Review for the nonlinear acoustics final

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These problems, based on Dr. Hamilton's lectures, address the major topics of the latter half of the course, corresponding to HW6-HW8. Good luck on the exam!

1 Rankine-Hugoniot relations

(a) Name the quantity that is conserved when f and g, as defined below, are substituted into equation (1.1).

$$\frac{\partial f}{\partial t} + \frac{\partial g}{\partial x} = 0\,. \tag{1.1}$$

	f	g
(i)	ρ	ρu
(ii)	ρu	$\rho u^2 + P$
(iii)	$\frac{1}{2}\rho u^2 + \rho e$	$\frac{1}{2}\rho u^3 + \rho u e + P u$

(i): Mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

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(ii): Momentum

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial P}{\partial x} = 0$$

(iii): Energy

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho u^3 + \rho u e + P u\right) = 0$$

In summary, where equations references correspond to [2]:

		f	g
mass	(4-76)	ρ	ρu
momentum	(4-77)	ρu	$\rho u^2 + P$
energy	(4-78)	$\frac{1}{2}\rho u^2 + \rho e$	$\frac{1}{2}\rho u^3 + \rho u e + P u$

(b) Write equation (1.1) in integral form by integrating from x_1 to x_2. Write the result such that the quantity $g(x_1, t) - g(x_2, t)$ appears on one side of the equation. Call this quantity I.

Integrating equation (1.1) from x_1 to x_2 gives

$$\frac{d}{dt}\int_{x_1}^{x_2} f(x,t) + g(x_2) - g(x_1) = 0$$

Moving the gs to the other side gives

$$I = \frac{d}{dt}\int_{x_1}^{x_2} f(x,t) = g(x_1, t) - g(x_2, t)$$

(c) Let a discontinuity exist at $x_{\text{sh}}(t)$ in the result from part (b). Split the integral I from part (b) into $I_1 + I_2$, to account for the discontinuity. Use the notation $x_1 < x_{\text{sh}}^- < x_{\text{sh}}^+ < x_2$. *Hint: the upper limit of I_1 should be x_{sh}^-, and the lower limit of I_2 should be x_{sh}^+, where $x_{sh}^{\pm} = x_{sh} \pm \epsilon$, $\epsilon \to 0$.*

$$I_1 = \frac{d}{dt}\int_{x_1}^{x_{\mathsf{sh}}^-(t)} f(x,t)dx$$

and

$$I_2 = \frac{d}{dt}\int_{x_{\mathsf{sh}}^+(t)}^{x_2} f(x,t)dx$$

(d) Evaluate the integrals I_1 and I_2. *Hint: Note that for an arbitrary function $q(x,t)$,*

$$\frac{d}{dt}\int_{x_i}^{x^\pm} q(x,t)dx = q(x^\pm,t)\frac{dx^\pm}{dt} - q(x_i,t)\frac{dx_i}{dt} + \int_{x_i}^{x^\pm}\frac{\partial q}{\partial t}dx.$$

Note that $dx_1(t)/dt = dx_2(t)/dt = 0$. Also, denote $dx_{sh}^-/dt = U_{sh}$.

Using the rule suggested,

$$I_1 = f(x_{\mathsf{sh}}^-,t)\frac{dx_{\mathsf{sh}}^-}{dt} - f(x_1,t)\frac{dx_1}{dt} + \int_{x_1}^{x_{\mathsf{sh}}^-}\frac{\partial f}{\partial t}dx$$

Noting that $\partial x_1(t)/\partial t = 0$ and denoting $dx_{\mathsf{sh}}^-/dt = U_{\mathsf{sh}}$,

$$I_1 = f(x_{\mathsf{sh}}^-,t)U_{\mathsf{sh}} + \int_{x_1}^{x_{\mathsf{sh}}^-}\frac{\partial f}{\partial t}dx$$

Similarly,

$$I_2 = -f(x_{\mathsf{sh}}^+,t)\frac{dx_{\mathsf{sh}}^+}{dt} + f(x_2,t)\frac{dx_2}{dt} - \int_{x_2}^{x_{\mathsf{sh}}^-}\frac{\partial f}{\partial t}dx$$

$$I_2 = -f(x_{\mathsf{sh}}^+,t)U_{\mathsf{sh}} - \int_{x_2}^{x_{\mathsf{sh}}^+}\frac{\partial f}{\partial t}dx$$

(e) Take the limit of I_1, as found in the previous part, as $x_1 \to x_{\text{sh}}^-$. Similarly, take the limit of I_2 as $x_2 \to x_{\text{sh}}^+$. Note that the integral vanishes in both cases.

Taking the limits gives

$$\lim_{x_1 \to x_{\text{sh}}^-} I_1 = U_{\text{sh}} f(x_{\text{sh}}^-, t)$$

and

$$\lim_{x_2 \to x_{\text{sh}}^+} I_2 = -U_{\text{sh}} f(x_{\text{sh}}^+, t)$$

(f) Use the above result, as well as the result of part (b), to show that as $x_1 \to x_{\text{sh}}^-$ and $x_2 \to x_{\text{sh}}^+$,

$$g(x_{\text{sh}}^-, t) - g(x_{\text{sh}}^+, t) = U_{\text{sh}}[f(x_{\text{sh}}^-, t) - f(x_{\text{sh}}^+, t)] \tag{1.2}$$

From part (b), $I = g(x_1, t) - g(x_2, t)$. Taking the appropriate limits and noting that $\lim_{x_1 \to x_{\text{sh}}^-} I_1 = U_{\text{sh}} f(x_{\text{sh}}^-, t)$ and $\lim_{x_2 \to x_{\text{sh}}^+} I_2 = -U_{\text{sh}} f(x_{\text{sh}}^+, t)$ gives the desired result,

$$g(x_{\text{sh}}^-, t) - g(x_{\text{sh}}^+, t) = U_{\text{sh}}[f(x_{\text{sh}}^-, t) - f(x_{\text{sh}}^+, t)]$$

(g) Rewrite equation (1.2) by letting the subscript a correspond to "ahead of the shock," x_{sh}^+, and by letting the subscript b correspond to "behind the shock," x_{sh}^-.

