagation problems in ocean acoustics" (1)

application to

Ocean Acoustics (PE - parabolic eqn

In cyl. coords. (ignore ϕ dependence), and for $k\rho \gg 1$, let



$$p_\omega(\rho,z) = q_\omega(\rho,z)\frac{e^{ik\rho}}{\sqrt{\rho}} \quad (1)$$

where
$$i2k\frac{\partial q_\omega}{\partial \rho} + \frac{\partial^2 q_\omega}{\partial z^2} = 0 \quad \text{(the "PE")} \quad (2)$$

BC's at $z=0,D$ produce modes. Let

$$q_\omega(\rho,z) = R(\rho)\,Z(z)$$

(2): $$i2k\frac{R'}{R} = -\frac{Z''}{Z} \triangleq k_z^2$$

$$\boxed{\begin{array}{l}\text{e.g.,}\\ Z_n = \sin k_{zn}z\\ k_{zn} = \frac{(2n-1)\pi}{2D}\\ n = 1,2,\ldots\end{array}}$$

so $$Z_n'' + k_{zn}^2 Z_n = 0 \quad (Z_n'' = -k_{zn}^2 Z_n)$$

Eigenfunctions thus identical to those of Helmholtz eqn. Then

$$R_n' + \frac{ik_{zn}^2}{2k}R_n = 0$$

so
$$R_n = A_n e^{-(ik_{zn}^2/2k)\rho}$$

from (1)

$$p_n(\rho,z) = A_n Z_n(z)\frac{1}{\sqrt{\rho}}\exp\left\{i\left(k - \frac{k_{zn}^2}{2k}\right)\rho\right\}$$

Soln of Helmholtz eqn. for $k\rho \gg 1$:

$$p_n(\rho,z) = A_n Z_n(z)\frac{1}{\sqrt{\rho}}\exp\left\{i(k^2 - k_{zn}^2)^{1/2}\rho\right\}$$

Mode shapes correct but phase speeds $c_{pn} = \omega/k_{\rho n}$ different:

$$\text{HE:} \quad c_{pn} = \frac{c_0}{\sqrt{1-k_{zn}^2/k^2}}$$

$$\text{PE:} \quad c_{pn} = \frac{c_0}{1-k_{zn}^2/2k^2}$$

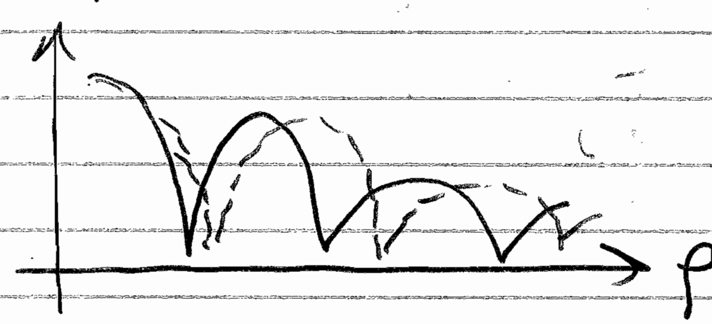
agreement improves with increasing k/k_{zn} (ω/ω_n^c)

This affects "transmission loss":

$$TL = -20\log_{10}\frac{|p_\omega(\rho)|}{p_{ref}}$$

when multiple modes propagate and thus interfere.

2 modes:

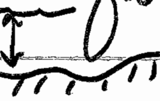


Usually $c_0 = c(z)$, for which PE becomes

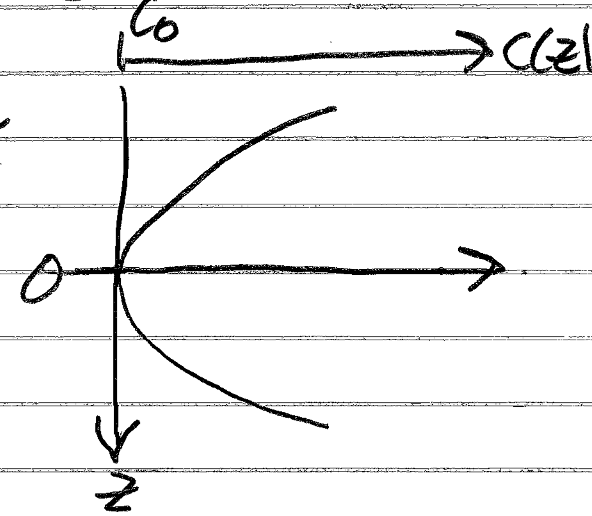
$$i2k_0\frac{\partial q_\omega}{\partial \rho} + \frac{\partial^2 q_\omega}{\partial z^2} + (n^2-1)k_0^2 q_\omega = 0$$

where $k_0 = \frac{\omega}{c_0}$, $n(z) = \frac{c_0}{c(z)}$

* Generally applied to range dependent channels, so $q_\omega(\rho,z) \neq R(\rho)Z(z)$, e.g., $D = D(\rho)$.



Example SOFAR channel



$$c^2(z) \simeq c_0^2\left(1+\frac{z^2}{h^2}\right), \quad |z| \ll h$$

$$n^2(z) = \left(1+\frac{z^2}{h^2}\right)^{-1}$$

$$\simeq 1-\frac{z^2}{h^2}$$

$$2ik_0\frac{\partial q_\omega}{\partial \rho} + \frac{\partial^2 q_\omega}{\partial z^2} - \frac{k_0^2}{h^2}z^2 q_\omega = 0$$

$$2ik_0\frac{R'}{R} = -\frac{Z''}{Z} + \frac{k_0^2}{h^2}z^2 = \gamma$$

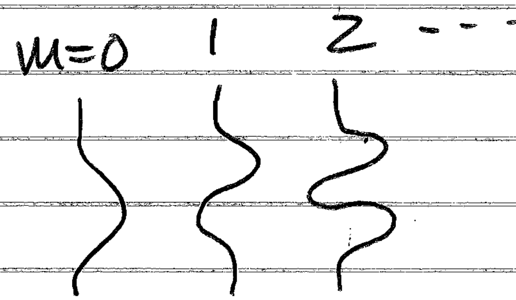
$$Z'' + \left(\gamma - \frac{k_0^2}{h^2}z^2\right)Z = 0$$

Sol'n: $\gamma \equiv \gamma_m = 2m+1, \quad m = 0,1,2\ldots$

(bounded at $z=\pm\infty$)

$$Z(\xi) = e^{-\xi^2/2}H_m(\xi)$$

H_m = Hermite polynomial
$= 1, \quad m=0$
$= 2\xi, \quad m=1$
$= 4\xi^2-2, \quad m=2$
$\vdots$



$\xi = \text{const} \cdot z$