Let $f(t) = F e^{j(\omega t + \phi_F)} = \tilde{F} e^{j\omega t}$ (i.e., $\tilde{F} = F e^{j\phi_F}$)

and $g(t) = G e^{j(\omega t + \phi_G)} = \tilde{G} e^{j\omega t}$ (i.e., $\tilde{G} = G e^{j\phi_G}$).

Show that $\langle \mathrm{Re}\, f \; \mathrm{Re}\, g \rangle = \frac{1}{2} \mathrm{Re}(\tilde{F} \tilde{G}^*) = \frac{1}{2} \mathrm{Re}(\tilde{F}^* \tilde{G})$.

Note that $\mathrm{Re}\, f = F \cos(\omega t + \phi_F)$

$\mathrm{Re}\, g = G \cos(\omega t + \phi_G)$.

Then

$$\langle \mathrm{Re}\, f \; \mathrm{Re}\, g \rangle = FG \langle \cos(\omega t + \phi_F) \cos(\omega t + \phi_G) \rangle. \qquad (1)$$

Ⓐ

$$= FG \langle \cos(2\omega t + \phi_F + \phi_G) + \sin(\omega t + \phi_F) \sin(\omega t + \phi_G) \rangle.$$

Ⓑ

$$= FG \langle \cos(2\omega t + \phi_F + \phi_G) - \tfrac{1}{2} \cos(2\omega t + \phi_F + \phi_G) + \cos(\phi_F - \phi_G)$$

$$= \frac{1}{2} FG \langle \cos(2\omega t + \phi_F + \phi_G) + \cos(\phi_F - \phi_G) \rangle \qquad (2)$$

Because $\langle \cos(2\omega t \cdots) \rangle = 0$.

$$= \frac{1}{2} FG \cos(\phi_F - \phi_G)$$

$$= \frac{1}{2} \mathrm{Re}\left[FG\, e^{j(\phi_F - \phi_G)} \right]$$

$$= \frac{1}{2} \mathrm{Re}(\tilde{F} \tilde{G}^*) = \frac{1}{2} \mathrm{Re}(\tilde{F}^* \tilde{G}).$$

Step Ⓐ : Use $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

i.e., $\cos A \cos B = \cos(A+B) + \underline{\sin A \sin B}$,

where $A = \omega t + \phi_F$ and $B = \omega t + \phi_G$.

Step Ⓑ : Use $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

for the underlined term above.

This comes from

$$\begin{aligned} \cos A+B &= \cos A \cos B - \sin A \sin B \\ - \quad \cos A-B &= \cos A \cos B + \sin A \sin B \\ \hline -\cos(A-B) + \cos(A+B) &= -2 \sin A \sin B. \end{aligned}$$

or $\frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] = \sin A \sin B$