$$g_b - g_a = U_{\text{sh}}(f_b - f_a)$$

(h) Write the above result using the jump notation, $[q] = q_b - q_a$.

$$[g] = U_{\text{sh}}[f]$$

(i) Define $v = u - U_{\text{sh}}$ and use the table from part (a) to derive the so-called Rankine-Hugoniot relations. *Hint: for the conservation of* *momentum and energy, some rearrangement is required. Just see your notes.*

For the conservation of mass,

$$[\rho u] = U_{\mathsf{sh}}[\rho].$$

Invoking the definition $v = u - U$ gives

$$\boxed{[\rho v] = 0}$$

For the conservation of momentum,

$$[\rho u^2 + P] = U_{\mathsf{sh}}[\rho u]\,.$$

After some rearrangement (see class notes):

$$\boxed{[\rho v^2 + P] = 0}$$

For the conservation of energy,

$$\left[\frac{1}{2}\rho u^3 + \rho u e + P u\right] = U_{\mathsf{sh}}\left[\frac{1}{2}\rho u^2 + \rho e\right]\,.$$

Defining $h = e + P/\rho =$ enthalpy per unit mass, one finds

$$\boxed{\left[\frac{1}{2}v^2 + h\right] = 0}$$

(j) What did we find in class to be the order of the entropy jump across the shock, for an arbitrary fluid?

It was found that the order of the entropy jump across the shock is $\mathcal{O}(\epsilon^3)$.

(k) What did we find in class to be the order of the reflection from the shock front?

The reflection from the shock front is also $\mathcal{O}(\epsilon^3)$.

(l) What do the previous two parts imply about the shock at quadratic order?

A shock at quadratic order is simple and isentropic.

2 Weak shock speed

(a) Denoting $v = u - U_{\text{sh}}$ and $Q = \rho u$, use the Rankine-Hugoniot relation $[\rho v] = 0$ to show that

$$U_{\text{sh}} = \frac{[Q]}{[\rho]}\,. \tag{2.1}$$

By the conservation of mass, $[\rho v] = 0$,

$$[\rho u - \rho U_{\mathsf{sh}}] = 0$$
$$[\rho u] = [\rho] U_{\mathsf{sh}}$$

Solving for U_{sh} gives

$$\boxed{U_{\mathsf{sh}} = \frac{[\rho u]}{[\rho]} = \frac{[Q]}{[\rho]}}$$

(b) Taylor expand $[Q]$ in $[\rho]$ to $\mathcal{O}(\epsilon^3)$ and combine with equation (2.1) to show that

$$U_{\text{sh}} = Q'_a + \frac{1}{2} Q''_a [\rho] + \mathcal{O}(\epsilon^2)\,. \tag{2.2}$$

$$[Q] = Q'_a [\rho] + \frac{1}{2} Q''_a [\rho]^2 + \mathcal{O}(\epsilon^3)$$

Combining with equation (2.1) gives

$$\boxed{U_{\mathsf{sh}} = Q'_a + \frac{1}{2} Q''_a [\rho] + \mathcal{O}(\epsilon^2)}$$

(c) Noting that $[Q'] = Q''_a[\rho] + \mathcal{O}(\epsilon^2)$ (the first-order Taylor expansion of $[Q']$ in ρ), substitute Q''_a into equation (2.2) to show that

$$U_{\text{sh}} = Q'_a + \frac{1}{2}[Q'] + \mathcal{O}(\epsilon^2)\,. \tag{2.3}$$

Then write $[Q'] = Q'_b - Q'_a$ to write equation (2.3) as

$$U_{\text{sh}} = \frac{1}{2}(Q'_a + Q'_b) + \mathcal{O}(\epsilon^2)\,. \tag{2.4}$$

By inspection, $U_{\mathsf{sh}} = Q'_a + \frac{1}{2}[Q'] + \mathcal{O}(\epsilon^2)$.. Writing $[Q'] = Q'_b - Q'_a$ gives $U_{\mathsf{sh}} = Q'_a + \frac{1}{2}(Q'_b - Q'_a) + \mathcal{O}(\epsilon^2) =$

$$U_{\mathsf{sh}} = \frac{1}{2}(Q'_a + Q'_b) + \mathcal{O}(\epsilon^2)\,.$$

(d) Note that

$$\begin{aligned} Q' = \frac{d(\rho u)}{d\rho} &= u + \rho\frac{du}{d\rho} \\ &= u + c \\ &= u + c_0 + \frac{B}{2A}u + \mathcal{O}(\epsilon^2) \\ &= c_0 + \beta u\,, \end{aligned} \tag{2.5}$$

where the simple-wave relation $du = \frac{c}{\rho}d\rho$ has been used. Combine equation (2.5) with equation (2.4) to show that

$$U_{\text{sh}} = c_0 + \frac{\beta}{2}(u_a + u_b) + \mathcal{O}(\epsilon^2)\,. \tag{2.6}$$

Substituting $Q' = c_0 + \beta u$ into $U_{\mathsf{sh}} = \frac{1}{2}(Q'_a + Q'_b) + \mathcal{O}(\epsilon^2)$ gives

$$U_{\mathsf{sh}} = \frac{1}{2}(c_0 + \beta u_a + c_0 + \beta u_b) + \mathcal{O}(\epsilon^2) \tag{2.7}$$

Rearranging gives the result

$$U_{\mathsf{sh}} = c_0 + \frac{\beta}{2}(u_a + u_b) + \mathcal{O}(\epsilon^2)\,.$$

Energy dissipation at a shock front was very involved an is not included in this review. See class notes for the derivation leading to dE/dt, which is cubic in the pressure jump. dT/dt is also cubic in the pressure jump. See also the applications to HIFU discussed in class.

