Getting my mind around spherically converging waves

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This is a confusing topic, and these notes summarize different peoples' approaches to the topic.

Dr. Blackstock's discussion[1]

Consider a bursting balloon enforcing the initial conditions

$$p(r,0) = A[H(r) - H(r - r_0)] \tag{1}$$
$$u(r,0) = 0\,. \tag{2}$$

For reasons not well explained, the volume velocity $q = Su$ must vanish at $r = 0$ (because it's spherically symmetric sound?):

$$\lim_{r\to 0} q = \lim_{r\to 0} Su = \lim_{r\to 0} 4\pi r^2 u = 0 \tag{3}$$

The velocity potential ϕ will be used.[2] Since the sound obeys the spherically symmetric wave equation, the velocity potential is of the form

$$\phi = \frac{f(r - c_0 t)}{r} + \frac{g(r + c_0 t)}{r}\,.$$

The pressure is therefore

$$p(r,t) = -\rho_0 \phi_t = \rho_0 c_0 \frac{f'(r - c_0 t) - g'(r + c_0 t)}{r}\,, \tag{4}$$

[1] pages 121-124 "Fundamentals of Physical Acoustics"

[2] Recall that $p = -\rho_0 \phi_t$ and $u = \phi_r$

and the particle velocity is

$$u(r,t) = \phi_r = -\frac{f(r-c_0t)+g(r+c_0t)}{r^2} + \frac{f'(r-c_0t)+g'(r+c_0t)}{r} \,. \quad (5)$$

Applying the initial condition given by equation (2) on equation (5) gives

$$\frac{f(r)+g(r)}{r^2} = \frac{f'(r)+g'(r)}{r}$$

This equality is guaranteed if $g(r) = -f(r)$, because this implies that $g'(r) = -f'(r)$.[3] Therefore, equation (4) becomes

$$p(r,t) = \rho_0 c_0 \frac{f'(r-c_0t)+f'(r+c_0t)}{r} \,, \quad (6)$$

and equation (5) becomes

$$u(r,t) = -\frac{f(r-c_0t)-f(r+c_0t)}{r^2} + \frac{f'(r-c_0t)-f'(r+c_0t)}{r} \,.$$

The volume velocity is therefore

$$\begin{aligned} q &= Su = 4\pi r^2 u \\ &= -4\pi[f(r-c_0t)-f(r+c_0t)] + 4\pi r[f'(r-c_0t)-f'(r+c_0t)] \,. \end{aligned} \quad (7)$$

The condition given by equation (3) is applied to equation (7):

$$\begin{aligned} &\lim_{r\to 0} q = -4\pi[f(-c_0t)-f(c_0t)] = 0 \\ \implies\ & f(-c_0t) = f(c_0t) \,. \end{aligned} \quad (8)$$

Taking the derivative of equation (8) gives

$$-f'(-c_0t) = f'(c_0t) \,, \quad (9)$$

i.e., that f' is **odd**.

Meanwhile, the initial condition given by equation (1) is applied to equation (6):

$$A[H(r)-H(r-r_0)] = 2\rho_0 c_0 \frac{f'(r)}{r}$$

[3] The converse is not necessarily true.

Solving the above for $f'(r)$ gives

$$f'(r) = \frac{rA[H(r) - H(r - r_0)]}{2\rho_0 c_0} \tag{10}$$

Enforcing equation (9) (the oddness of f') on equation (10) requires that f' is defined for $-r$ as well as $+r$. This can be achieved using the rectangle function:[4]

$$f'(r) = \frac{rA}{2\rho_0 c_0}\,\text{rect}\left(\frac{r}{2r_0}\right)$$

Therefore,

$$f'(r \pm c_0 t) = \frac{A}{2\rho_0 c_0}(r \pm c_0 t)\,\text{rect}\left(\frac{r \pm c_0 t}{2r_0}\right) \tag{11}$$

Substituting equation (11) into equation (6) gives the solution:

$$\boxed{p(r,t) = \frac{A}{2r}\left[(r - c_0 t)\,\text{rect}\left(\frac{r - c_0 t}{2r_0}\right) + (r + c_0 t)\,\text{rect}\left(\frac{r + c_0 t}{2r_0}\right)\right]}$$

