Review for the nonlinear acoustics midterm

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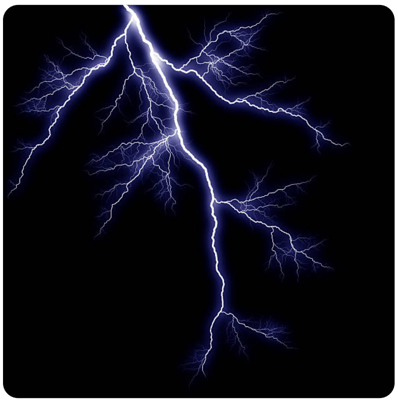

These conceptual and analytical problems, which are based on Dr. Hamilton's lectures, address the major topics of the course thus far. Best wishes on the exam!

1 Gauge functions & linear lossy theory

(a) Use the linearized 1D momentum equation $\rho_0 \partial u/\partial t + \partial p/\partial x = 0$ to show that

$$\frac{u}{c_0} = \frac{p}{\rho_0 c_0^2} = \frac{\rho'}{\rho_0}. \tag{1.1}$$

Hint: Start by integrating the momentum equation,

$$u = -\frac{1}{\rho_0}\int \frac{\partial p}{\partial x} dt,$$

and then let $p = f(t - x/c_0)$. *Note that* $\partial f/\partial x = -\frac{1}{c_0}\partial f/\partial t$.

Letting $p = f(t - x/c_0)$, the integral form of the momentum equation is

$$u = -\frac{1}{\rho_0}\int \frac{\partial f}{\partial x} dt.$$

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Noting that $\partial f/\partial x = -\frac{1}{c_0}\partial f/\partial t$ gives

$$u = \frac{1}{\rho_0 c_0}\int \frac{\partial f}{\partial t}dt = \frac{f}{\rho_0 c_0},$$

which upon replacing $f = p$, noting that $p = \rho' c_0^2$, and dividing through by c_0, gives the desired result,

$$\boxed{\frac{u}{c_0} = \frac{p}{\rho_0 c_0^2} = \frac{\rho'}{\rho_0}.}$$

(b) What is the definition of the acoustic mach number ϵ?

Letting p_0 (or u_0) characterize the sound pressure (or particle velocity), the acoustic mach number is, by equation (1.1),

$$\boxed{\epsilon = \frac{u_0}{c_0} = \frac{p_0}{\rho_0 c_0^2} = \frac{\rho'}{\rho_0}.}$$

(c) What is the order of

- $a\mathcal{O}(\epsilon^n)$, where a is a constant?
- $\mathcal{O}(\epsilon^n) + \mathcal{O}(\epsilon^m)$, for $n < m$?
- $\mathcal{O}(\epsilon^n)\mathcal{O}(\epsilon^m)$?
- $[\mathcal{O}(\epsilon^n)]^m$?
- $\nabla^2 p = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}$?
- $\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}$?

$$
\begin{aligned}
a\mathcal{O}(\epsilon^n) &= \mathcal{O}(\epsilon^n) \\
\mathcal{O}(\epsilon^n) + \mathcal{O}(\epsilon^m) &= \mathcal{O}(\epsilon^n) \\
\mathcal{O}(\epsilon^n)\mathcal{O}(\epsilon^m) &= \mathcal{O}(\epsilon^{n+m}) \\
[\mathcal{O}(\epsilon^n)]^m &= \mathcal{O}(\epsilon^{nm}) \\
\nabla^2 p = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} &= \mathcal{O}(\epsilon) \\
\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} &= \mathcal{O}(\epsilon^2)
\end{aligned}
$$

(d) What does the "acoustic Stokes number" $\eta = \mu\omega/\rho_0 c_0^2$ characterize? It which governing equation does it arise and aid the ordering of terms? What is the order of μ, the shear viscosity?

The acoustic Stokes number characterizes the importance of ``viscous stresses in a plane progressive sound wave, relative to the fluctuating pressure" [2]. It arises in the momentum equation and is a small but important correction to lossless theory. Since ω, ρ_0, and c_0 are $\mathcal{O}(1)$, μ is $\mathcal{O}(\eta)$.

(e) How does a term of $\mathcal{O}(\epsilon^2)$ compare to a term of $\mathcal{O}(\eta\epsilon)$?

$\mathcal{O}(\epsilon^2)$ terms are given the same importance as $\mathcal{O}(\eta\epsilon)$ terms. This follows Lighthill (1956) [2].

(f) What is the order of the Prandtl number $\mathrm{Pr} = \mu C_P/\kappa$, and what does it characterize? It which governing equation does it manifest? What is the order of κ which appears in the definition of the Prandtl number?

The Prandtl number is $\mathcal{O}(1)$ is related to heat conduction and arises in the entropy equation. Since $C_P = \mathcal{O}(1)$ and $\mu = \mathcal{O}(\eta)$, $\kappa = \mathcal{O}(\eta)$ also.

(g) Show that the temperature perturbation T' is $\mathcal{O}(\epsilon)$ by expanding the ideal gas law, $P = R\rho T$, where $P = P_0 + p$, $\rho = \rho_0 + \rho'$, and $T = T_0 + T'$.