3 Landau's equal-area rule

(a) Note that the area under a shock is given by

$$A = \int_{u_a}^{u_b} (x - x_{\text{sh}}) du \,. \tag{3.1}$$

Write dA/dt using the rule

$$\frac{d}{dt}\int_{u_a}^{u_b} q(u,t)du = q(u_b,t)\frac{du_b}{dt} - q(u_a,t)\frac{du_a}{dt} + \int_{u_a}^{u_b} \frac{\partial q}{\partial t} du.$$

Hint: let q above $= x - x_{sh}$.

$$\frac{dA}{dt} = (x - x_{\mathsf{sh}})\Big|_{u_b} \frac{du_b}{dt} - (x - x_{\mathsf{sh}})\Big|_{u_a} \frac{du_a}{dt} + \int_{u_a}^{u_b} \left[\frac{\partial x}{\partial t} - \frac{\partial x_{\mathsf{sh}}}{\partial t}\right] dx. \tag{3.2}$$

(b) Noting that $x = x_{\text{sh}}$ at $u = u_a$ and $u = u_b$, show that

$$\frac{dA}{dt} = \int_{u_a}^{u_b} \left[\frac{dx}{dt} - \frac{dx_{\text{sh}}}{dt}\right] du \,. \tag{3.3}$$

This is done by inspection. The first two terms on the RHS of equation (3.2) are zero, giving the result.

(c) Identify dx/dt in equation (3.3) to be the finite amplitude propagation speed, $c_0 + \beta u + \mathcal{O}(\epsilon^2)$, and identify dx_{sh}/dt to be $U_{\text{sh}} = c_0 + \frac{\beta}{2}(u_a + u_b) + \mathcal{O}(\epsilon^2)$, by equation (2.6). Perform the integral in equation (3.3) over u to show that $dA/dt = 0$, i.e., $A =$ constant $= A_+ - A_-$.

$$\begin{aligned}
\frac{dA}{dt} &= \int_{u_a}^{u_b} \left[c_0 + \beta u - c_0 - \frac{\beta}{2}(u_a + u_b)\right] du + \mathcal{O}(\epsilon^2) \\
&= \beta \int_{u_a}^{u_b} \left[u - \frac{1}{2}(u_a + u_b)\right] du + \mathcal{O}(\epsilon^2) \\
&= \frac{\beta}{2}[u_b^2 - u_a^2 - (u_a + u_b)(u_b - u_a)] \\
&= \frac{\beta}{2}[u_b^2 - u_a^2 + u_a^2 - u_b^2] = 0
\end{aligned}$$

The time derivative of A is 0, so $A =$ constant. Since $A = 0$ at $t = \bar{t}$, $A = A_+ + A_-$ for all time. The location of the shocks is therefore determined by setting $A_- = A_+$.

4 Blackstock's weak-shock method

(a) The retarded shock time is $\tau_{\text{sh}} = t_{\text{sh}} - x/c_0$. Calculate $d\tau_{\text{sh}}/dx$ by define the shock slowness to be $1/U_{\text{sh}} = dt_{\text{sh}}/dx = [c_0 + \frac{\beta}{2}(u_a + u_b)]^{-1}$.
Answer:

$$\frac{d\tau_{\text{sh}}}{dx} = -\frac{\beta}{2\rho_0 c_0^3}(p_a + p_b) + \mathcal{O}(\epsilon^2) \tag{4.1}$$

$$\begin{aligned}
\frac{d\tau_{\text{sh}}}{dx} &= \frac{d}{dx}(t_{\text{sh}} - x/c_0) \\
&= \frac{1}{U_{\text{sh}}} - \frac{1}{c_0} \\
&= [c_0 + \frac{\beta}{2}(u_a + u_b)]^{-1} - \frac{1}{c_0} + \mathcal{O}(\epsilon^2) \\
&= \frac{1}{c_0}[1 - \frac{\beta}{2c_0}(u_a + u_b)] - \frac{1}{c_0} + \mathcal{O}(\epsilon^2) \\
&= -\frac{\beta}{2c_0^2}(u_a + u_b) + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Using the linear impedance relationship, the speed of the shock is written in terms of pressure:

$$\boxed{\frac{d\tau_{\text{sh}}}{dx} = -\frac{\beta}{2\rho_0 c_0^3}(p_a + p_b) + \mathcal{O}(\epsilon^2)}$$

(b) From where are p_a and p_b obtained?

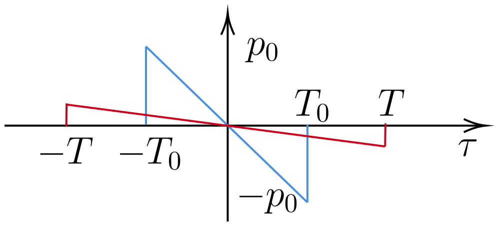
p_a and p_b are obtained from the approximate nonlinear lossless solution discussed in the first half of the course,

$$\boxed{p = f(\phi), \qquad \phi = \tau + \frac{\beta x f(\phi)}{\rho_0 c_0^3}} \tag{4.2}$$

(c) N-wave example: Use the Blackstock weak shock method to find $p_{\mathrm{sh}}(x)$ for the boundary condition

$$f(t) = \begin{cases} -p_0 t/T_0, & |t| < T_0 \\ 0, & |t| > T \end{cases} \quad , \tag{4.3}$$

which is prescribed at $x = 0, \phi = \tau$.



First, replace t with ϕ to describe the waveform for $|t| < T$:

$$f(\phi) = -\frac{p_0}{T_0}\phi$$

By equation (4.2), the phase of the implicit solution is

$$\phi = \tau - \frac{\beta p_0}{\rho_0 c_0^3 T_0} x\phi$$

Denoting $b = \beta p_0/\rho_0 c_0^3 T_0$, the phase can be written as

$$\phi = \tau - bx\phi$$

Solving for ϕ gives

$$\phi = \frac{\tau}{1 + bx}.$$

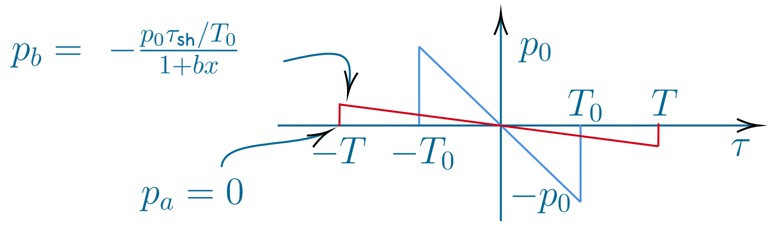
Then, the pressure solution is

$$p(x, \tau) = \begin{cases} \frac{p_0 \tau/T_0}{1+bx}, & |\tau| < T_0 \\ 0, & |\tau| > T \end{cases} . \tag{4.4}$$

