Analytical solution for a focused vortex beam radiated by a Gaussian source

Chirag A. Gokani
Yuqi Meng
Michael R. Haberman
Mark F. Hamilton

Applied Research Laboratories
Walker Department of Mechanical Engineering
The University of Texas at Austin

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Acoustic Manipulation and Atmospheric Propagation
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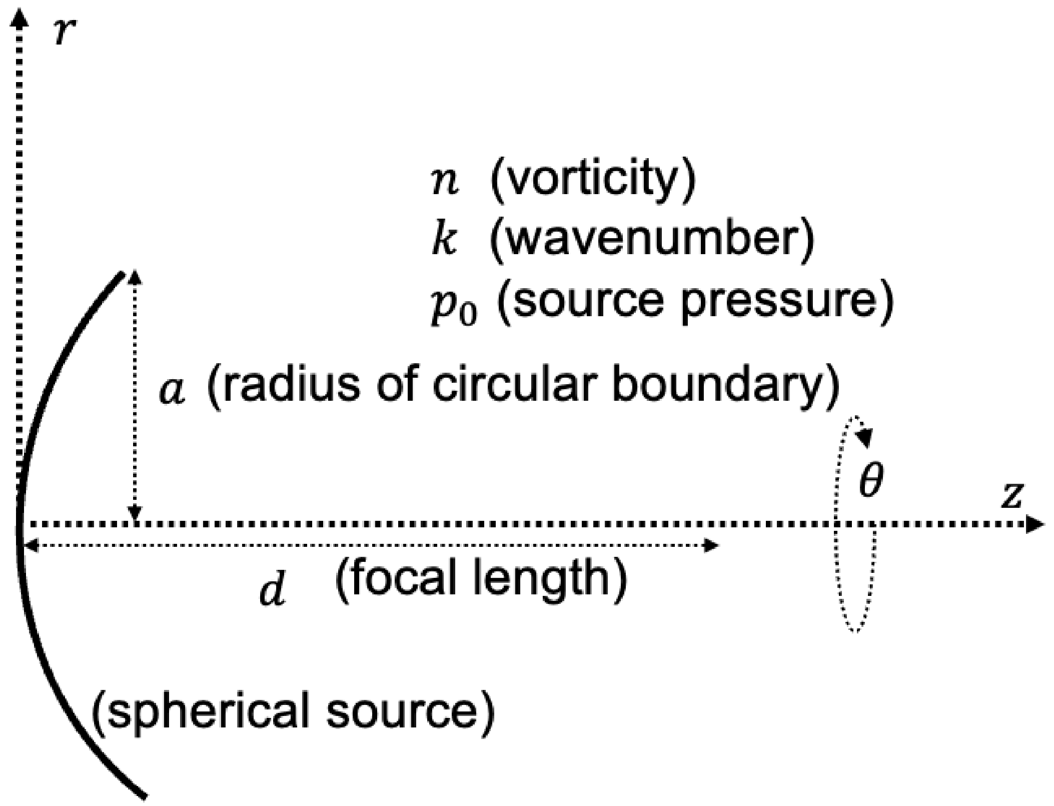
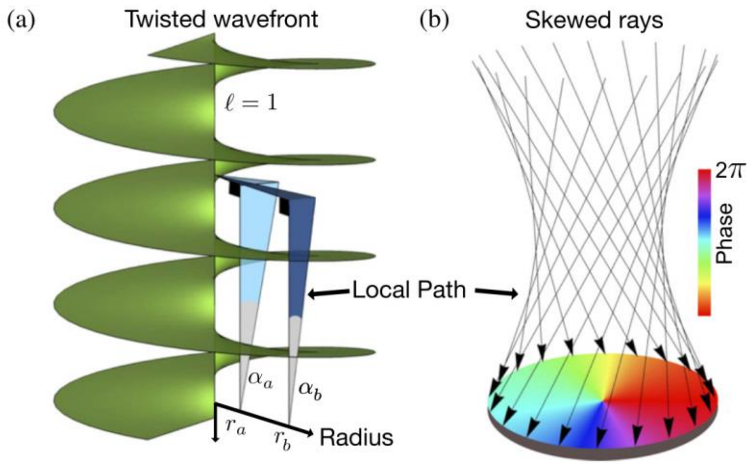
Outline

1. Background
2. Previous work
3. Fresnel approximation
4. Analytical solution
5. Comparison to solution of Helmholtz equation
6. Field plots
7. Scaling law for focal beamwidth
8. Global maximum



Background

- *Vorticity*: number of equal-phase wavefronts that fit about the propagation axis



Richard et al., New J. Phys. **22**, 063021 (2020)

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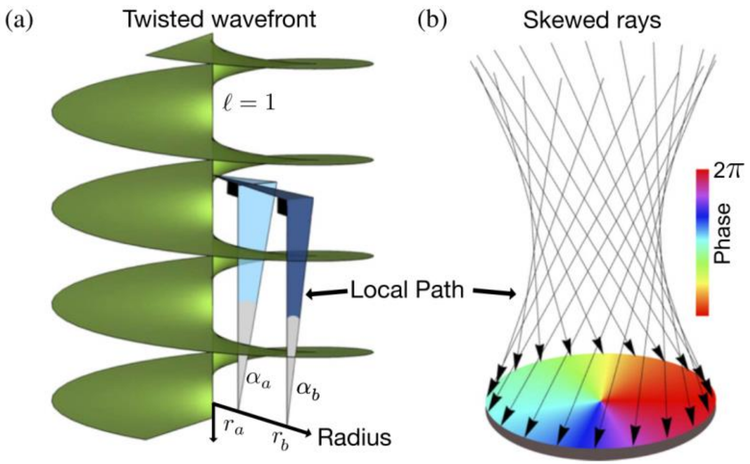

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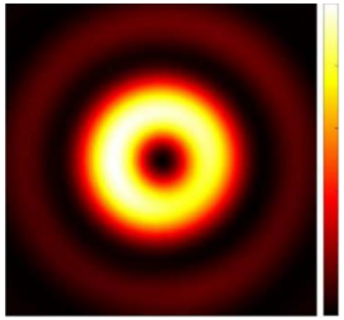

Baudoin et al.,
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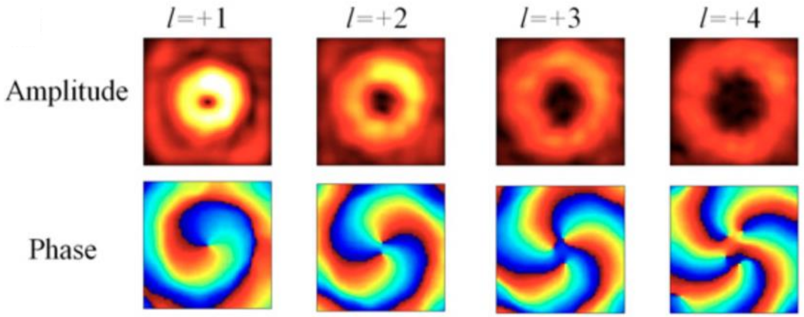
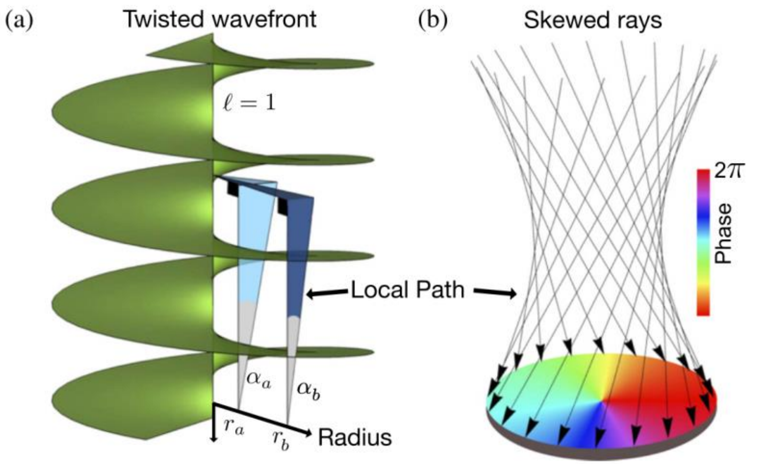


Shi et al., Proc. NAS. **114**, 28 (2017)

Background

- *Vorticity*: number of equal-phase wavefronts that fit about the propagation axis
- Characterized by central null
- Focused beams for radial and axial particle manipulation
- Unfocused beams for communication



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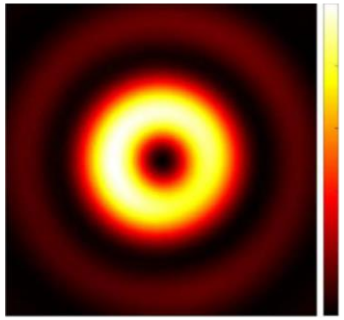

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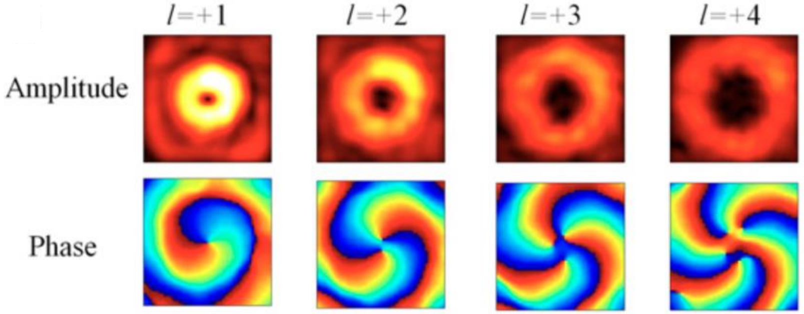

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Background

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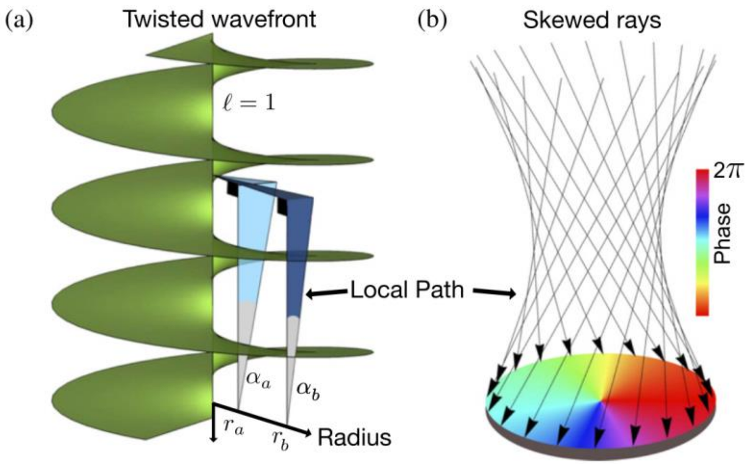
- Generated using array of transducers, or using a transducer, phase plate, active piezoelectric metasurface