Expanding the equation of state as suggested gives

$$\begin{aligned} P &= R\rho T \\ P_0 + p &= R(\rho_0 + \rho')(T_0 + T') \\ &= R(\rho_0 T_0 + \rho_0 T' + \rho' T_0 + \rho' T')\,. \end{aligned}$$

Equating the terms of $\mathcal{O}(\epsilon)$ gives $p = R(\rho_0 T' + \rho' T_0)$. Since R and ρ_0 are $\mathcal{O}(1)$, T' in the first term is $\mathcal{O}(\epsilon)$, because the product $R\rho_0 T'$ must be $\mathcal{O}(\epsilon)$.

(h) Recall the entropy equation,

$$\rho_0(T_0 + T')\frac{\partial s'}{\partial t} = \kappa \nabla^2 T'.$$

What is the order of the entropy perturbation s'?

Since $\kappa = \mathcal{O}(\eta)$ and $T' = \mathcal{O}(\epsilon)$, the right-hand side of the entropy equation is $\mathcal{O}(\eta\epsilon)$. Meanwhile, the lowest-order term on the left-hand side is $\rho_0 T_0 \dot{s}'$. Since $\rho_0 T_0 = \mathcal{O}(1)$, the entropy perturbation s' is $\mathcal{O}(\eta\epsilon)$.

(i) Derive the attenuation coefficient from the lossy linear wave equation below to $\mathcal{O}(\eta)$. What is the next highest-order term in the attenuation coefficient?

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\delta}{c_0^4}\frac{\partial^3 p}{\partial t^3} \tag{1.2}$$

Substituting $p \propto \exp[j(\omega t - \tilde{k}x)]$ gives

$$\begin{aligned} \tilde{k}^2 &= \omega^2\left(\frac{1}{c_0^2} - \frac{j\omega\delta}{c_0^4}\right) \\ \implies \tilde{k} &= \frac{\omega}{c_0}(1 - j\omega\delta/c_0^2)^{1/2} \\ &= \frac{\omega}{c_0}(1 - j\omega\delta/2c_0^2) + \mathcal{O}(\eta^2) \\ &= k - j\omega^2\delta/2c_0^3 + \mathcal{O}(\eta^2) \end{aligned}$$

The imaginary part is read off as the attenuation coefficient

$$\boxed{\alpha = \frac{\delta\omega^2}{2c_0^3} + \mathcal{O}(\eta^3)\,.}$$

The higher order term in the attenuation coefficient is determined to be $\mathcal{O}(\eta^3)$ because the $\mathcal{O}(\epsilon^2)$ term is real and contributes to k.

(j) Derive the progressive wave equation of $\mathcal{O}(\eta\epsilon)$ from the linear lossy wave equation

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\delta}{c_0^4}\frac{\partial^3 p}{\partial t^3} \tag{1.3}$$

Hint: start by transforming coordinates

$$(\eta x, t - x/c_0) \mapsto (x_1, \tau)$$

$\partial/\partial x$ is first transformed:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial x} = \eta\frac{\partial}{\partial x_1} - \frac{1}{c_0}\frac{\partial}{\partial \tau} \tag{i}$$

Next, $\partial/\partial t$ is transformed:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial \tau}\,. \tag{ii}$$

Applying equation (i) to $\partial/\partial x$ gives

$$\frac{\partial^2}{\partial x^2} = \eta^2\frac{\partial^2}{\partial x_1^2} - \frac{2\eta}{c_0}\frac{\partial^2}{\partial \tau \partial x_1} + \frac{1}{c_0^2}\frac{\partial}{\partial \tau^2}\,, \tag{iii}$$

and applying equation (ii) to $\partial/\partial t$ gives

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \tau^2}. \tag{iv}$$

Substituting equations (i)–(iv) into equation (1.3) gives

$$\eta^2\frac{\partial^2 p}{\partial x_1^2} - \frac{2\eta}{c_0}\frac{\partial^2 p}{\partial \tau \partial x_1} = -\frac{\delta}{c_0^4}\frac{\partial^3 p}{\partial \tau^3}$$

The first term on the left-hand side above is thrown out because it is $\mathcal{O}(\eta^2\epsilon)$. The substitution $x = \eta x_1$ is also made. Integrating over τ and rearranging gives

$$\boxed{\frac{\partial p}{\partial x} = \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial \tau^2}\,.}$$

(k) On a plot of pressure vs. retarded time τ, what is the meaning of "to the left of $\tau = 0$" and "to the right of $\tau = 0$"? What is strange about the solution to the linear lossy progressive wave equation derived in the previous question?

``To the left of $\tau = 0$" means ``faster than the (linear) speed of sound c_0," because if $t - x/c_0 < 0$, then $x/t > c_0$. Similarly, ``to the right of $\tau = 0$" means ``slower than the (linear) speed of sound c_0," because if $t - x/c_0 > 0$, then $x/t < c_0$. What is strange about the solution to the linear lossy progressive wave equation is that for an impulsive source condition $p(0, \tau) = \delta(\tau)$, the solution is a Gaussian in τ, which is an infinite waveform. That is, this solution ``tickles the ends of the universe" [1], even after propagating only a finite distance x.