It is desired to find $p_{\mathsf{sh}}(x)$. Note from the above that at the head shock,

$$p_b = -\frac{p_0 \tau_{\mathsf{sh}}/T_0}{1 + bx} \tag{i}$$

and $$p_a = 0\,. \tag{ii}$$



Evidently, to find $p_{\text{sh}}(x)$, τ_{sh} must be found. At the head shock, $\tau_{\text{sh}} = -T$. Invoking equation (4.1) gives

$$\begin{aligned}\frac{d\tau_{\text{sh}}}{dx} = -\frac{dT}{dx} &= -\frac{\beta}{2\rho_0 c_0^3}(p_a + p_b)\\ &= \frac{\beta}{2\rho_0 c_0^3}\frac{p_0\tau_{\text{sh}}/T_0}{1+bx}\\ &= -\frac{1}{2}\frac{\beta p_0}{\rho_0 c_0^3}\frac{T/T_0}{1+bx}\\ &= -\frac{b}{2}\frac{T}{1+bx}\end{aligned}$$

Integrating the above gives $T = A\sqrt{1+bx}$. Since $T = T_0$ at $x = 0$, $A = T_0$. Therefore, equation (i) becomes

$$p_{\text{sh}} = \frac{p_0}{\sqrt{1+bx}} \tag{4.5}$$

5 Blackstock's bridging function

(a) From the development of the Fubini solution, which expands the pressure as a Fourier sine series $P(\sigma,\theta) = \sum_{n=1}^{\infty} B_n(\sigma)\sin n\theta$, where $\sigma = x/\bar{x}$ and $\theta = \omega\tau$, it was found that the expansion coefficients B_n are given by the sum $B_n^{(1)} + B_n^{(2)}$, where

$$B_n^{(1)} = -\frac{2}{n\pi}\cos(n\theta)\sin(\Phi)\Big|_{\theta,\Phi=0}^{\theta,\Phi=\pi} = 0 \tag{5.1}$$

$$B_n^{(2)} = \frac{2}{n\pi}\int_{\Phi=0}^{\Phi=\pi}\cos n\theta\cos\Phi d\Phi = \frac{2}{n\sigma}J_n(n\sigma) \tag{5.2}$$

where $\Phi = \theta + \sigma \sin \Phi$. For $\sigma < 1$, why are the limits on θ and Φ above equal?

For $\sigma < 1$, the two limits need to be assessed: (1) when $\theta = 0$, the transcendental equation for Φ is $\Phi = \sigma \sin \Phi$, for which $\Phi = 0$ is the only solution; and (2) when $\theta = \pi$, the transcendental equation for Φ is $\Phi = \pi + \sigma \sin \Phi$, for which $\Phi = \pi$ is the only solution. Evidently, the limits on θ and Φ are the same. For graphical solutions of the transcendental equation, see **here**.

(b) For $\sigma > 1$, what is Φ when $\theta = \pi$? What are the two possibilities for Φ at $\theta = 0$? Noting that $P_{\text{sh}} = P_b$, what is the correct choice for Φ?

$\Phi = \pi$ when $\theta = \pi$ as before, but at $\theta = 0$, Φ can either be 0 or Φ_{sh}. $\Phi = \Phi_{\text{sh}}$ is the correct choice because at that point, the waveform is multivalued, and its physical value is $P_{\text{sh}} = P_b$.

(c) Given how Φ and θ have different limits at $\theta = 0$, how do equations (5.1) and (5.2) change for $\sigma > 1$? (Qualitative answer is sufficient...the math is a bit confusing)

Qualitatively, the upper limit on both Φ and θ remain the same, and the lower limit for θ stays the same, but the lower limit for Φ changes from 0 to P_{sh}.

Quantitatively, using the relations $P_{\text{sh}} = \sin \Phi_{\text{sh}} = \sin(\sigma \sin \Phi_{\text{sh}}) = \sin(\sigma P_{\text{sh}})$ and $\cos \Phi d\Phi = (d\Phi - d\theta)/\sigma$,

$$
\begin{aligned}
B_n^{(1)} &= \frac{2}{n\pi} \sin \Phi_{\text{sh}} = \frac{2P_{\text{sh}}}{n\pi} \\
B_n^{(2)} &= \frac{2}{n\pi\sigma} \int_{\Phi_{\text{sh}}}^{\Phi=\pi} \cos n\theta d\Phi - \int_0^{\pi} \cos n\pi d\theta \\
&= \frac{2}{n\pi\sigma} \int_{\Phi_{\text{sh}}}^{\pi} \cos[n(\Phi - \sigma \sin \Phi)] d\Phi
\end{aligned}
$$

Adding the two together gives the Blackstock bridging function.

6 Nonlinearity in multiple dimensions

(a) 1D spreading is modeled by adding a term mp/r to the LHS of the Burgers equation with no absorption, i.e., $\delta = 0$:

$$\frac{\partial p}{\partial r} + \frac{m}{r}p = \pm\frac{\beta p}{\rho_0 c_0^3}\frac{\partial p}{\partial \tau}. \tag{6.1}$$

where τ is now $t \mp (r - r_0)/c_0$. What is m for 1D spherical spreading? What is m for 1D cylindrical spreading? What is some restrictions on this formulation?

$m = 1$ for spherical spreading and $m = 1/2$ for cylindrical spreading. The cylindrical case applies only for large kr_0. All of this generally applies only to diverging waves.

(b) Introduce

$$q = \left(\frac{r}{r_0}\right)^m p$$

and calculate $\partial p/\partial r$ and $\partial p/\partial \tau$ in terms of q.