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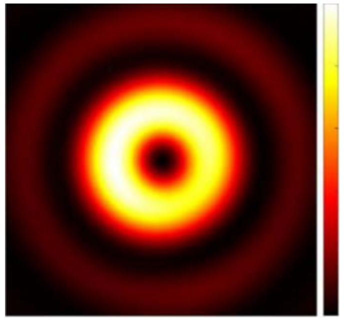

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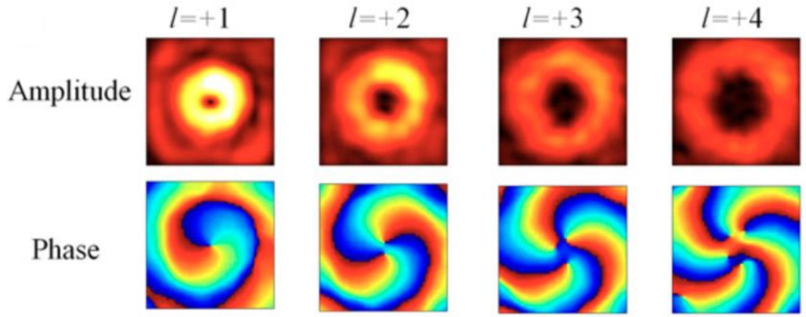

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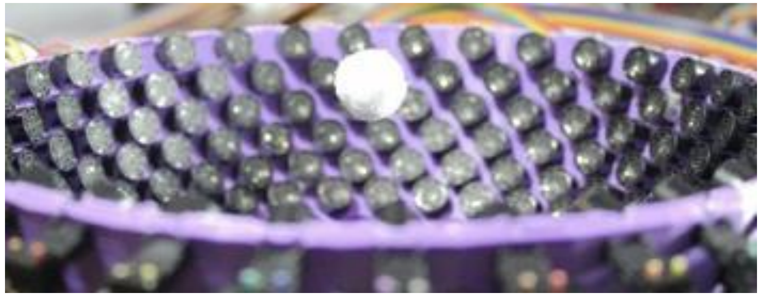
Li et al., Wiley (2022).



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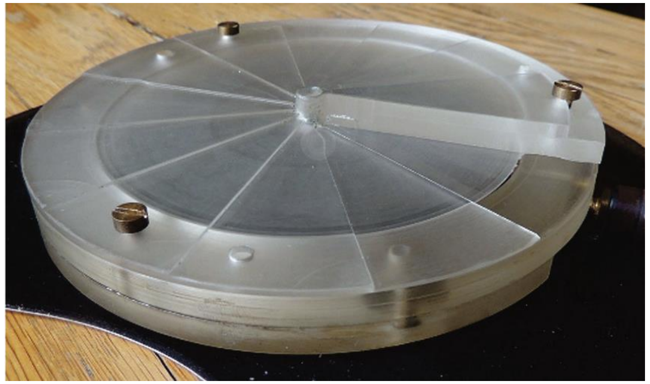
Phys. Rev. Lett. **120,** 044301 (2018).



Terzi et al.,
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Previous work

- Gaussian beams studied for analytical ease
- Analytical solutions & scaling laws provide physical intuition

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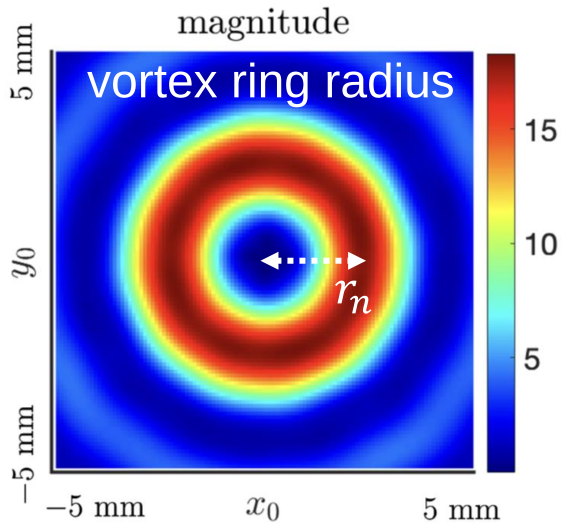

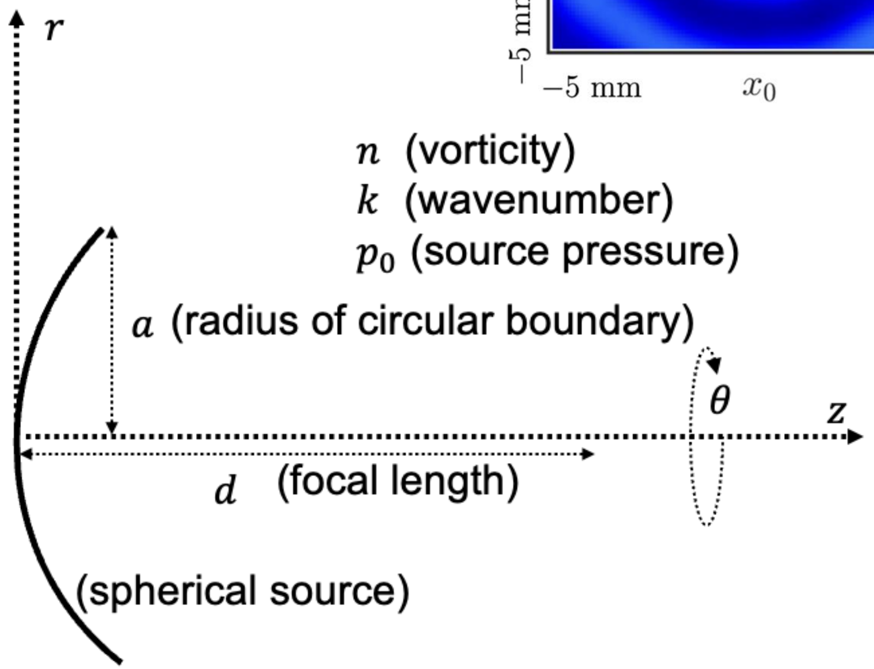

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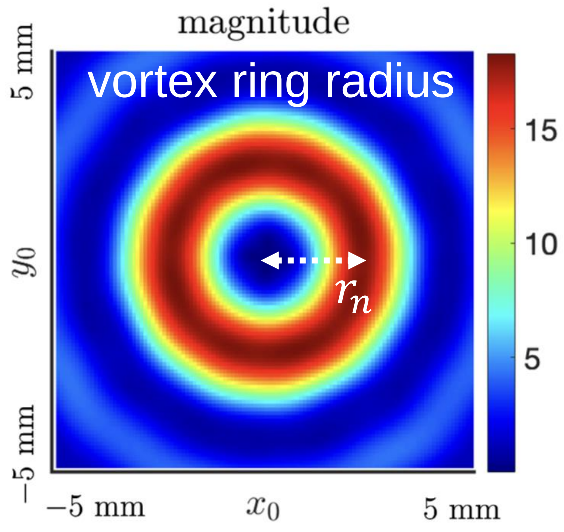
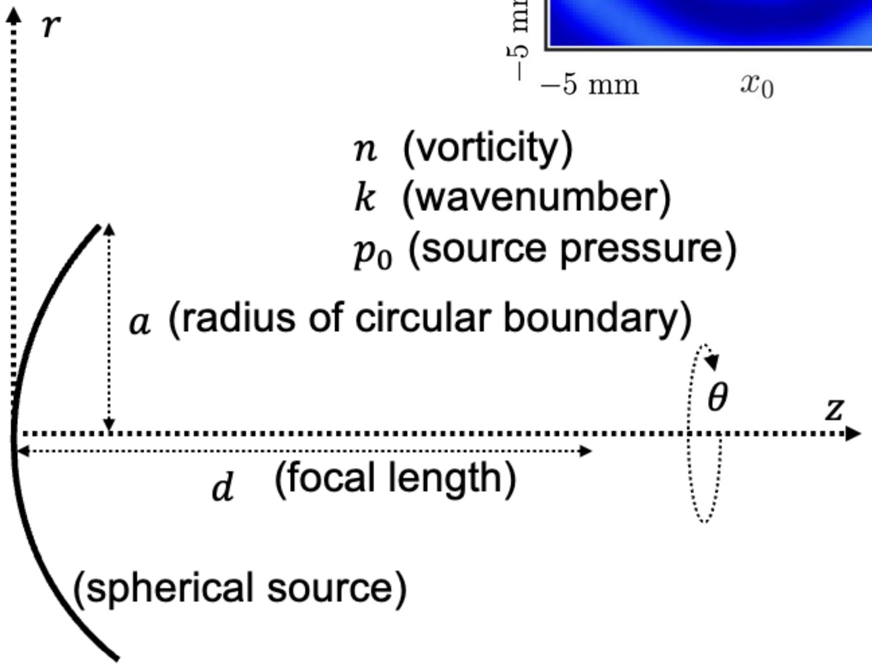
$$r_n = \frac{\alpha'_n a}{2\pi} \simeq \frac{(n + 0.81n^{1/3})a}{2\pi}$$

- Scaling law for focused optical uniform vortex beam**

$$r_n = \frac{d}{ka}(1.29 + 0.13n)$$





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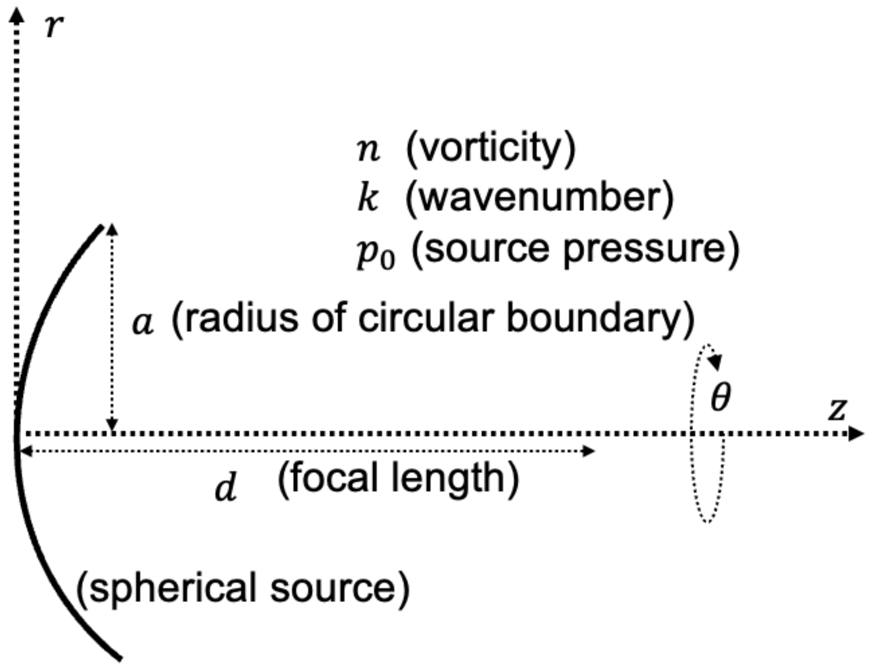
** Curtis et al., "Structure of Optical Vortices." PRL **90**, 13 (2003).