(l) *Coordinate transformation practice:* Write the partial derivatives with respect to the spherical coordinates[1] $\partial/\partial r$, $\partial/\partial\theta$, and $\partial/\partial\psi$ as partial derivatives with respect to Cartesian coordinates x, y, z.

Noting that

$$\frac{\partial x}{\partial r} = \cos\psi\sin\theta, \quad \frac{\partial y}{\partial r} = \sin\psi\sin\theta, \quad \frac{\partial z}{\partial r} = \cos\theta,$$
$$\frac{\partial x}{\partial \psi} = -r\sin\psi\sin\theta, \quad \frac{\partial y}{\partial \psi} = r\cos\psi\sin\theta, \quad \frac{\partial z}{\partial \psi} = 0,$$
$$\frac{\partial x}{\partial \theta} = r\cos\psi\cos\theta, \quad \frac{\partial y}{\partial \theta} = r\sin\psi\cos\theta, \quad \frac{\partial z}{\partial \theta} = -r\sin\theta,$$

the partial derivatives with respect to the spherical coordinates become

$$\frac{\partial}{\partial r} = \cos\psi\sin\theta\frac{\partial}{\partial x} + \sin\psi\sin\theta\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z}$$
$$\frac{\partial}{\partial \psi} = -r\sin\psi\sin\theta\frac{\partial}{\partial x} + r\cos\psi\sin\theta\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial \theta} = r\cos\psi\cos\theta\frac{\partial}{\partial x} + r\sin\psi\cos\theta\frac{\partial}{\partial y} - r\sin\theta\frac{\partial}{\partial z}$$

[1] θ is polar and ψ is azimuthal, i.e., $x = r\cos\psi\sin\theta$, $y = r\sin\psi\sin\theta$, and $z = r\cos\theta$.

(m) *Fun (but fictitious) coordinate transformation problem:* The spherically symmetric scalar wave equation for light is[2]

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0\,, \tag{1.4}$$

where c is the speed of light. Meanwhile, Einstein's general theory of relativity can be solved exactly to show that at a radial coordinate r from a non-rotating mass, time dilates as

$$t' = t\sqrt{1 - \frac{2GM}{rc^2}}, \tag{1.5}$$

where t is the time far away from the mass, G is the gravitational constant, and M is the mass of the object [3]. Denote $r_s = 2GM/c^2$ for convenience.[3] Write equation (1.4) in the coordinates $(r, t') = (r, t\sqrt{1 - r_s/r})$. *Hint: follow a procedure similar to problem (j). It helped me to introduce an auxiliary coordinate $r' = r$ to keep the "old" and "new" coordinates straight while performing the coordinate transformation.*

The partial derivatives become

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial r'}\frac{\partial r'}{\partial r} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial r} = \frac{\partial}{\partial r'} + \frac{r_s t}{2r^2}(1 - r_s/r)^{-1/2}\frac{\partial}{\partial t'} \tag{v}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial}{\partial r'}\frac{\partial r'}{\partial t} = (1 - r_s/r)^{1/2}\frac{\partial}{\partial t'} \tag{vi}$$

$$\frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial r'^2} + \frac{r_s t}{r^2}(1 - r_s/r)^{-1/2}\frac{\partial^2}{\partial t'\partial r'} + \frac{r_s^2 t^2}{4r^4}(1 - r_s/r)^{-1}\frac{\partial^2}{\partial t'^2} \tag{vii}$$

$$\frac{\partial^2}{\partial t^2} = (1 - r_s/r)\frac{\partial^2}{\partial t'^2} \tag{viii}$$

Substituting equations (v), (vii), and (viii) into equation (1.4) gives

$$\frac{\partial^2 p}{\partial r'^2} + \frac{r_s t}{r^2}(1 - r_s/r)^{-1/2}\frac{\partial^2 p}{\partial t'\partial r'} + \frac{r_s^2 t^2}{4r^4}(1 - r_s/r)^{-1}\frac{\partial^2 p}{\partial t'^2} + \frac{2}{r}\left[\frac{\partial p}{\partial r'} + \frac{r_s t}{2r^2}(1 - r_s/r)^{-1/2}\frac{\partial p}{\partial t'}\right] - \frac{1}{c^2}(1 - r_s/r)\frac{\partial^2 p}{\partial t'^2} = 0\,.$$

Substituting $r' = r$ and $t = t'(1 - r_s/r)^{-1/2}$ results in the spherically symmetric wave equation in the vicinity of non-rotating mass in (r, t') coordinates:

[2] p is used as the wave variable for familiarity.