Writing $p = \left(\frac{r_0}{r}\right)^m q$ and taking the derivatives w.r.t. r gives $\frac{\partial p}{\partial r} = -m(r_0/r)^{m-1} r_0 r^{-2} q + (r_0/r)^m \frac{\partial q}{\partial r}$, or

$$\boxed{\frac{\partial p}{\partial r} = (r_0/r)^m \frac{\partial q}{\partial r} - \frac{m}{r}\left(\frac{r_0}{r}\right)^m q}$$

Meanwhile, taking the derivative with respect to τ gives

$$\boxed{\frac{\partial p}{\partial \tau} = \left(\frac{r_0}{r}\right)^m \frac{\partial q}{\partial \tau}}$$

(c) Write equation (6.1) in terms of q. *Answer:*

$$\frac{\partial q}{\partial r} = \pm\left(\frac{r_0}{r}\right)^m \frac{\beta q}{\rho_0 c_0^3}\frac{\partial q}{\partial \tau} \tag{6.2}$$

Inserting the above-found derivatives and the definition of q into equation (6.1) gives

$$(r_0/r)^m \frac{\partial q}{\partial r} - \frac{m}{r}\left(\frac{r_0}{r}\right)^m q + \frac{m}{r}\left(\frac{r_0}{r}\right)^m q = \pm \frac{\beta p}{\rho_0 c_0^3}\left(\frac{r_0}{r}\right)^{2m} \frac{\partial q}{\partial \tau}.$$

Dividing by $(r_0/r)^m$ and canceling the second and third terms on the LHS gives the desired result provided above.

(d) With the intention of getting rid of the factor of $(r_0/r)^m$ altogether from equation (6.2), choose $z(r)$ such that

$$\frac{\partial q}{\partial r} = \frac{\partial q}{\partial z}\frac{dz}{dr} = \pm \left(\frac{r_0}{r}\right)^m \frac{\partial q}{\partial z}.$$

Integrate to find z for $m = 1$ and $m = 1/2$.

$$z = \int dz = \pm \int_{r_0}^{r} \left(\frac{r_0}{r'}\right)^m dr'$$

For $m = 1$,

$$\boxed{z = \pm r_0 \ln(r/r_0)}$$

For $m = 1/2$,

$$\boxed{z = \pm 2(\sqrt{r} - \sqrt{r_0})\sqrt{r_0}}$$

(e) Write equation (6.1) in terms of the stretched coordinates q and z.

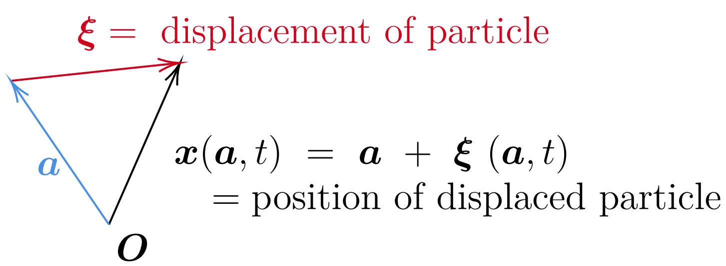
$$\boxed{\frac{\partial q}{\partial z} = \frac{\beta q}{\rho_0 c_0^3}\frac{\partial q}{\partial \tau}}$$

7 Radiation force

(a) What is the distinction between Eulerian and Lagrangian coordinates? Why does the distinction dissolve in linear theory?

Crudely put, Eulerian coordinates are fixed in space, while Lagrangian coordinates move with the particle. That is why Eulerian coordinates are referred to as "spatial coordinates," while Lagrangian coordinates are referred to as "material coordinates." Eulerian coordinates can be intuitively thought of as sitting on the bank of a river and watching a fallen leaf go by, and Lagrangian coordinates can be thought of as swimming alongside the leaf in the river.

(b) Let $\boldsymbol{a}$ be the position of a particle at rest, $\boldsymbol{\xi}$ be the displacement of the particle from $\boldsymbol{a}$, and $\boldsymbol{x}$ be the position of the displaced particle:



Then, the transformations between a Lagrangian quantity $q_{\mathrm{L}}(\boldsymbol{a},t)$ (can be a scalar, vector, or tensor) and Eulerian quantity $q_{\mathrm{E}}(\boldsymbol{x},t)$ are

$$q_{\mathrm{L}}(\boldsymbol{a},t) = q_{\mathrm{E}}(\boldsymbol{a},t) + \boldsymbol{\xi}(\boldsymbol{a},t)\cdot\boldsymbol{\nabla}_{\boldsymbol{a}} q_{\mathrm{E}}(\boldsymbol{a},t) \tag{7.1}$$

$$q_{\mathrm{E}}(\boldsymbol{x},t) = q_{\mathrm{L}}(\boldsymbol{x},t) - \boldsymbol{\xi}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla}_{\boldsymbol{x}} q_{\mathrm{L}}(\boldsymbol{x},t)\,. \tag{7.2}$$

Why can the coordinates in which the gradients in the above equations are evaluated be neglected?

Those corrections ($\boldsymbol{\nabla}_x$ vs $\boldsymbol{\nabla}_a$) are discarded because they are of higher order.

(c) Resolve Westervelt's paradox, which says that for $\dot{X}(t) = u_0 \sin\omega t$ at $x = X(t)$, $u_{\mathrm{E}} = u_0\sin(\omega t - kx)$ and $\langle u_{\mathrm{E}}\rangle = -\langle u^2\rangle/c_0 = -u_0^2/2c_0$. Do so by calculating $\langle u_{\mathrm{L}}\rangle$ in Lagrangian coordinates. *Hint: use equation (7.1).*

Letting the quantity q in $q_{\mathrm{L}}(\boldsymbol{a},t) = q_{\mathrm{E}}(\boldsymbol{a},t) + \boldsymbol{\xi}(\boldsymbol{a},t)\cdot\boldsymbol{\nabla} q_{\mathrm{E}}(\boldsymbol{a},t)$ be the scalar particle velocity particle velocity u. Then,

$$\boldsymbol{u}_{\mathrm{L}}(\boldsymbol{a},t) = \boldsymbol{u}_{\mathrm{E}}(\boldsymbol{a},t) + \xi(\boldsymbol{a},t)\frac{d\boldsymbol{u}_{\mathrm{E}}(\boldsymbol{a},t)}{dx}\,, \tag{7.3}$$

or in 1D,

$$u_{\mathrm{L}}(a,t) = u_{\mathrm{E}}(a,t) + \xi(a,t)\frac{du_{\mathrm{E}}(a,t)}{dx}\,, \tag{7.4}$$