Fresnel approximation

- Velocity source interchangeable with pressure source for $ka \gg 1$*

$$p(r,\theta,z) = -\frac{ikp_0}{2\pi}\frac{e^{ikz}}{z}\int_0^{2\pi}\int_0^{\infty} p_0(r_0,\theta_0,0)\, e^{\frac{ik}{2z}[r^2+r_0^2-2rr_0\cos(\theta+\theta_0)]} r_0\,dr_0\,d\theta_0$$



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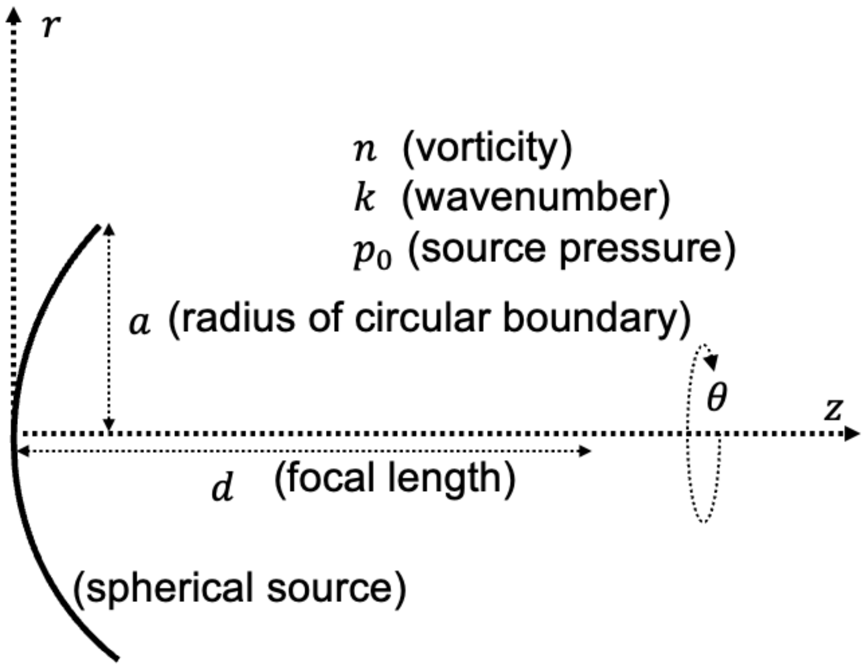
- Source condition:

$$p_0(r,\theta,0) = p_0 \underbrace{e^{-ikr^2/2d}}_{\text{focused}} \underbrace{e^{-r^2/a^2}}_{\text{Gaussian}} \underbrace{e^{in\theta}}_{\text{vortex}}$$



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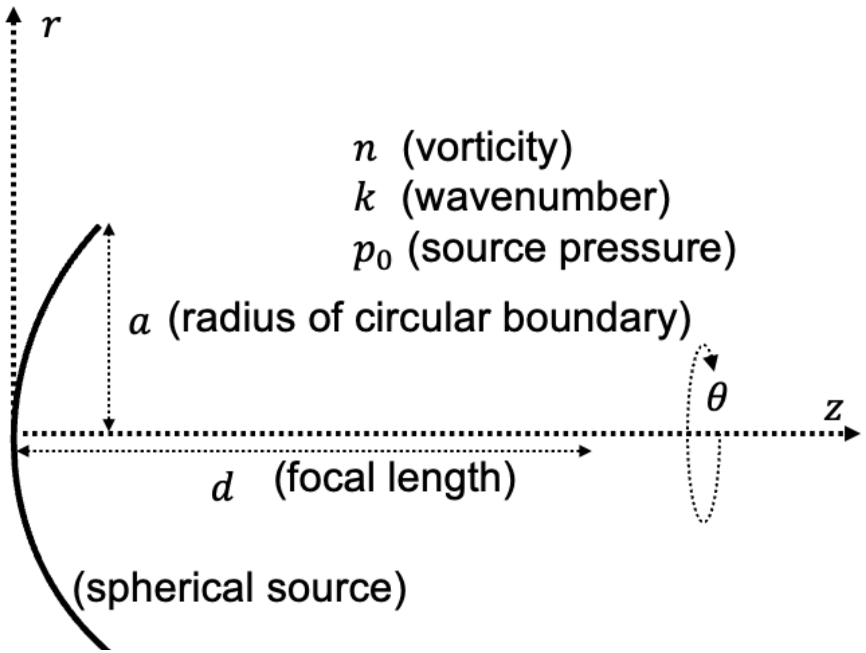

- Radial integral over Bessel function:

$$p(r,\theta,z) = -ikp_0\frac{e^{ikz}}{z}e^{ikr^2/2z}e^{-il(\theta+\pi/2)}\int_0^{\infty}\exp\left\{\left[-\frac{1}{a^2}+\frac{ik}{2}\left(\frac{1}{z}-\frac{1}{d}\right)\right]r_0^2\right\}J_n(krr_0/z)r_0dr_0$$

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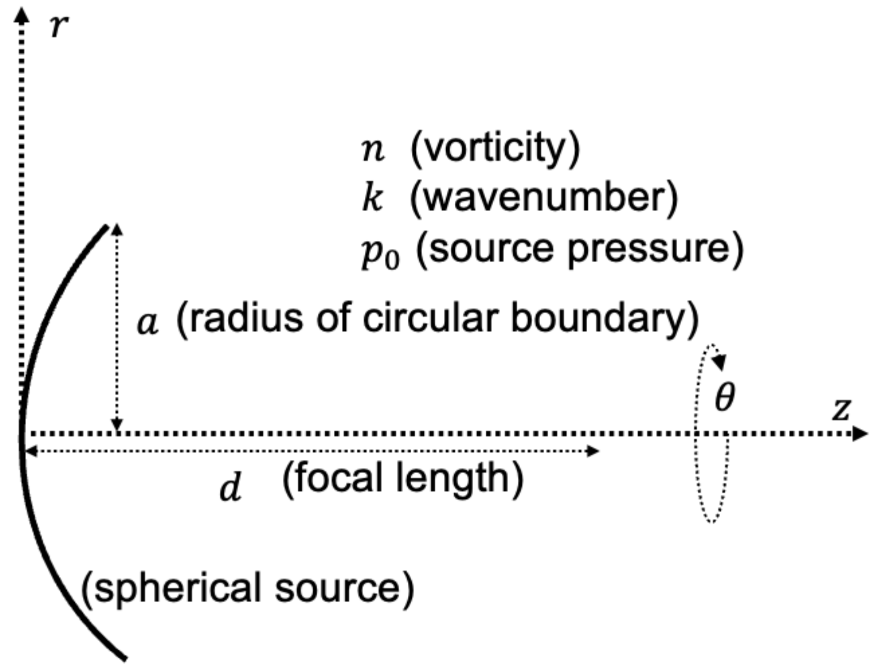
Analytical solution

- Closed-form solution is

$$p(r,\theta,z) = -i\sqrt{8\pi}\,p_0 e^{ikz} e^{ikr^2/2z} e^{-in(\theta+\pi/2)}\,\frac{z}{kr^2}\,\chi^{3/2} e^{-\chi}\left[I_{(n-1)/2}(\chi) - I_{(n+1)/2}(\chi)\right]$$

where

$$\chi = \frac{1}{8}\left(\frac{kar}{z}\right)^2\left[1 - \frac{ika^2}{2}\left(\frac{1}{z} - \frac{1}{d}\right)\right]^{-1}$$



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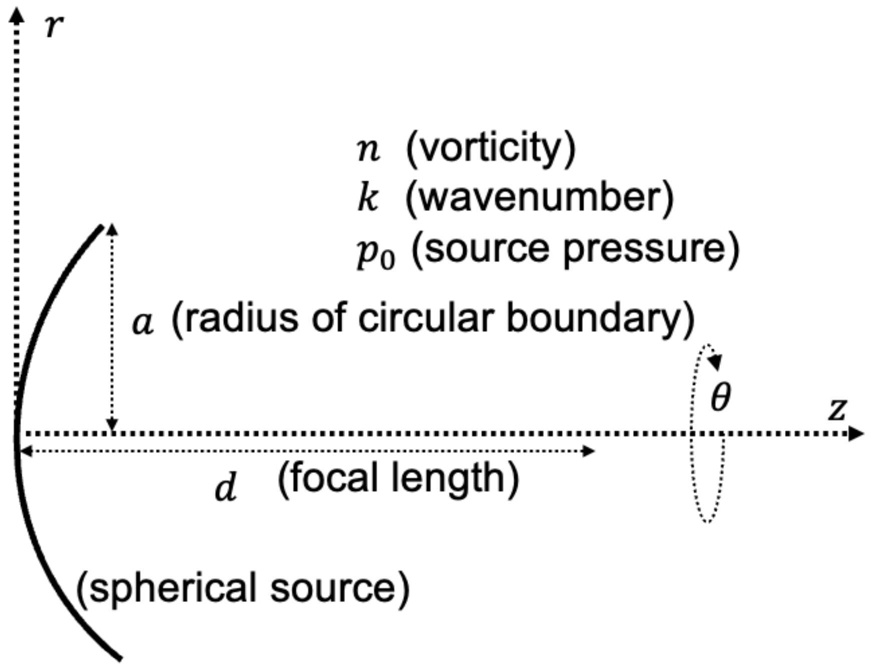
$I_m(\chi) = i^{-m}J_m(i\chi)$ = modified Bessel function