[3] r_s is the Schwarzschild radius.

$$\frac{\partial^2 p}{\partial r^2} + \frac{r_s t'}{r^2}(1 - r_s/r)^{-1}\frac{\partial^2 p}{\partial t'\partial r} + \frac{r_s^2 t'^2}{4r^4}(1 - r_s/r)^{-2}\frac{\partial^2 p}{\partial t'^2}$$
$$+ \frac{2}{r}\left[\frac{\partial p}{\partial r} + \frac{r_s t'}{2r^2}(1 - r_s/r)^{-1}\frac{\partial p}{\partial t'}\right] - \frac{1}{c^2}(1 - r_s/r)\frac{\partial^2 p}{\partial t'^2} = 0\,.$$

2 Lossless nonlinear theory

(a) Show that the so-called Poisson solution

$$u = g[x - (c + u)t]$$

satisfies the exact lossless nonlinear equation for progressive waves,

$$\frac{\partial u}{\partial t} + (c + u)\frac{\partial u}{\partial x} = 0. \tag{2.1}$$

Hint: for notational ease, denote $y = x - v(u)t$, where $v(u) = c + u$. Note that $\partial y/\partial x = 1 - v'(u)\,t\,\partial u/\partial x$ and $\partial y/\partial t = -v - v'(u)\,t\,\partial u/\partial t$.

The progressive wave equation, where $c + u$ has been denoted v, is

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$$

Using the suggested substitutions, the derivatives in the exact lossless nonlinear progressive wave equation are evaluated:

$$\frac{\partial u}{\partial t} = g'\frac{\partial y}{\partial t} = -g' \cdot \left(v - v'\,t\,\frac{\partial u}{\partial t}\right)$$
$$\implies \frac{\partial u}{\partial t} = -\frac{g'v}{1 + g'v't}$$
$$\frac{\partial u}{\partial x} = g'\frac{\partial y}{\partial x} = g' \cdot \left(1 - v'\,t\,\frac{\partial u}{\partial x}\right)$$
$$\implies \frac{\partial u}{\partial x} = \frac{g'}{1 + g'v't}$$

Substituting $\partial u/\partial t$ and $\partial u/\partial x$ into the progressive wave equation gives 0 on both sides of the equation,

$$-\frac{g'v}{1+g'v't}+v\frac{g'}{1+g'v't}=0\,,$$

i.e., the Poisson solution indeed satisfies the lossless nonlinear progressive wave equation.

(b) Recalling the adiabatic sound speed $c^2=\gamma P/\rho$ and the adiabatic gas law, show that

$$\left(\frac{\rho}{\rho_0}\right)=\left(\frac{c}{c_0}\right)^{2/(\gamma-1)}. \tag{2.2}$$

Show that the differential $d\rho$ is

$$d\rho=\frac{2}{\gamma-1}\rho_0(c/c_0)^{2/(\gamma-1)}c^{-1}\,dc\,. \tag{2.3}$$

Substituting $P=c^2\rho/\gamma$ and $P_0=c_0^2\rho_0/\gamma$ into the adiabatic gas law $P/P_0=(\rho/\rho_0)^\gamma$ gives

$$\frac{P}{P_0}=(c/c_0)^2\rho/\rho_0=(\rho/\rho_0)^\gamma.$$

Dividing by ρ/ρ_0 gives and taking the square root and

$$c/c_0=(\rho/\rho_0)^{(\gamma-1)/2}.$$

Inverting for ρ/ρ_0 gives

$$\left(\frac{\rho}{\rho_0}\right)=\left(\frac{c}{c_0}\right)^{2/(\gamma-1)}.$$

Writing $\rho=\rho_0(c/c_0)^{2/(\gamma-1)}$, the differential is calculated to be

$$d\rho=\frac{d\rho}{dc}dc=\frac{2}{\gamma-1}\rho_0(c/c_0)^{2/(\gamma-1)}c^{-1}\,dc\,.$$

(c) Recall that the quantity λ equals u for a progressive wave, because this condition sets the Riemann invariant J_- (which corresponds to backward-traveling waves) to 0. Noting that $d\lambda = (c/\rho)\,d\rho$, use equation (2.3) to show that

$$c + u = c_0 + \beta u \quad \text{where} \quad \beta = \frac{1}{2}(\gamma + 1) \tag{2.4}$$

Using equation (2.3) and simplifying gives

$$\begin{aligned} d\lambda &= \frac{c}{\rho} d\rho = \frac{c}{\rho}\frac{2}{\gamma - 1}\rho_0 (c/c_0)^{2/(\gamma-1)} c^{-1}\, dc \\ &= \frac{1}{\rho_0 (c/c_0)^{2/(\gamma-1)}} \frac{2}{\gamma - 1}\rho_0 (c/c_0)^{2/(\gamma-1)}\, dc = \frac{2}{\gamma - 1}\, dc\,. \end{aligned}$$

Integrating the above result gives

$$\lambda = \int d\lambda = \frac{2}{\gamma - 1} c + \text{ constant}$$

Noting that $\lambda = 0$ for $c = c_0$ sets the constant equal to $-2c_0/(\gamma - 1)$. Finally, using the fact that $\lambda = u$ for a progressive wave gives $c = c_0 + (\gamma - 1)/2u$. Adding u to both sides and identifying $\beta = \frac{1}{2}(\gamma + 1)$ gives

$$\boxed{c + u = c_0 + \beta u\,.}$$

(d) Derive the Earnshaw solution for a boundary value problem, in which the time variation $u(0, t)$ at the face of the piston is known.