ξ, the separation between positions a and x, can be found by integrating u_{E} over time: $\xi = \int u_{\mathrm{E}} dt = -\frac{u_0}{\omega}\cos(\omega\tau - kx)$. Also note that $\frac{du_{\mathrm{E}}}{dx} = -ku_0\cos(\omega\tau - kx)$. Taking the time average of u_{L} gives

$$\begin{aligned}\langle u_{\mathrm{L}}\rangle &= \langle u_{\mathrm{E}}\rangle + \frac{u_0^2 k}{\omega}\langle \cos^2\omega\tau\rangle \\ &= -\frac{u_0^2}{2c_0} + \frac{u_0^2}{2c_0} = 0\end{aligned}$$

The paradox is resolved: in the reference frame that moves with the piston, there is no d.c. flow through the piston.

(d) Calculate the mean excess pressure in Eulerian coordinates. *Hint: Start with the linearized momentum equation*

$$\rho_0\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} p = -\boldsymbol{\nabla}\mathcal{L} \tag{7.5}$$

where $\mathcal{L} = \frac{1}{2}\rho_0 u^2 - p^2/2\rho_0 c_0^2$ *is the Lagrangian (leave everything in terms of* $\mathcal{L}$*). Then let* $\boldsymbol{u} = \boldsymbol{\nabla}\phi$*, i.e., irrotational, and integrate over volume. Call the constant of integration* $g(t)$ *on the right-hand side. Finally take the time average and call* $\langle g(t)\rangle \equiv C$*. Answer:*

$$\langle p_{\mathrm{E}}\rangle = -\langle\mathcal{L}\rangle + C \tag{7.6}$$

Letting $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, the momentum equation becomes

$$\boldsymbol{\nabla}\left(\rho_0\frac{\partial\phi}{\partial t} + p\right) = -\boldsymbol{\nabla}\mathcal{L} \tag{7.7}$$

By the gradient theorem,

$$\rho_0\frac{\partial\phi}{\partial t} + p = -\mathcal{L} + g(t)$$

Taking the time average gives

$$\boxed{\langle p_{\mathrm{E}}\rangle = -\langle\mathcal{L}\rangle + C}$$

(e) Calculate the mean excess pressure in Lagrangian coordinates by using equation (7.1). *Hint: After using equation (7.1), take the time average, and use the momentum equation to write* $\langle \boldsymbol{\xi} \cdot \boldsymbol{\nabla} p \rangle = -\rho_0 \langle \boldsymbol{\xi} \cdot \partial \boldsymbol{u}/\partial t \rangle$. *Further note that one can write* $\partial^2 \boldsymbol{\xi}/\partial t^2$ *as* $2\partial\boldsymbol{\xi}/\partial t \cdot \partial\boldsymbol{\xi}/\partial t + 2\boldsymbol{\xi} \cdot \partial^2\boldsymbol{\xi}/\partial t^2$. *Answer:*

$$\langle p_{\mathrm{L}} \rangle = \langle p_{\mathrm{E}} \rangle + \rho_0 \langle u^2 \rangle \tag{7.8}$$

By equation (7.1),

$$p_{\mathrm{L}} = p_{\mathrm{E}} + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} p$$

Taking the time average,

$$\langle p_{\mathrm{L}} \rangle = \langle p_{\mathrm{E}} \rangle + \langle \boldsymbol{\xi} \cdot \boldsymbol{\nabla} p \rangle \tag{7.9}$$

Now use the hint, $\langle \boldsymbol{\xi} \cdot \boldsymbol{\nabla} p \rangle = -\rho_0 \langle \boldsymbol{\xi} \cdot \partial \boldsymbol{u}/\partial t \rangle$. By the definition of $\boldsymbol{u} = \dot{\boldsymbol{\xi}}$, this quantity can be written as $-\rho_0 \langle \boldsymbol{\xi} \cdot \partial^2\boldsymbol{\xi}/\partial t^2 \rangle$. Further noting that $\partial^2\boldsymbol{\xi}/\partial t^2 = 2\partial\boldsymbol{\xi}/\partial t \cdot \partial\boldsymbol{\xi}/\partial t + 2\boldsymbol{\xi} \cdot \partial^2\boldsymbol{\xi}/\partial t^2$ gives $\langle \boldsymbol{\xi} \cdot \boldsymbol{\nabla} p \rangle = \rho_0 \langle u^2 \rangle$. Upon all these considerations, equation (7.9) becomes

$$\boxed{\langle p_{\mathrm{L}} \rangle = \langle p_{\mathrm{E}} \rangle + \rho_0 \langle u^2 \rangle}$$

(f) Define $V = p^2/2\rho_0 c_0^2$, $K = \rho_0 u^2/2$. Then the energy is $\mathcal{E} = K + V$ and the Lagrangian is $\mathcal{L} = K - V$. By equation (7.8), $\langle p_{\mathrm{L}} \rangle = \langle p_{\mathrm{E}} \rangle + 2\langle K \rangle$. Combine this result and the new notation with equation (7.6) to find $\langle p_{\mathrm{L}} \rangle$.

Combining equation (7.6), $\langle p_{\mathrm{E}} \rangle = -\langle \mathcal{L} \rangle + C$, with $\langle p_{\mathrm{E}} \rangle = \langle p_{\mathrm{L}} \rangle - 2\langle K \rangle$ gives $\langle p_{\mathrm{L}} \rangle = -\langle \mathcal{L} \rangle + 2\langle K \rangle + C$. Writing $-\langle \mathcal{L} \rangle = \langle V \rangle - \langle K \rangle$ gives

$$\boxed{\langle p_{\mathrm{E}} \rangle = \langle V \rangle - \langle K \rangle + C}$$

and

$$\boxed{\langle p_{\mathrm{L}} \rangle = \langle V \rangle + \langle K \rangle + C}$$

(g) Show that in the linear limit, the Eulerian and Lagrangian excess pressures are equal.