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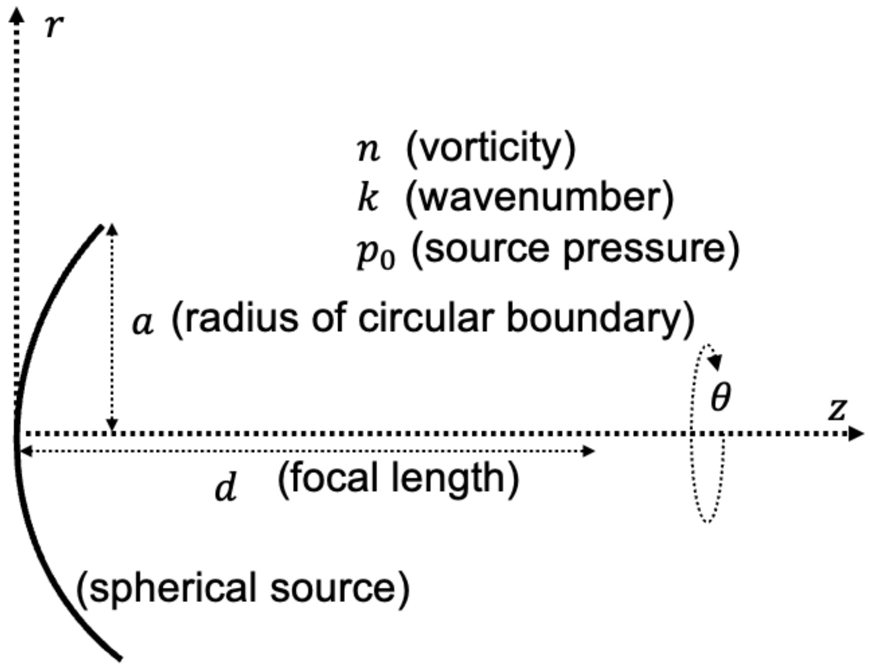

- standard vortex beam angular dependence
- For $n > 0$, $p = 0$ as $r = 0$: on-axis null

Analytical solution

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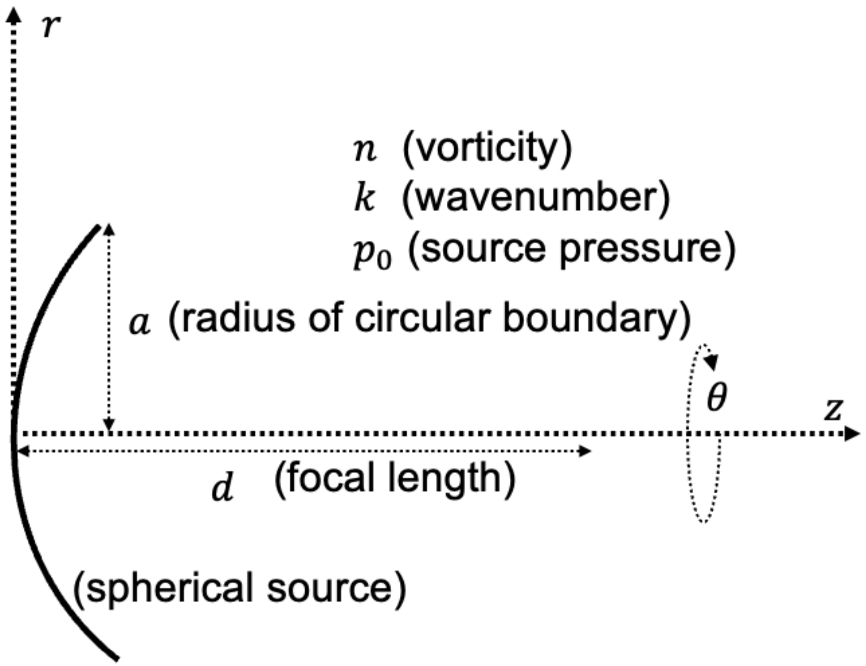

- Magnitude given by

$$\left|p(r,z)\right| = \sqrt{8\pi}\, p_0 \frac{z}{kr^2} \left| \chi^{3/2} e^{-\chi} \left[I_{(n-1)/2}(\chi) - I_{(n+1)/2}(\chi) \right] \right|$$

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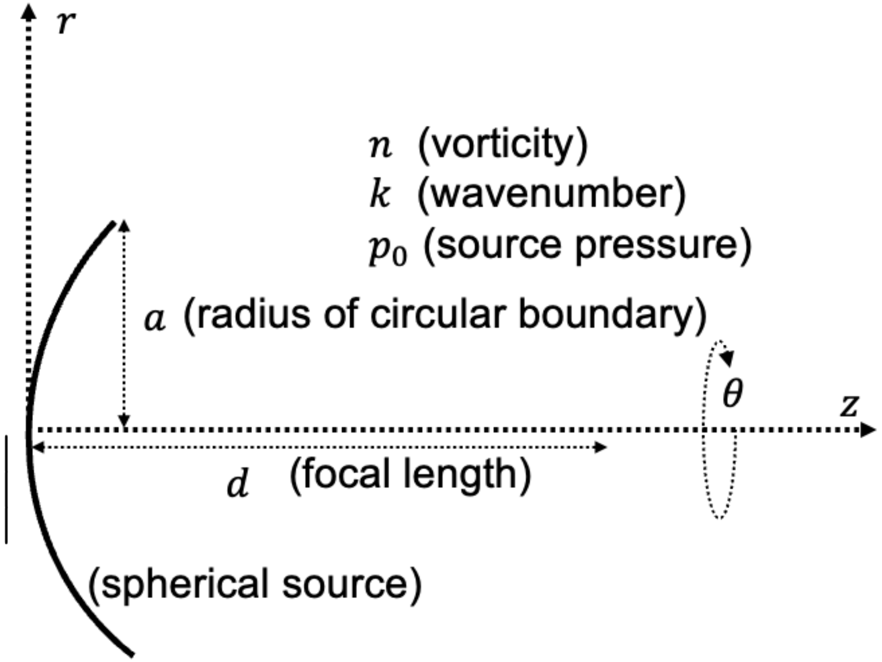

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- Returns focused Gaussian beam solution for $n = 0$:

$$p(r,z) = \frac{p_0 e^{ikz}}{1 - (1 - iG^{-1})z/d} \exp\left[- \frac{(1+iG)(r/a)^2}{1 - (1 - iG^{-1})z/d} \right]$$

where $G = ka^2/2d$

Comparison with solution of Helmholtz equation

Fresnel approximation compared with exact solution of the Helmholtz equation via Fourier acoustics:

$$p(x,y,z) = \iint_{-\infty}^{\infty} \hat{p}_0(k_x,k_y) e^{i(k_x x + k_y y + k_z z)} \frac{dk_x}{2\pi}\frac{dk_y}{2\pi} \qquad k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

where

$$\hat{p}_0(k_x,k_y) = p_0 \iint_{-\infty}^{\infty} \exp\left(\underbrace{-\frac{x^2+y^2}{a^2}}_{\text{Gaussian}} \underbrace{- \frac{ik}{2d}(x^2+y^2)}_{\text{focused}} + \underbrace{in \tan^{-1}\frac{y}{x}}_{\text{vortex}} \right) e^{-i(k_x x + k_y y)} dx\, dy$$

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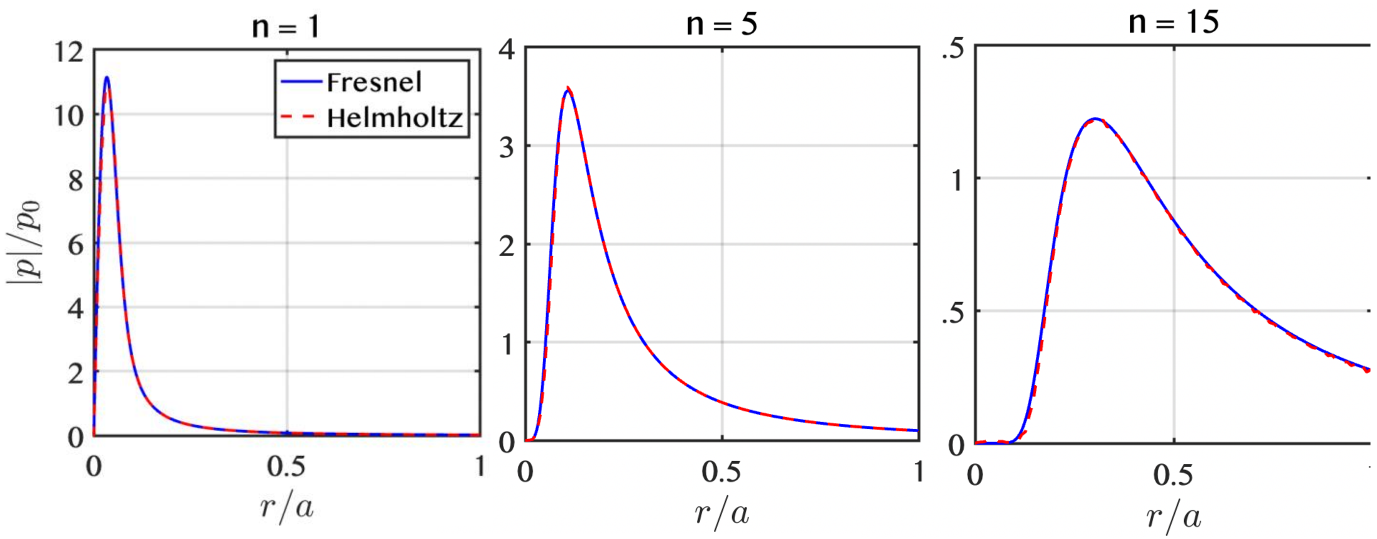
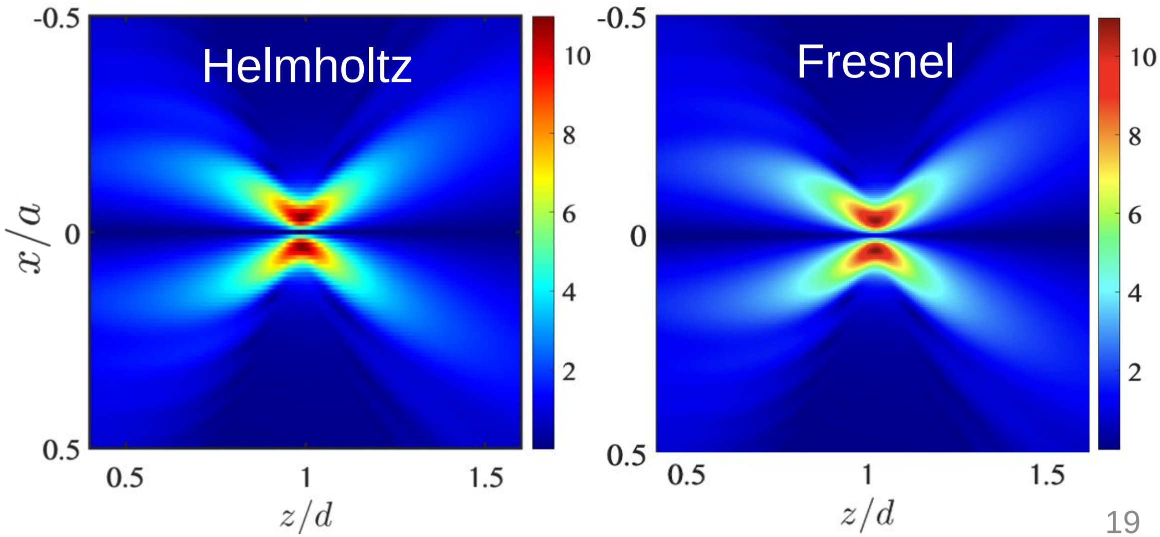
Focal plane