Let X denote the position of the piston. Suppose the disturbance $X(\phi)$ is the position of the piston face at the time the acoustic disturbance leaves the piston face. The speed of the piston is therefore $u = \dot{X}(\phi)$. Meanwhile, at $t > \phi$, the disturbance will have traveled to position $x > X(\phi)$. Therefore,

$$\text{the speed of disturbance } = \frac{x - X(\phi)}{t - \phi} \tag{ix}$$

According to equation (2.4), the disturbance propagates at $c_0 + \beta\dot{X}(\phi)$. Therefore, equation (ix) becomes

$$c_0 + \beta\dot{X}(\phi) = \frac{x - X(\phi)}{t - \phi} \, .$$

Solving for ϕ gives the implicit solution:

$$\boxed{u = \dot{X}(\phi), \quad \phi = t - \frac{x - X(\phi)}{c_0 + \beta\dot{X}(\phi)} \, .}$$

(e) What is the Earnshaw solution for a sinusoidal boundary condition, $u(0,t) = u_0 \sin \omega t$?[4]

Given the boundary condition of the velocity of the piston, the boundary condition of the position of the piston can be found by integration:

$$\begin{aligned} X(t) &= -(u_0/\omega)\cos\omega t + \text{ constant} \\ &= (u_0/\omega)(1 - \cos\omega t) \, , \end{aligned}$$

where the constant has been determined by setting the displacement of the piston to 0 at $t = 0$, i.e., $X(0) = 0$. The Earnshaw solution is therefore

$$u = u_0 \sin\omega\phi, \quad \phi = t - \frac{x - (u_0/\omega)(1 - \cos\omega\phi)}{c_0 + \beta u_0 \sin\omega t}$$

(f) Derive the Earnshaw solution for an initial value problem (homework problem 2-7), in which the spatial variation at the face of the boundary is known, e.g., $u(x,0)$.

See homework problem 2-7, part (a), for the solution.

(g) What is the Earnshaw solution for a linear boundary condition, $u(x,0) = m_0 x$?

See homework problem 2-7, part (b), for the solution.

[4]This was not done in the lecture; see page 70 of [2].

3 Lossless nonlinear approximate theory

(a) Reduce the exact progressive nonlinear lossless wave equation is given by equation (2.1), repeated below for convenience,

$$\frac{\partial u}{\partial x} + \frac{1}{c+u}\frac{\partial u}{\partial t} = 0 \tag{2.1}$$

to $\mathcal{O}(\epsilon^2)$ in $(x, \tau) = (x, t - x/c_0)$ coordinates. *Hint: Expand* $v(u) = c + u = v(0) + v'(0)u + \mathcal{O}(\epsilon^2) = c_0 + \beta u + \mathcal{O}(\epsilon^2)$ *and then perform a binomial expansion of* $(c_0 + \beta_0 u)^{-1}$. *Answer:*

$$\frac{\partial u}{\partial x} = \frac{\beta}{c_0^2} u \frac{\partial u}{\partial \tau} \tag{3.1}$$

The first order expansion of $c + u$ is $c_0 + \beta u$, as hinted. The denominator of the second term becomes $1/c_0 - \beta u/c_0^2 + \mathcal{O}(\epsilon^2)$. Substituting this into equation (2.1) gives

$$\frac{\partial u}{\partial x} + \frac{1}{c_0}\frac{\partial u}{\partial t} = \frac{\beta u}{c_0^2}\frac{\partial u}{\partial t} \tag{x}$$

Next, noting that

$$\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial x} - \frac{1}{c_0}\frac{\partial}{\partial \tau} \quad \text{and} \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau},$$

equation (x) becomes

$$\boxed{\frac{\partial u}{\partial x} = \frac{\beta u}{c_0^2}\frac{\partial u}{\partial \tau}}$$

(b) Verify that $u = f(\tau + \beta x u/c_0^2)$ solves equation (3.1).

See homework problem 2-6 for the solution. I had a typo in my solution, so I have re-worked it here. Note that

$$\frac{\partial u}{\partial x} = f'\left(\frac{\beta u}{c_0^2} + \frac{\beta}{c_0^2}\frac{\partial u}{\partial x}\right) \quad \Longrightarrow \quad \frac{\partial u}{\partial x} = \frac{f'\beta u}{c_0^2}\frac{1}{1 - \beta x f'/c_0^2}$$

$$\frac{\partial u}{\partial \tau} = f'\left(1 + \frac{\beta x}{c_0^2}\frac{\partial u}{\partial \tau}\right) \quad \Longrightarrow \quad \frac{\partial u}{\partial \tau} = f'\frac{1}{1 - \beta x f'/c_0^2}.$$

Multiplying $\partial u/\partial \tau$ by $\beta u/c_0^2$ as it appears on the right-hand side of equation (3.1) results in

$$\frac{\beta u}{c_0^2}\frac{\partial u}{\partial \tau} = \frac{f'\beta u}{c_0^2}\frac{1}{1-\beta x f'/c_0^2}$$

which equals the left-hand side, $\partial u/\partial x$. Therefore, $u = f(\tau + \beta x u/c_0^2)$ satisfies equation (3.1).