This is trivial. All the quantities in both expressions for the Eulerian and Lagrangian excess pressures are quadratic, so in the linear limit, the Eulerian and Lagrangian excess pressures vanish identically.

(h) Show that the Lagrangian radiation pressure $\langle p_{\mathrm{L}} \rangle$ on a surface normal to and in contact with the fluid motion is constant. What is remarkable about this result? *Hint: Take the time average of Newton's second law, which in Lagrangian coordinates reads* $\rho_0 \partial^2 \xi / \partial t^2 = -\partial p_{\mathrm{L}} / \partial a$.

The time average of Newton's second law is $\langle \rho_0 \partial^2 \xi / \partial t^2 \rangle = -\langle \partial p_{\mathrm{L}} / \partial a \rangle$. The LHS vanishes because the time average of a time derivative vanishes for a periodic wave. Thus $-\langle \partial p_{\mathrm{L}} / \partial a \rangle = 0$ and

$$\boxed{p_{\mathrm{L}} = \text{constant}}$$

(i) Calculate $\langle p_{\mathrm{E}} \rangle$ and $\langle p_{\mathrm{L}} \rangle$ in a standing wave, to accuracy of a constant of integration C. The standing wave is given by $p = p_0 \cos kx \sin \omega t$, or equivalently $u = \frac{p_0}{\rho_0 c_0} \sin kx \cos \omega t$. *Hint: Recall from Acoustics I that* $V = p^2 / 2\rho_0 c_0^2$ *and* $K = \rho_0 u^2 / 2$ *and write the answer in terms of* $\mathcal{E} = V + K$ *and* $\mathcal{L} = K - V$.

Using the definition of the potential energy density gives

$$V = \frac{p_0^2}{2\rho_0 c_0^2} \cos^2 kx \sin^2 \omega t \,.$$

Taking the time average and using the double angle trigonometric identity gives

$$\langle V \rangle = \frac{p_0^2}{8\rho_0 c_0^2} (1 + \cos 2kx)$$

Meanwhile the kinetic energy density is

$$K = \frac{p_0^2}{2\rho_0 c_0^2} \sin^2 kx \cos^2 \omega t \,.$$

Taking the time average gives

$$\langle K \rangle = \frac{p_0^2}{8\rho_0 c_0^2} (1 - \cos 2kx) \,.$$

These results are inserted in $\langle p_\mathrm{E}\rangle$ and $\langle p_\mathrm{L}\rangle$ calculated previously [see part (f)]:

$$\boxed{p_\mathrm{E} = \frac{p_0^2}{4\rho_0 c_0^2}\cos(2kx) + C}$$

Similarly,

$$\boxed{p_\mathrm{L} = \frac{p_0^2}{4\rho_0 c_0^2} + C}$$

(j) Determine the constant of integration C by invoking the conservation of mass. Specifically, require that

$$\int_x^{x+\lambda} \langle \rho'_\mathrm{E}\rangle\, dx = 0, \qquad \text{at } \mathcal{O}(\epsilon^2)\,,$$

where[1] $\rho' = (p - VB/A)/c_0^2$. *Answer:*

$$C = \frac{B}{2A}\frac{p_0^2}{4\rho_0 c_0^2}$$

See notes for the full solution. We had to use equation (3-39) from [2], a nonlinear pressure-density relationship, to show this. I am not comfortable with that step, and therefore I omit writing this solution in these notes.

(k) Given that the radiation force on an object of volume V is $\boldsymbol{F}_\mathrm{rad} = -\langle V\boldsymbol{\nabla}p\rangle$, calculate the radiation force exerted on a ping-pong ball of radius R by a standing pressure wave $p(x,t) = p_0 \cos kx \sin \omega t$ in a closed tube. Assume that the ball is perfectly rigid and that $kR \ll 1$. *Hint: Reduce the problem to 1D, i.e.,* $\boldsymbol{F}_{rad} = -\langle V\partial p/\partial x\rangle$ *and assume that the ping-pong ball has sufficient inertia such that Eulerian radiation pressure* $\langle p_\mathrm{E}\rangle$ *found in the previous problem can be used. Also note that since the ball is rigid, its volume is constant. All these considerations result in the radiation force being given by*

$$F_\mathrm{rad} = -\frac{4\pi R^3}{3}\frac{d\langle p_\mathrm{E}\rangle}{dx}$$

[1] See equation (3-39) of [2].

Inserting

$$\langle p_E \rangle = \frac{p_0^2}{4\rho_0 c_0^2}\left(\frac{B}{2A} + \cos 2kx\right)$$

into

$$F_{\text{rad}} = -\frac{4\pi R^3}{3}\frac{d\langle p_E \rangle}{dx}$$

gives

$$\boxed{F_{\text{rad}} = -\frac{2\pi p_0 R^3}{3\rho_0 c_0^2}\sin 2kx} \tag{7.10}$$

(l) Define $\langle \mathcal{P} \rangle$ to be the time-averaged momentum density, given by $\langle$momentum/volume$\rangle$. Show that

$$\langle \mathcal{P} \rangle = \frac{\langle \mathcal{E} \rangle}{c_0} \quad \text{for} \quad f(x - c_0 t)$$

and

$$\langle \mathcal{P} \rangle = -\frac{\langle \mathcal{E} \rangle}{c_0} \quad \text{for} \quad f(x + c_0 t).$$

Hint: write momentum/volume as $\rho' u$, and use linear relations $\rho' = p/c_0^2$ and $u = p/\rho_0 c_0$. Then note that $\mathcal{E} = p^2/\rho_0 c_0^2$.