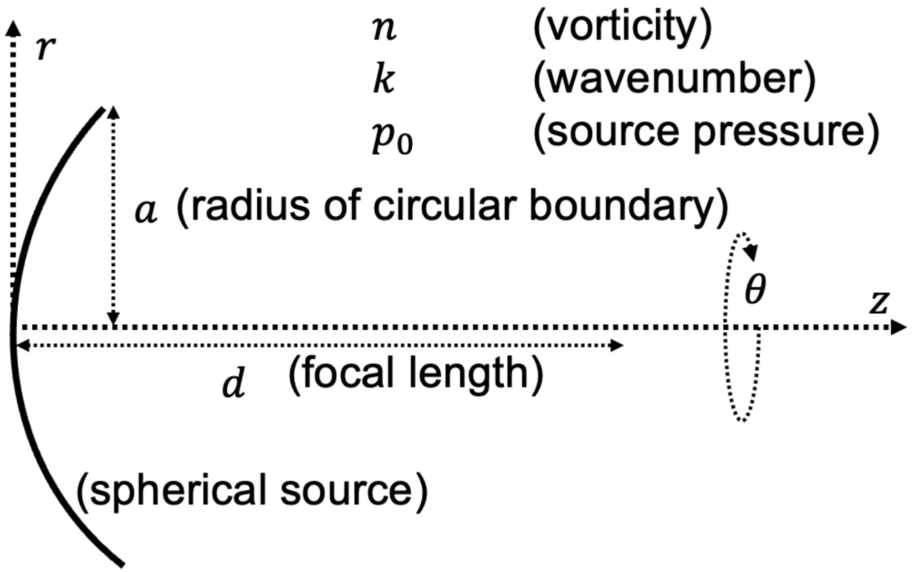


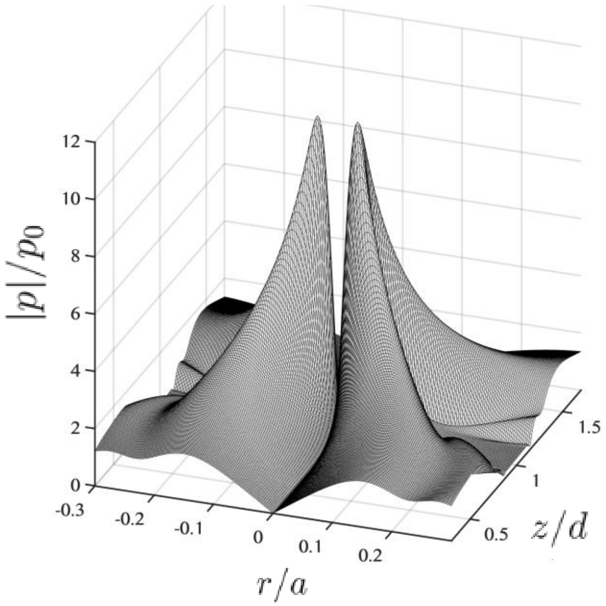
Full field for $n = 1$



Field plots

$ka = 500,\ d/a = 10$





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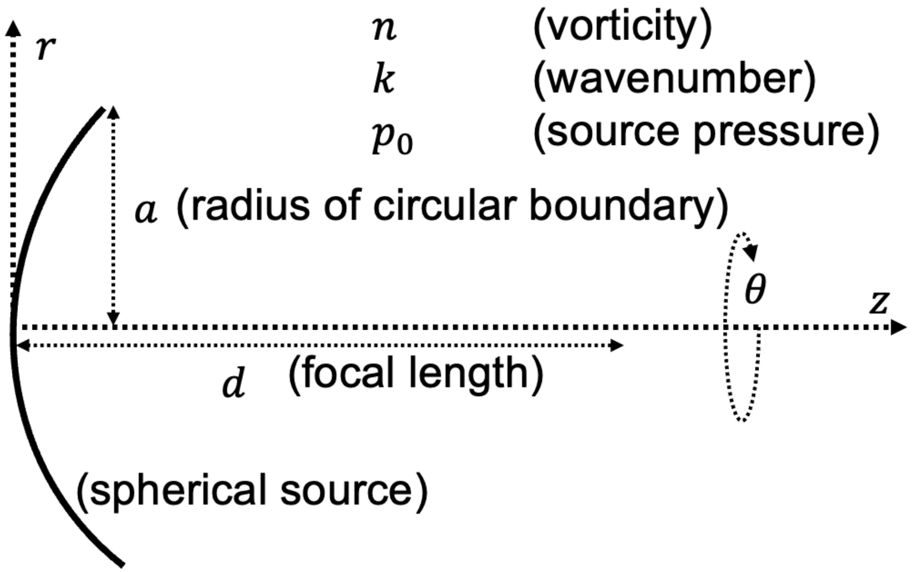

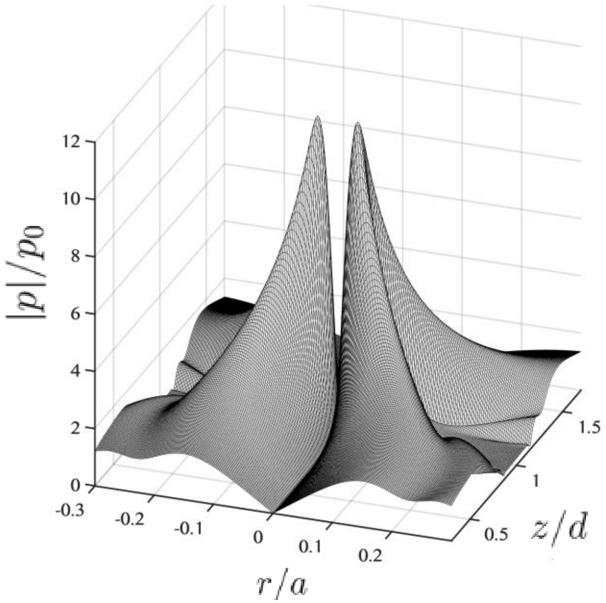

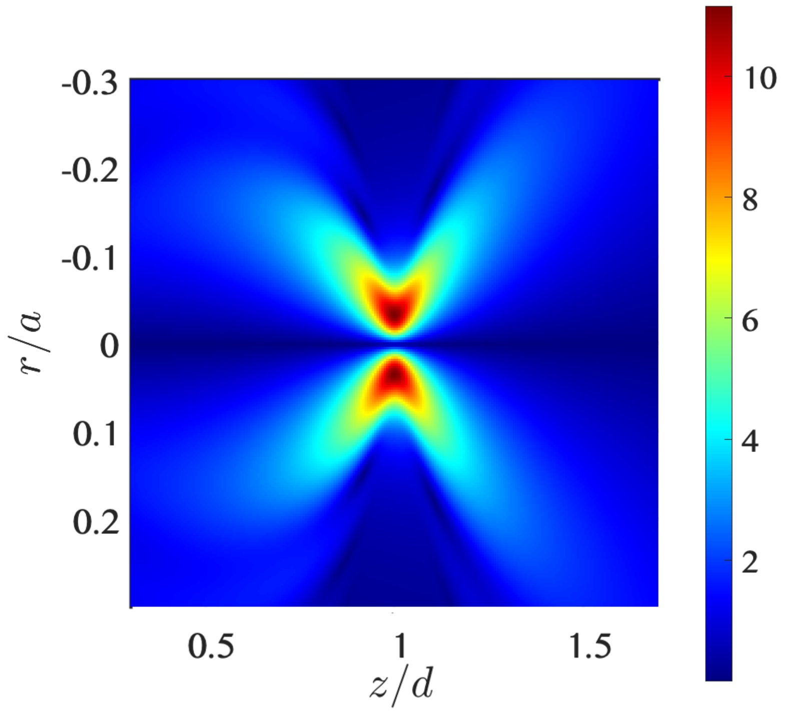

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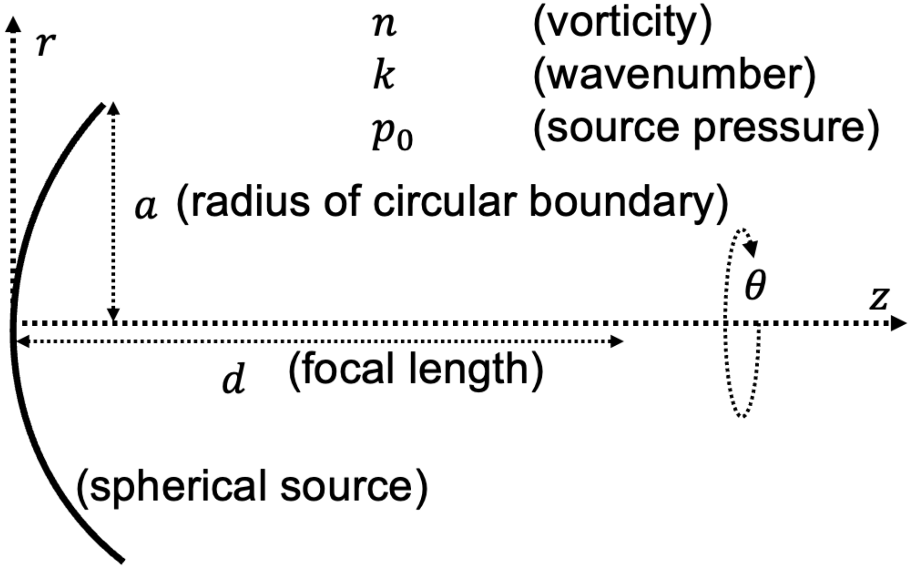