(c) Use $u = f(\tau + \beta x u/c_0^2)$ to calculate $\bar{x}$, the shock-forming distance. *Hint: Find* $x_{\text{vt}} = x$ *such that* $\partial u/\partial \tau = \infty$ *(the condition for a vertical tangent line). The smallest value of* x_{vt} *is* $\bar{x}$*.*

The condition for a shock

$$\frac{\partial u}{\partial \tau} = \frac{f'}{1-\beta x_{\text{vt}} f'/c_0^2} = \infty \tag{3.2}$$

reveals that $\beta x_{\text{vt}} f'/c_0^2 = 1$, or

$$x_{\text{vt}} = \frac{c_0^2}{\beta f'}\,.$$

The smallest value of x_{vt} is $\bar{x}$, which occurs for the largest value of f', f'_{max}. Therefore, the shock forming distance is

$$\boxed{\bar{x} = \frac{c_0^2}{\beta f'_{\text{max}}}\,.}$$

(d) What is the shock-forming distance $\bar{x}$ for a sinusoidal source $f(t) = u_0 \sin \omega t$? Express the distance in terms of β, ϵ, and k.

Noting that $f'_{\text{max}} = \omega u_0$, the result from the previous question $\bar{x} = c_0^2/(\beta f'_{\text{max}})$ is used, giving

$$\boxed{\bar{x} = \frac{1}{\beta \epsilon k}}$$

where it has been noted that $c_0/\omega = k$ and $u_0/c_0 = \epsilon$.

(e) Perform a perturbation solution for

$$\frac{\partial u}{\partial x} = \frac{\beta}{2c_0^2}\frac{\partial u^2}{\partial \tau}$$

for $u = u_0 \sin \omega\tau$. For what range of $\sigma = x/\bar{x}$ is this solution valid? *Hint: Substitute the power series in* ϵ

$$\frac{u}{c_0} = \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \dots$$

where $\epsilon \ll 1$ *into the evolution equation, and then match orders on both sides of the equation.*

Note that

$$\frac{u^2}{c_0^2} = \epsilon^2 v_1^2 + 2\epsilon^3 v_1 v_2 + \mathcal{O}(\epsilon^4).$$

Upon making the suggested substitution, the approximate non-linear progressive wave equation becomes

$$\frac{\partial}{\partial x}(\epsilon v_1 + \epsilon^2 v_2) = \frac{\beta}{2c_0}\frac{\partial}{\partial \tau}(\epsilon^2 v_1^2 + 2\epsilon^3 v_1 v_2)\,.$$

Now the orders on either side of the equation above are matched. There is no $\mathcal{O}(\epsilon)$ term on the right-hand side, so

$$\frac{\partial v_1}{\partial x} = 0 \qquad \Longrightarrow \qquad v_1 = \sin \omega\tau\,.$$

Matching the $\mathcal{O}(\epsilon^2)$ terms on the left- and right-hand sides and noting $k = \omega/c_0$ gives

$$\frac{\partial v_2}{\partial x} = \frac{\beta}{2c_0}\frac{\partial}{\partial \tau}\sin^2 \omega\tau = \frac{\beta}{4c_0}\frac{\partial}{\partial \tau}(1 - \cos 2\omega\tau) = \frac{\beta k}{2}\sin 2\omega\tau$$

Integrating over x gives v_2:

$$v_2 = \frac{\beta k x}{2}\sin 2\omega\tau$$

Therefore, to $\mathcal{O}(\epsilon^2)$,

$$\frac{u}{c_0} = \epsilon \sin \omega\tau + \epsilon^2 \frac{\beta k x}{2}\sin 2\omega\tau.$$

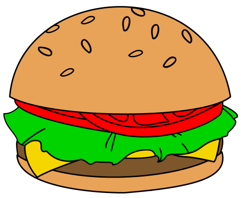
Replacing ϵ with u/c_0 and solving for u,

$$u = u_0 \sin \omega\tau + \frac{u_0^2 \beta k x}{2c_0} \sin 2\omega\tau + \mathcal{O}(\epsilon^3)$$

(f) What problem does the Fubini solution address? For what σ is it valid? To what solution did Blackstock "bridge" the Fubini solution?

The Fubini solution provides an explicit solution to the lossless $\mathcal{O}(\epsilon^2)$ nonlinear progressive equation, equation (3.1). It is valid until a shock forms, i.e., $\sigma = x/\bar{x} < 1$. Blackstock ``bridged" the Fubini solution to the Fay solution.

4 Burgers equation



Comic relief

$$\frac{\partial p}{\partial x} - \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial \tau^2} = \frac{\beta p}{\rho_0 c_0^3}\frac{\partial p}{\partial \tau} \tag{4.1}$$

1. Why did the Burgers equation go to the fast food restaurant?
   To get a quadratic meal deal!
2. Why was the Burgers equation feeling down?
   Because it felt like it was getting grilled by all those derivatives!
3. Why did the Burgers equation get stuck in traffic?
   Because of its nonlinear convection term, it just couldn't get past all the other terms!