Dr. Hamilton's discussion[5]

Consider a sphere of radius r_0. At $r = r_0$, the incident pressure wave is given by $p_{\text{in}}(t)$. The pressure solution is therefore of the form

$$p = \frac{r_0}{r}p_{\text{in}}(t + r/c_0) + \frac{F(t - r/c_0)}{r}, \tag{12}$$

where F corresponds to the wave emerging through the focus. The goal of what follows is to determine F in terms of p_{in}. First, apply the momentum equation for a spherical wave, $\rho_0 \dot{u} = -p_r$, to equation (12):

$$\rho_0 \frac{\partial u}{\partial t} = \frac{F'(t - r/c_0) - r_0 p'_{\text{in}}(t + r/c_0)}{c_0 r} + \frac{F(t - r/c_0) + r_0 p_{\text{in}}(t + r/c_0)}{r^2}$$

[4] $\text{rect}\left(\frac{x - x_0}{w}\right) = H(x - x_0 + w/2) - H(x - x_0 - w/2)$

[5] from Acoustics I lecture notes. Dr. Hamilton's discussion is a bit more general than Dr. Blackstock's.

Solving the above for u by integration over time gives

$$
\begin{aligned}
u &= -\frac{1}{\rho_0}\int \frac{\partial p}{\partial t}\,dt \\
&= \frac{F(t-r/c_0) - r_0 p_{\mathsf{in}}(t+r/c_0)}{\rho_0 c_0 r} + \frac{\tilde{F}(t-r/c_0) + r_0 \tilde{p}_{\mathsf{in}}(t+r/c_0)}{\rho_0 r^2}, \qquad (13)
\end{aligned}
$$

where $\tilde{p}$ is the antiderivative of p, and $\tilde{F}$ is the antiderivative of F. When the boundary condition $\lim_{r\to 0} q = \lim_{r\to 0} 4\pi r^2 u = 0$ is applied to equation (13), the first term of equation (13) vanishes, and the second term gives

$$
\frac{4\pi}{\rho_0}[\tilde{F}(t) + r_0 \tilde{p}_{\mathsf{in}}(t)] = 0
$$

Solving the above for $\tilde{F}(t)$ gives

$$
\tilde{F}(t) = -r_0 \tilde{p}_{\mathsf{in}}(t) \qquad \Longrightarrow \qquad F(t) = -r_0 p_{\mathsf{in}}(t)
$$

Substituting $F(t) = -r_0 p_{\mathsf{in}}(t)$ into equation (12) gives the solution

$$
\boxed{p = \frac{r_0}{r} p_{\mathsf{in}}(t + r/c_0) - \frac{r_0}{r} p_{\mathsf{in}}(t - r/c_0),} \qquad (14)
$$

The first term corresponds to the incoming wave, and the second term corresponds to the outgoing wave.

What happens at $r = 0$ (the focus)? The limit of equation (14) is taken in that limit:

$$
\begin{aligned}
\lim_{r\to 0} p &= \lim_{r\to 0} \frac{r_0}{r}\left[p_{\mathsf{in}}(t + r/c_0) - p_{\mathsf{in}}(t - r/c_0)\right] \\
&= \lim_{r\to 0} \frac{r_0}{r}\left[p_{\mathsf{in}}(t) + \frac{r}{c_0} p'_{\mathsf{in}}(t) - p_{\mathsf{in}}(t) + \frac{r}{c_0} p'_{\mathsf{in}}(t)\right] \\
&= \frac{2r_0}{c_0} p'_{\mathsf{in}}(t)
\end{aligned}
$$

In the second equality above, the function is Taylor expanded to first order, and the higher-order terms are dropped. The conclusion is that the pressure at the center of the sphere is proportional to the **time derivative** of the incident pressure.

$$
\boxed{p(r = 0, t) = \frac{2r_0}{c_0} p'_{\mathsf{in}}(t)}
$$