Following the hints, for $f(x - c_0 t)$,

$$\boxed{\langle \mathcal{P} \rangle = \langle \rho' u \rangle = \frac{\langle p^2 \rangle}{\rho_0 c_0^3} = \frac{\langle \mathcal{E} \rangle}{c_0}}$$

while for $f(x + c_0 t)$,

$$\boxed{\langle \mathcal{P} \rangle = -\langle \rho' u \rangle = -\frac{\langle p^2 \rangle}{\rho_0 c_0^3} = -\frac{\langle \mathcal{E} \rangle}{c_0}}$$

(m) In class, the momentum flux (time-averaged momentum per unit time per unit area) was found to be given by $J = c_0 \langle \mathcal{P} \rangle = \langle \mathcal{E} \rangle$, where $\langle \mathcal{P} \rangle$ is the time-averaged momentum density discussed in the previous part. At a 2-fluid interface, with the first fluid having parameters ρ_1 and c_1 and the second fluid having parameters ρ_2 and c_2, the net momentum flux J into the interface was identified to be the time-averaged Lagrangian pressure $\langle p_{\mathrm{L}} \rangle$. Use these relations to find $\langle p_{\mathrm{L}} \rangle$ in terms of the fluid parameters, the incident time averaged energy density $\langle \mathcal{E}_1 \rangle$, and the pressure reflection and transmission coefficients R and T. *Hint: Start with* $\langle p_{\mathrm{L}} \rangle = c_0 \langle \mathcal{P}_i - \mathcal{P}_r - \mathcal{P}_t \rangle$.

As suggested in the hint:

$$\begin{aligned}\langle p_{\mathrm{L}} \rangle &= c_0 \langle \mathcal{P}_i - \mathcal{P}_r - \mathcal{P}_t \rangle \\ &= \langle \mathcal{E}_i + \mathcal{E}_r - \mathcal{E}_t \rangle \ .\end{aligned}$$

Now divide through by $\mathcal{E}_i$:

$$\langle p_{\mathrm{L}} \rangle = \langle \mathcal{E}_i \rangle \left(1 + \frac{\langle \mathcal{E}_r \rangle}{\langle \mathcal{E}_i \rangle} - \frac{\langle \mathcal{E}_t \rangle}{\langle \mathcal{E}_i \rangle} \right) .$$

Recalling $\langle \mathcal{E} \rangle = \langle p^2 \rangle / \rho_0 c_0^2$ and the definitions of the reflection and transmission coefficients gives the result:

$$\boxed{\langle p_{\mathrm{L}} \rangle = \langle \mathcal{E}_i \rangle \left(1 + R^2 - \frac{\rho_1 c_1^2}{\rho_2 c_2^2} T^2 \right)}$$

(n) Given that $F_{\mathrm{rad}} \propto \langle \mathcal{P} \rangle$, is it possible to have acoustic radiation force in the linear limit?

It is not possible to have acoustic radiation force in the linear limit because $\langle \mathcal{P} \rangle$ is a quadratic quantity. Since $F_{\mathrm{rad}} \propto \langle \mathcal{P} \rangle$, $F_{\mathrm{rad}} = 0$ in the linear limit.

8 Streaming

(a) It was shown that the "full momentum equation" i.e., equation (3-2) of [2], can be written as

$$\frac{\partial(\rho \boldsymbol{u})}{\partial t} - \boldsymbol{F}' + \boldsymbol{\nabla} P = \mu \nabla^2 \boldsymbol{u} + (\mu_B + \mu/3) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) \tag{8.1}$$

$$\text{where} \ - \boldsymbol{F}' = \rho(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{u}(\boldsymbol{\nabla} \cdot \rho \boldsymbol{u}) \tag{8.2}$$

Take the time-average of equations (8.1) and (8.2) and denote $\boldsymbol{F} \equiv \langle \boldsymbol{F}' \rangle$ to show that

$$\boldsymbol{F} = \boldsymbol{\nabla} \langle P \rangle - \mu \nabla^2 \langle \boldsymbol{u} \rangle \tag{8.3}$$
$$\boldsymbol{F} = -\langle \rho(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{u}\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) \rangle \tag{8.4}$$

What assumption has been made about the fluid in equation (8.3)? What is another name for this assumption? Why does the $\langle \partial(\rho \boldsymbol{u})/\partial t \rangle = 0$?

(b) Drop the $\langle\ \rangle$ notation denoting "time average" and let the time averaging be implied on all wave quantities. Letting $P = P_0 + p_1 + p_2$, $\rho = \rho_0 + \rho_1 + \rho_2$, and $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$, the subscripts refer to the order of the term, show that the $\mathcal{O}(\epsilon^2)$ version of equations (8.3) and (8.4) are

$$\boldsymbol{F}_2 = \boldsymbol{\nabla} p_2 - \mu \nabla^2 \langle \boldsymbol{u}_2 \rangle \tag{8.5}$$
$$\boldsymbol{F}_2 = -\langle \rho_0(\boldsymbol{u}_1 \cdot \boldsymbol{\nabla})\boldsymbol{u}_1 + \boldsymbol{u}_1 \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}_1) \rangle \tag{8.6}$$

(c) Let $p_1 = p_0 e^{-\alpha x} \sin(\omega t - kx)$. Calculate $\boldsymbol{F}_2$ using equation (8.6). *Hint: use the linear relation* $p_1 \simeq \rho_0 c_0 u_1$.

(d) Take the limit of the above result as take the limit as $\alpha \ll k$. *Answer:* $F_{2l} = \alpha p_0^2/\rho_0 c_0^2$. What does this result say about the nature of acoustic streaming?

(e) In class, it was shown that in the presence of shocks,

$$F_{2f} = \frac{2\beta k P_{\mathrm{sh}}}{3\pi \rho_0^2 c_0^4},$$

the maximum value of which is $2\beta k p_0^3/3\pi \rho_0^2 c_0^4$. Show that the ratio of F_{2f} to F_{2l} is $2\Gamma/3\pi$, where $\Gamma = \beta \epsilon k/\alpha$ (the Gol'berg number).

References

[1] M. F. Hamilton, Lecture notes from Nonlinear Acoustics. University of Texas at Austin, (2023).

[2] M. F. Hamilton and D. T. Blackstock, "Nonlinear Acoustics." Acoustical Society of America, (2008).