(a) Why is equation (4.1) accurate to $\mathcal{O}(\epsilon^2)$?

The Burgers equation is cobbled together two equations that are accurate to $\mathcal{O}(\epsilon^2)$: the lossy linear progressive wave equation, given by equation (1.2) the lossless approximate nonlinear progressive wave equation, given by equation (3.1).

(b) What is an alternate way of writing the right-hand-side of equation (4.1)?

The right-hand side can alternatively be written as

$$\frac{\beta}{2\rho_0 c_0^3}\frac{\partial(p^2)}{\partial\tau}$$

(c) What is the Gol'berg number Γ? What do the limits $\Gamma < 1$ and $\Gamma \gg 1$ correspond to? In which case does a shock form?

The Gol'berg number is

$$\boxed{\Gamma = \frac{\beta\epsilon k}{\alpha} = \frac{\ell_a}{\bar{x}}.}$$

It characterizes the relationship between absorption and nonlinearity. $\Gamma < 1$ corresponds to absorption dominating nonlinearity, i.e., the sound is absorbed before a shock forms. $\Gamma \gg 1$ corresponds to nonlinearity dominating absorption, in which case a shock will form.

(d) Which of the following is a solution to equation (4.1) in the case that $\delta = 0$? Which is a solution in the case that $\beta = 0$?

$$p = f[\tau + (\beta p/\rho_0 c_0^3)x] \qquad p = p_0 \exp[j\omega\tau - (\delta\omega^2/2c_0^3)x] \tag{4.2}$$

$p = f[\tau + (\beta p/\rho_0 c_0^3)x]$ is the solution to equation (4.1) for $\delta = 0$ (lossless approximate nonlinear progressive wave equation), and $p = p_0 \exp[j\omega\tau - (\delta\omega^2/2c_0^3)x]$ is the solution to equation (4.1) for $\beta = 0$ (lossy linear approximate evolution equation).

(e) Assess second harmonic generation in the Burgers equation, using as as the first approximation

$$p_1 = p_0 e^{-\alpha_1 x}\sin\omega\tau.$$

where $\alpha_1 = \delta\omega^2/2c_0^3$. *Hint: In this case it is more convenient to write the Burgers equation as*

$$\frac{\partial p}{\partial x} - \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial \tau^2} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial (p^2)}{\partial \tau}\,.$$

As per the method of successive approximations, the first approximation is fed into the nonlinear operator, and the second order approximation is fed into the linear operator:

$$\frac{\partial p_2}{\partial x} - \frac{\delta}{2c_0^3}\frac{\partial^2 p_2}{\partial \tau^2} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial (p_1^2)}{\partial \tau}\,.$$

The right-hand side, upon feeding it $p_1 = p_0 e^{-\alpha_1 x}\sin\omega\tau$ and writing $\sin^2\omega\tau = (1-\cos 2\omega\tau)/2$, becomes

$$\frac{p_0^2\beta}{4\rho_0 c_0^3}e^{-2\alpha_1 x}\frac{\partial}{\partial \tau}(1-\cos 2\omega\tau) = \frac{p_0^2\beta\omega}{2\rho_0 c_0^3}e^{-2\alpha_1 x}\sin 2\omega\tau\,.$$

Anticipating that the second approximation has the same time-dependence as the right-hand side, i.e., $p_2 = q(x)\sin 2\omega\tau$, results in an ordinary differential equation,

$$\frac{dq_2}{dx} + \alpha_2 q_2 = \frac{p_0^2\beta\omega}{2\rho_0 c_0^3}e^{-2\alpha_1 x},$$

where $\alpha_2 = 4\alpha_1$. The particular solution is of the form $q_{2\mathsf{p}} = Ae^{-2\alpha_1 x}$, where the constant A is determined by substitution into the above:

$$A(-2\alpha_1 + \alpha_2) = \frac{p_0^2\beta\omega}{2\rho_0 c_0^3} \qquad \Longrightarrow \qquad A = \frac{1}{\alpha_2 - 2\alpha_1}\frac{p_0^2\beta\omega}{2\rho_0 c_0^3}\,.$$

The homogeneous solution is

$$q_{2\mathsf{h}} = Be^{-2\alpha_2 x}\,.$$

Since $q_2(0) = 0$, i.e., there is no second harmonic at the source $x = 0$, $B = -A$. Therefore, the second harmonic is fully determined:

$$p_2(x,t) = (q_{2\mathsf{p}} + q_{2\mathsf{h}})\sin\omega\tau = \frac{p_0^2\beta\omega}{2\rho_0 c_0^3}\left(\frac{e^{-2\alpha_1 x} - e^{-2\alpha_2 x}}{\alpha_2 - 2\alpha_1}\right)\sin 2\omega\tau$$

(f) The solution to the Burgers equation for a so-called Taylor shock is

$$p = \frac{\Delta p}{2}\left[1 + \tanh\left\{\frac{\beta \Delta p}{2\rho_0 \delta}(t' - t_0)\right\}\right] \tag{4.3}$$