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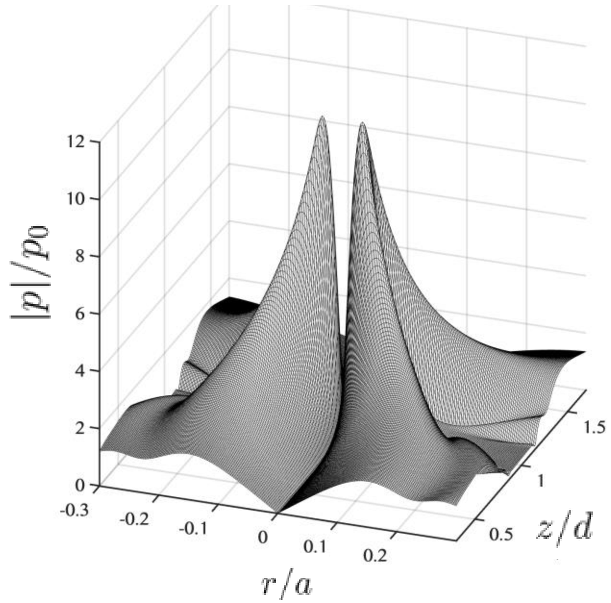
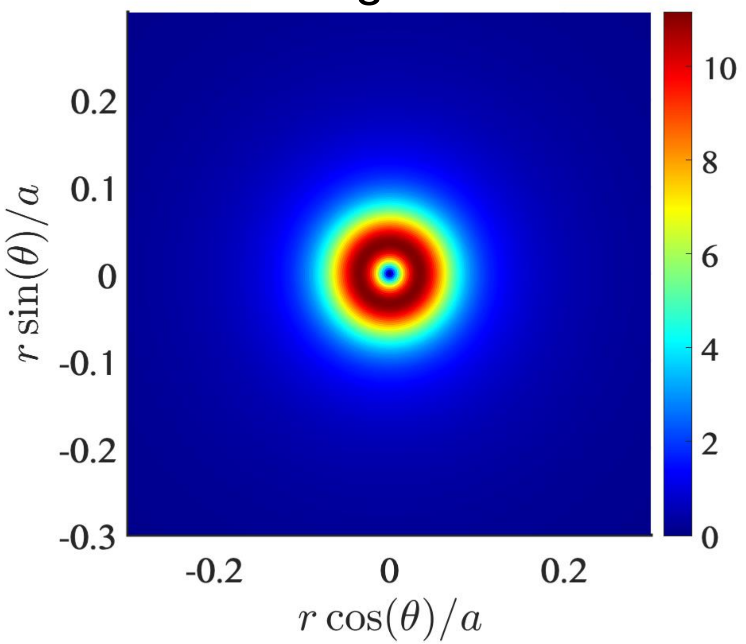


magnitude



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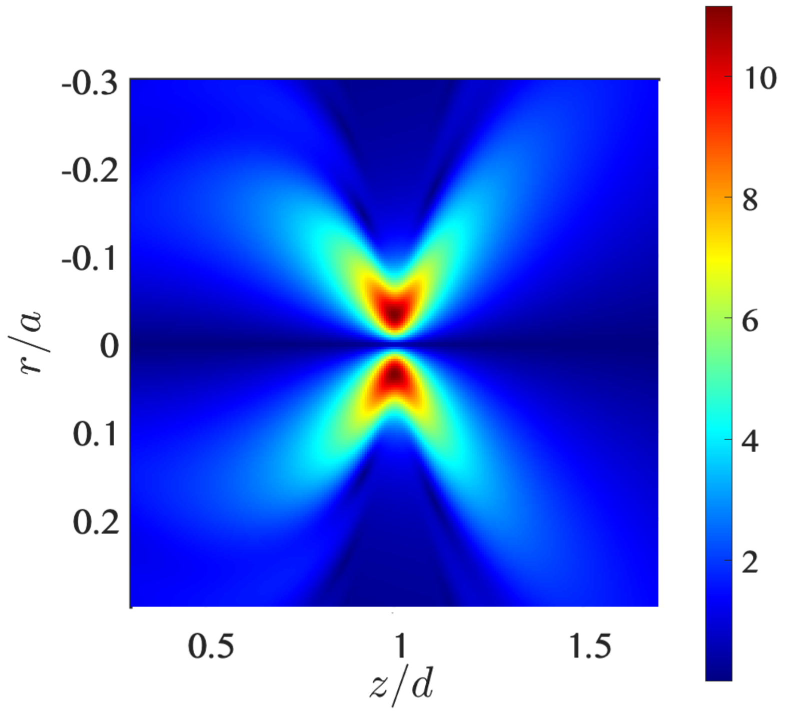

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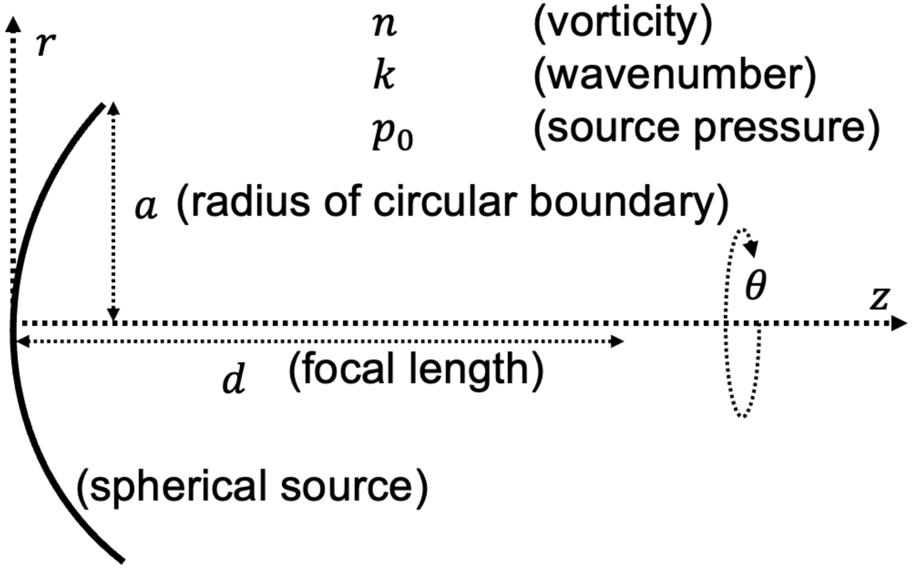

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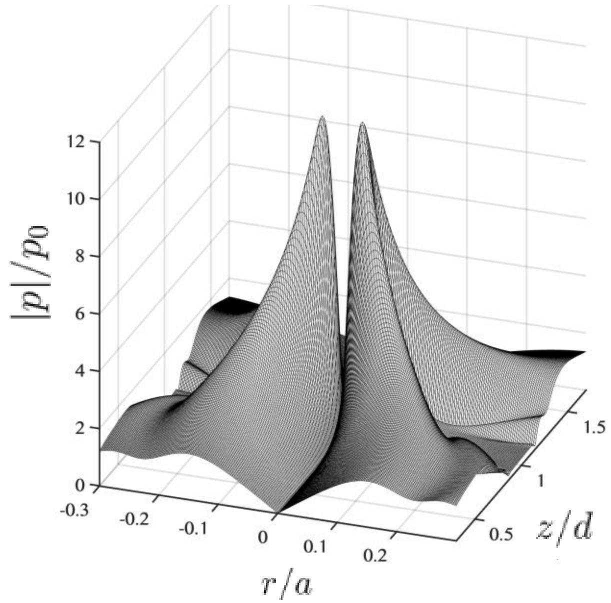

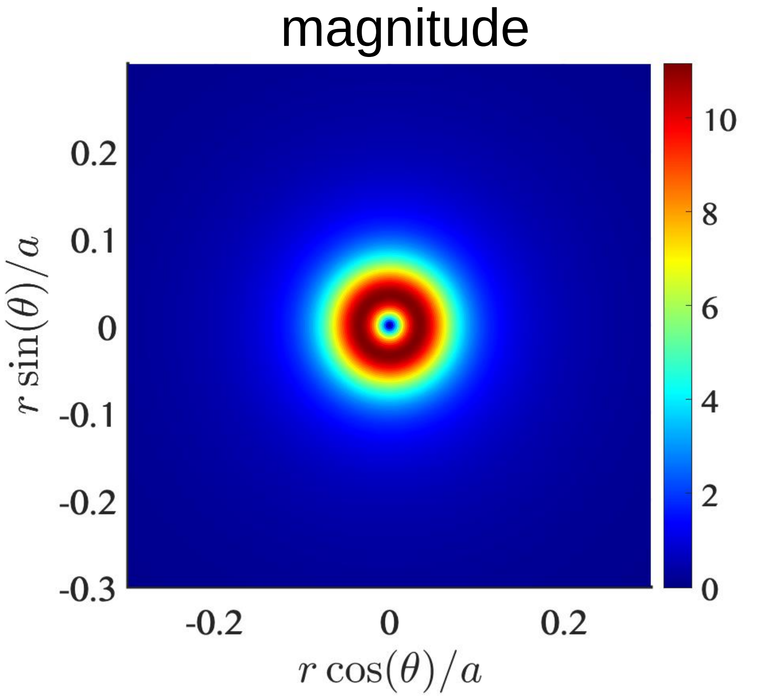

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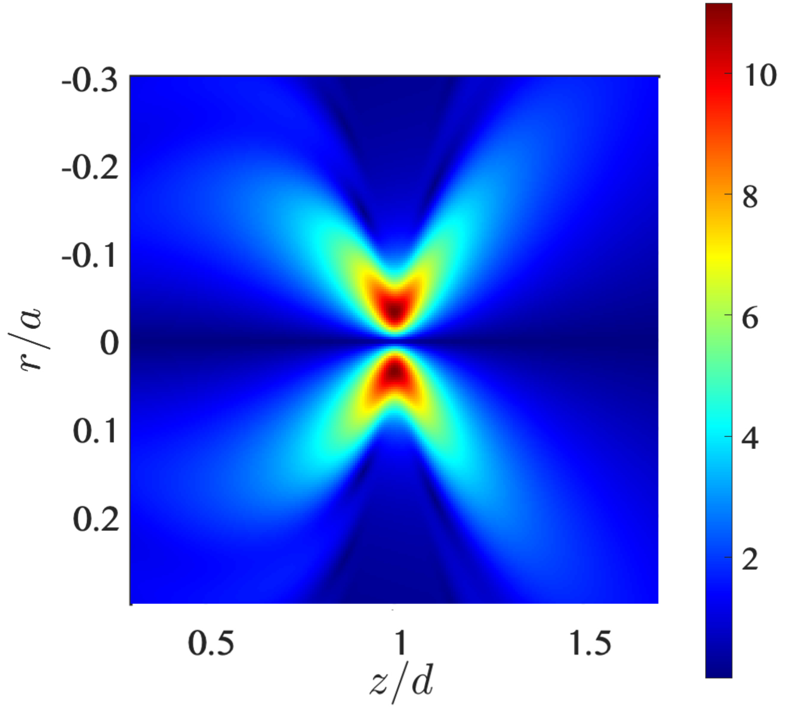

