

In an infinite medium, the total field  $\psi(\mathbf{r})$  (velocity potential, or pressure if you want) is given by the volume integral

$$\psi(\mathbf{r}) = \int_{V_0} f(\mathbf{r}_0) g_\omega(\mathbf{r}|\mathbf{r}_0) dv_0 \quad (1)$$

where  $f(\mathbf{r}_0)$  is the source strength, and where  $g_\omega(\mathbf{r}|\mathbf{r}_0)$  is the free-field Green's function.

Noting that the total field is also given by the Helmholtz-Kirchoff integral,

$$\psi(\mathbf{r}) = \iiint f_\omega(\mathbf{r}_0) g_\omega(\mathbf{r}_0|\mathbf{r}) dv_0 + \iint \left\{ g_\omega(\mathbf{r}_0|\mathbf{r}) \frac{\partial \psi(\mathbf{r}_0)}{\partial n_0} - \psi(\mathbf{r}_0) \frac{\partial g_\omega(\mathbf{r}_0|\mathbf{r})}{\partial n_0} \right\} dS_0, \quad (2)$$

Comparing equations (1) and (2) shows that the surface integral in (2) must vanish:

$$\begin{aligned} 0 &= \iint \left\{ g_\omega(\mathbf{r}_0|\mathbf{r}) \frac{\partial \psi(\mathbf{r}_0)}{\partial n_0} - \psi(\mathbf{r}_0) \frac{\partial g_\omega(\mathbf{r}_0|\mathbf{r})}{\partial n_0} \right\} dS_0 \\ &= \iint \left\{ \frac{e^{ikR}}{4\pi R} \frac{\partial \psi(\mathbf{r}_0)}{\partial n_0} - \psi(\mathbf{r}_0) \frac{\partial}{\partial n_0} \frac{e^{ikR}}{4\pi R} \right\} dS_0 \\ &= \iint \frac{e^{ikR}}{4\pi R} \left\{ \frac{\partial \psi(\mathbf{r}_0)}{\partial n_0} - ik\psi(\mathbf{r}_0) \right\} dS_0 \end{aligned}$$

The area of a sphere goes as  $R^2$ , so for the right-hand-side to go to 0, the integrand must go to 0 faster than  $R^{-2}$  as  $R \rightarrow \infty$ . Since  $\frac{e^{ikR}}{4\pi}$  is oscillatory, it does not affect the convergence to 0. The condition for convergence is therefore<sup>1</sup>

$$\begin{aligned} \lim_{R \rightarrow \infty} \left( \frac{1}{R} \right) \left( \frac{\partial \psi(\mathbf{r}_0)}{\partial n_0} - ik\psi(\mathbf{r}_0) \right) \div R^{-2} &= 0 \\ \lim_{R \rightarrow \infty} R \left( \frac{\partial \psi(\mathbf{r}_0)}{\partial n_0} - ik\psi(\mathbf{r}_0) \right) &= 0 \\ \lim_{R \rightarrow \infty} R \left( \frac{\partial}{\partial n_0} - ik \right) \psi(\mathbf{r}_0) &= 0 \end{aligned}$$

This is the so-called Sommerfeld radiation condition, which must be satisfied by any field in an infinite medium.

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<sup>1</sup>For two functions  $A(x)$  and  $B(x)$  that approach 0 as  $x \rightarrow \infty$ ,  $A(x) \rightarrow 0$  faster than  $B(x)$  if  $\lim_{x \rightarrow \infty} \frac{A(x)}{B(x)} \rightarrow 0$ .