Rewrite equation (4.3) by defining

$$t_{\text{rise}} = \frac{4\rho_0 \delta}{\beta \Delta p} \tag{4.4}$$

and use the result to show that

$$t_{\text{rise}}\left(\frac{\partial p}{\partial t'}\right)_{t'=t_0} = \Delta p,$$

Employing the definition of t_{rise}, equation (4.3) becomes

$$p = \frac{\Delta p}{2}\left\{1 + \tanh\left[\frac{2}{t_{\text{rise}}}(t' - t_0)\right]\right\}.$$

Taking the derivative with respect to t' and evaluating at $t' = t_0$, and solving for Δp gives

$$\boxed{t_{\text{rise}}\left(\frac{\partial p}{\partial t'}\right)_{t'=t_0} = \Delta p\,.}$$

(g) In what limit of viscosity δ does the Taylor shock become a step shock? *Hint: A step shock is defined by* $t_{\text{rise}} \to 0$.

By equation (4.4), the Taylor shock becomes a step shock in the limit that $\delta \to 0$.

(h) What is the name of the transformation that leads to the Fay solution? What equation does the Fay solution satisfy? What are the restrictions on this solution?

The transformation is called the ``Hopf-Cole transformation."The Fay solution satisfies the Burgers equation for $\Gamma \gg 1$ and $\sigma \gtrsim 3$.

(i) Show that $\sigma = x/\bar{x}$ can be written as

$$\sigma = \frac{\beta p_0 k x}{\rho_0 c_0^2} \tag{4.5}$$

by recalling the definition of the shock formation distance and the acoustic mach number.

The shock formation distance is $\bar{x} = 1/\beta\epsilon k$, and the acoustic mach number is $\epsilon = u_0/c_0 = p_0/\rho_0 c_0^2$. Therefore,

$$\boxed{\sigma = \beta\epsilon k x = \frac{\beta p_0 k x}{\rho_0 c_0^2}\,.}$$

(j) The Fay solution is

$$p = p_0 \sum_{n=1}^{\infty} B_n(\sigma) \sin n\omega\tau, \quad \text{where} \quad B_n(\sigma) = \frac{2A}{\sinh[nA(1+\sigma)]}.$$

where $A = \alpha\bar{x} = \alpha\rho_0 c_0^2/p_0\beta k$. Show that for $\sigma \gg 1$, the pressure becomes

$$p = \frac{4\rho_0 c_0^2 \alpha}{\beta k} \sum_{n=1}^{\infty} e^{-n\alpha x} \sin n\omega\tau\,,$$

where equation (4.5) has been invoked. What is this waveform called? *Hint: think about the color of Dr. Hamilton's hair.*

In the limit that $\sigma \gg 1$,

$$\sinh[nA(1+\sigma)] = \frac{e^{nA(1+\sigma)} - e^{-nA(1+\sigma)}}{2} \rightarrow e^{nA\sigma}/2$$

so

$$B_n \rightarrow 4Ae^{-nA\sigma}.$$

Therefore, the Fay solution becomes

$$p = 4Ap_0 \sum_{n=1}^{\infty} e^{-nA\sigma} \sin n\omega\tau$$

Making the substitution $A = \alpha\rho_0 c_0^2/p_0\beta k$ results in

$$\boxed{p = \frac{4\alpha\rho_0 c_0^2}{\beta k} \sum_{n=1}^{\infty} e^{-nA\sigma} \sin n\omega\tau}$$

Note that Dr. Hamilton's hair is white, and that white hair is associated with old age. Thus it is recalled that the $\sigma \gg 1$ limit of Fay solution is referred to as the ``old age" waveform.

(k) It was shown in class that for $\Gamma \to \infty$ (equivalently, $A \to 0$), the Fay solution reduces to

$$p = \frac{2p_0}{1+\sigma} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\omega\tau \,. \tag{4.6}$$

Combine equations (4.5) and (4.6) to show that for $\sigma \gg 1$,

$$p = \frac{2\rho_0 c_0^2}{\beta k x} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\omega\tau \,. \tag{4.7}$$

What is remarkable about equation (4.7)?

In the limit that $\sigma \gg 1$, equation (4.6) becomes

$$p = \frac{2p_0}{\sigma} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\omega\tau \,.$$

Writing $\sigma = \frac{\beta p_0 k x}{\rho_0 c_0^2}$ gives

$$\boxed{p = \frac{2\rho_0 c_0^2}{\beta k x} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\omega\tau \,.}$$

What makes this solution remarkable is that it does not depend on the source pressure p_0.

(l) What is the time-domain version of the Fay solution called?

The time-domain version of the Fay solution is called the Khokhlov solution.

References

[1] M. F. Hamilton, Lecture notes from Nonlinear Acoustics. University of Texas at Austin, (2023).

[2] M. F. Hamilton and D. T. Blackstock, “Nonlinear Acoustics.” Acoustical Society of America, (2008).

[3] “Gravitational time dilation,” Wikipedia.