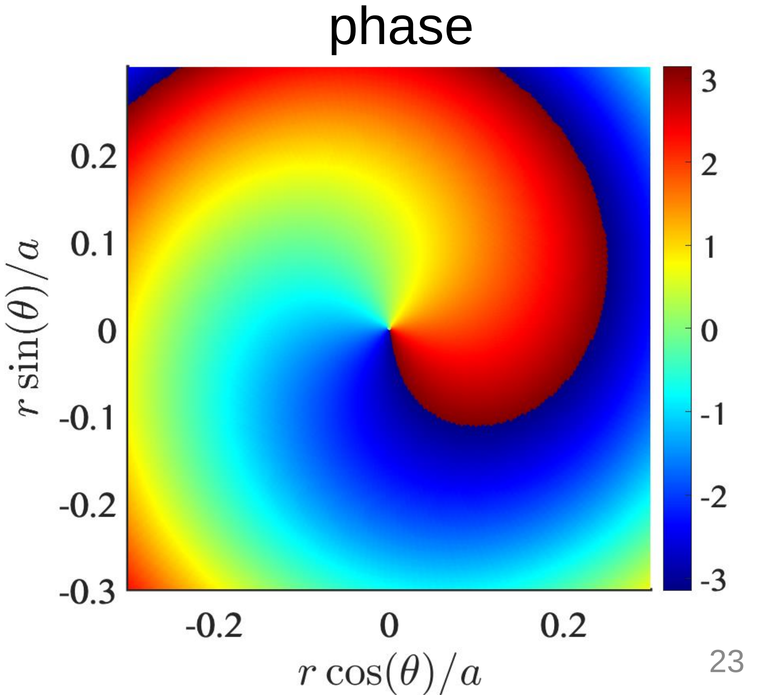

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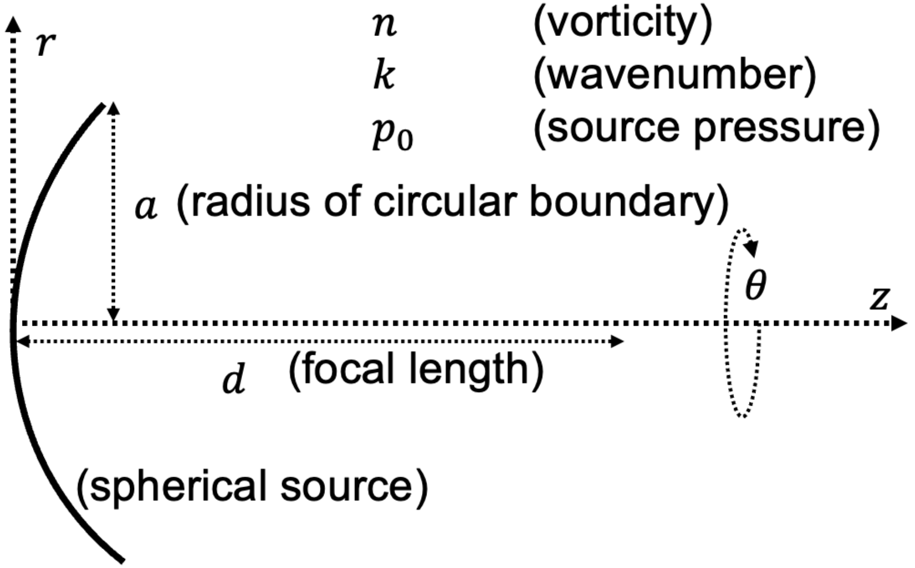

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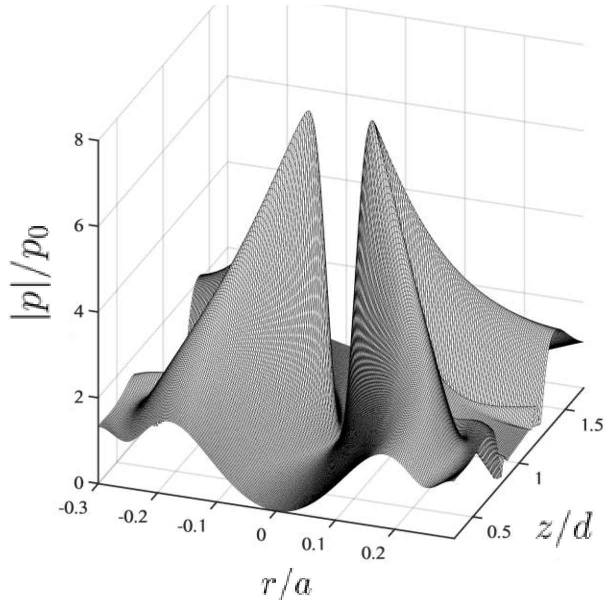
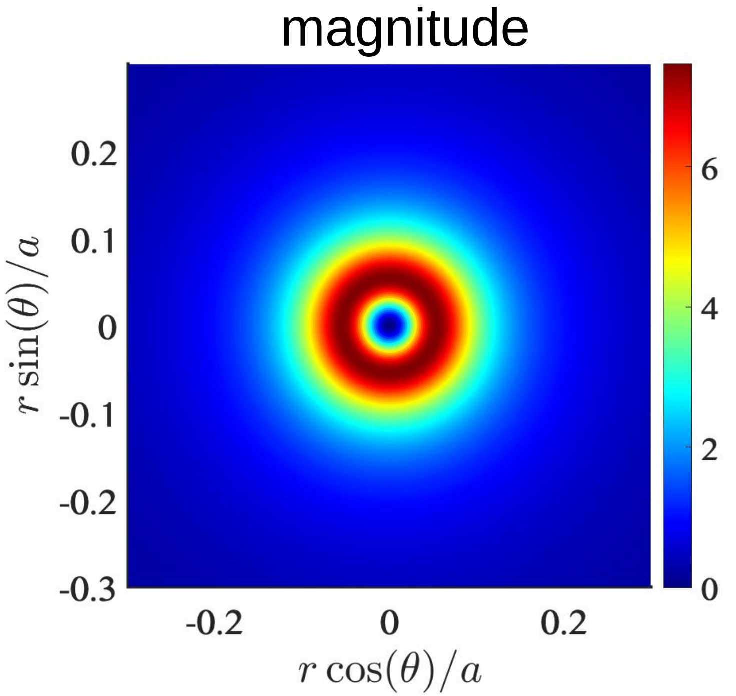
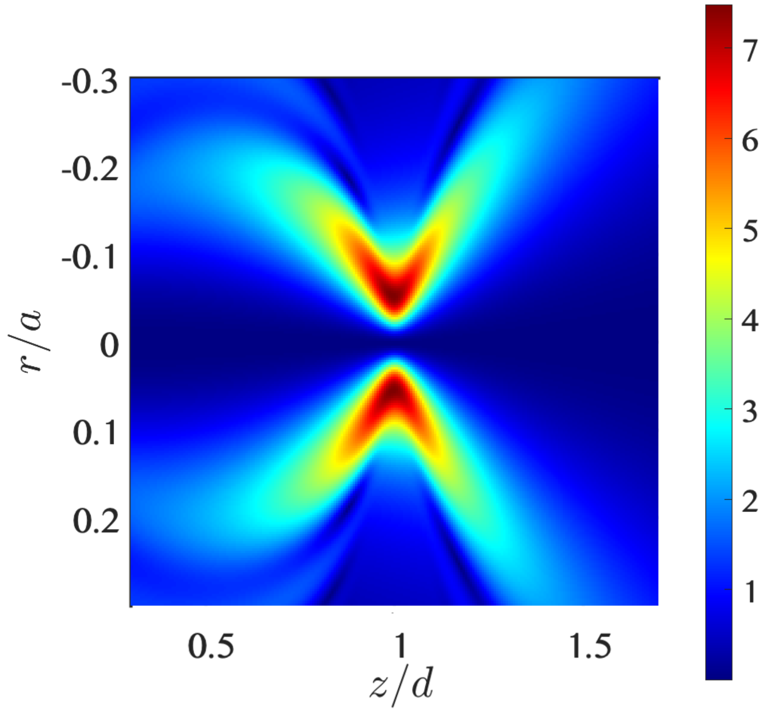
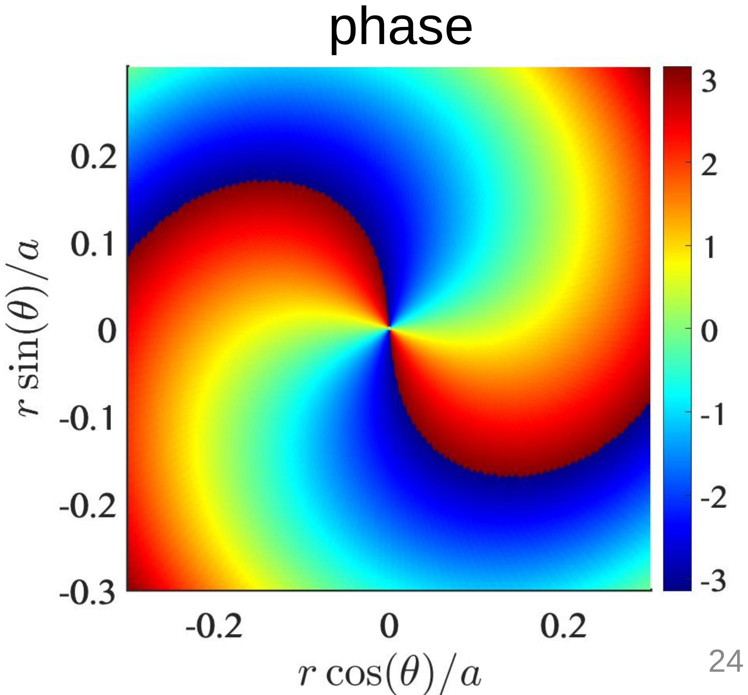


