Notating $R \equiv |\boldsymbol{r} - \boldsymbol{r}_0|$, show that[1]

$$g_\omega = \frac{1}{4\pi R} e^{ikR}$$

solves

$$\nabla^2 g_\omega(\boldsymbol{r}|\boldsymbol{r}_0) + k^2 g_\omega(\boldsymbol{r}|\boldsymbol{r}_0) = -\delta(\boldsymbol{r} - \boldsymbol{r}_0) \tag{1}$$

First, integrate both sides of equation (1) over a sphere of small radius a:

$$\iiint \nabla^2 g_\omega(\boldsymbol{r}|\boldsymbol{r}_0)\,\mathrm{d}V + \iiint k^2 g_\omega(\boldsymbol{r}|\boldsymbol{r}_0)\,\mathrm{d}V = -\iiint \delta(\boldsymbol{r} - \boldsymbol{r}_0)\,\mathrm{d}V \tag{2}$$

Note that the right-hand-side of equation (2) is just -1, by definition of the delta function. Meanwhile, the first integral on the left-hand-side of equation (2) can be written as

$$\begin{aligned}
\iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} g_\omega\,\mathrm{d}V &= \frac{1}{4\pi} \iiint \boldsymbol{\nabla} \cdot \hat{R} \left(-\frac{1}{R^2} + \frac{ik}{R} \right) e^{ikR} dV \\
&= \oiint \left(-\frac{1}{R^2} + \frac{ik}{R} \right) e^{ikR} dS \\
&= \frac{4\pi a^2}{4\pi} \left(-\frac{1}{a^2} + \frac{ik}{a} \right) e^{ika} \\
&= (-1 + ika) e^{ika} \\
&= (-1 + ika) \left(1 + ika - \frac{(ka)^2}{2!} - i\frac{(ka)^3}{3!} + \frac{(ka)^4}{4!} + \dots \right) \\
&\to -1 \text{ as } a \to 0
\end{aligned}$$

The second integral on the left-hand-side of equation (2) is evaluated in spherical coordinates[2], and the resulting radial integral is evaluated by parts:

$$\begin{aligned}
\frac{k^2}{4\pi} \iiint_V \frac{e^{ikR}}{R}\,\mathrm{d}V &= \frac{k^2}{4\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^{a} Re^{ikR} \sin\theta \,\mathrm{d}R\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= k^2 \int_0^a Re^{ikR} dR \\
&= -ikae^{ika} + e^{ika} - 1 \\
&= -ikae^{ika} + 1 + ika - \frac{(ka)^2}{2!} - i\frac{(ka)^3}{3!} + \frac{(ka)^4}{4!} \cdots - 1 \\
&\to 0 \text{ as } a \to 0
\end{aligned}$$

The left-hand-side and right-hand-side of equation (2) both equal -1, showing that g_ω solves equation (1).

[1] See Morse & Ingard, section 7.1

[2] Recall that the Jacobian in spherical coordinates is $R^2 \sin\theta$.