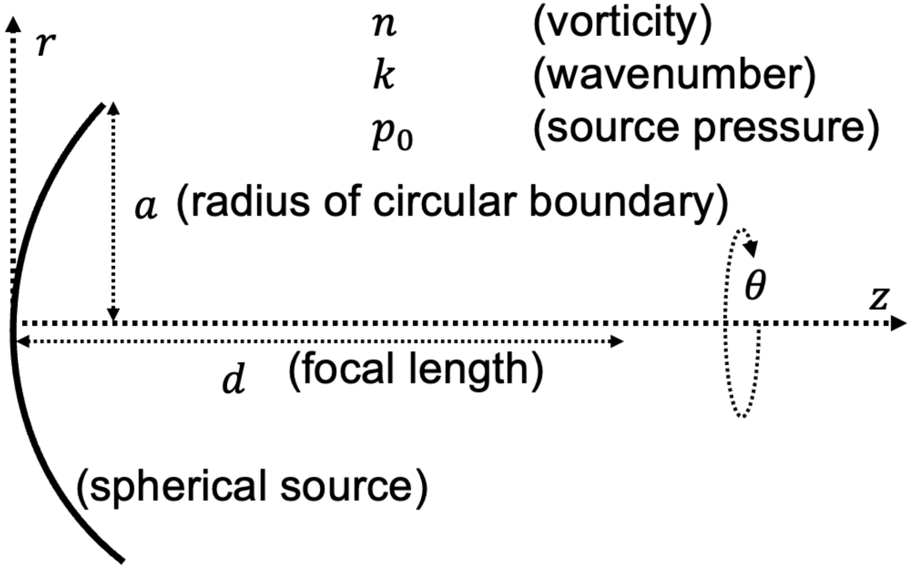


$n = 2$



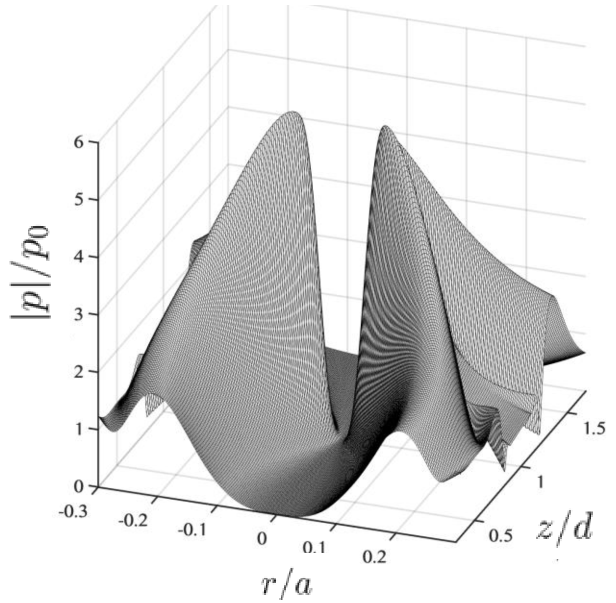


Field plots

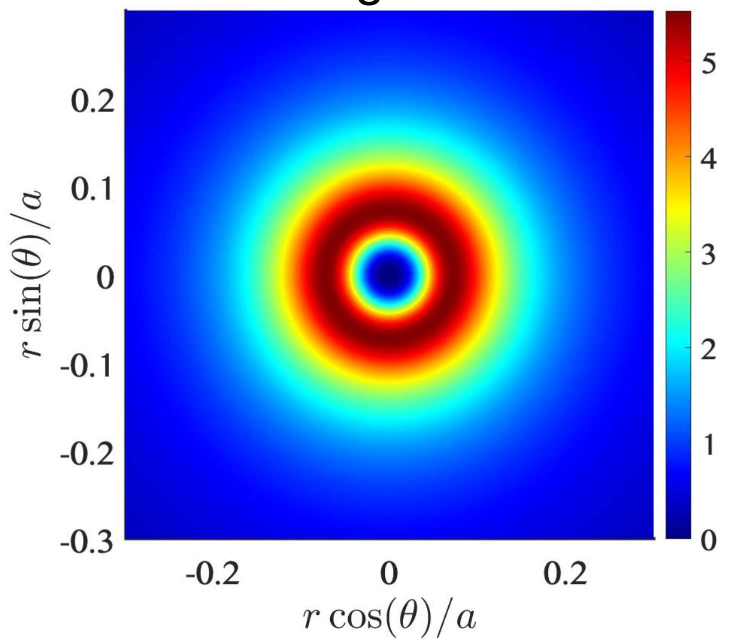


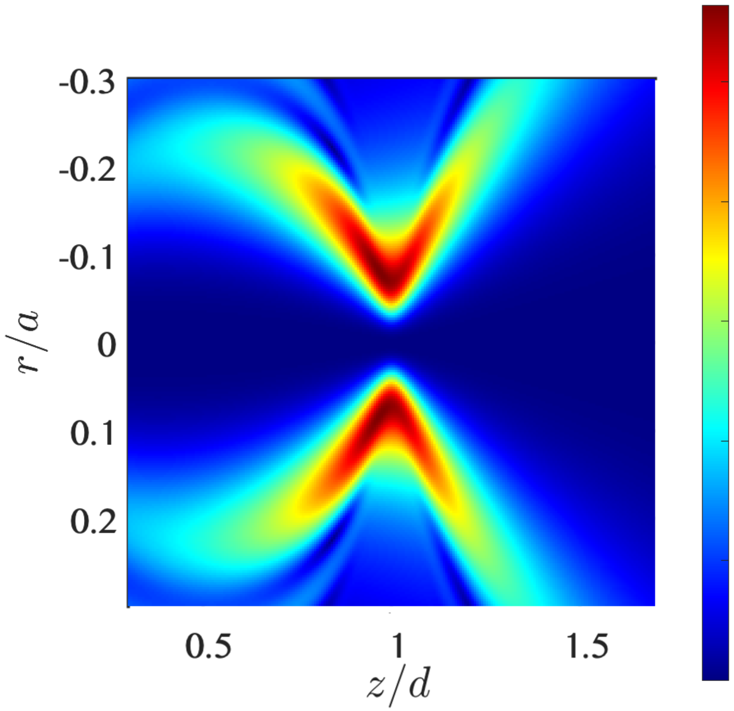
$$n = 3$$

$ka = 500,\ d/a = 10$

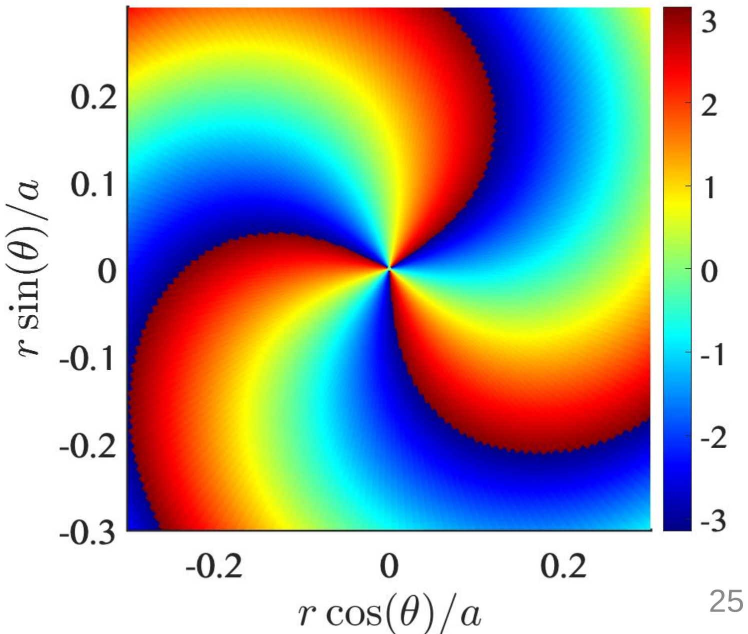


magnitude

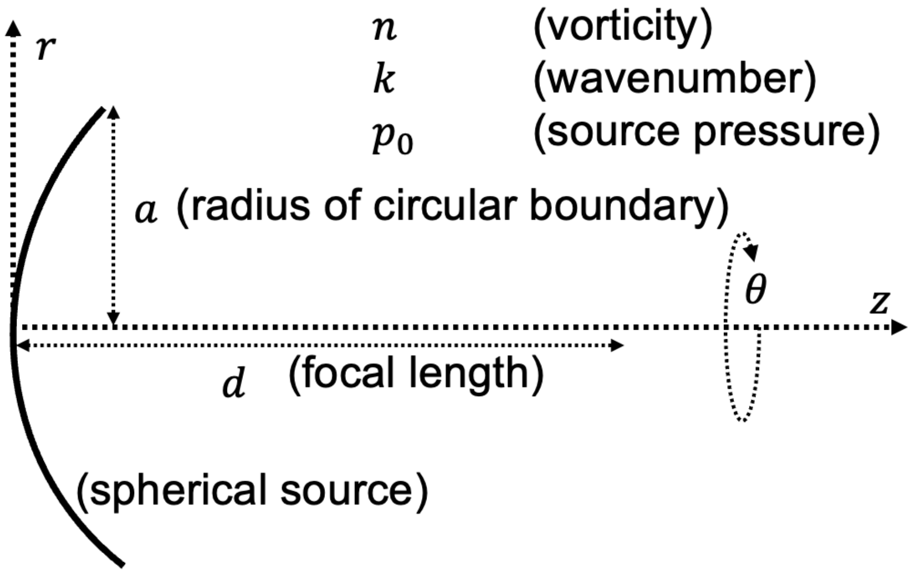




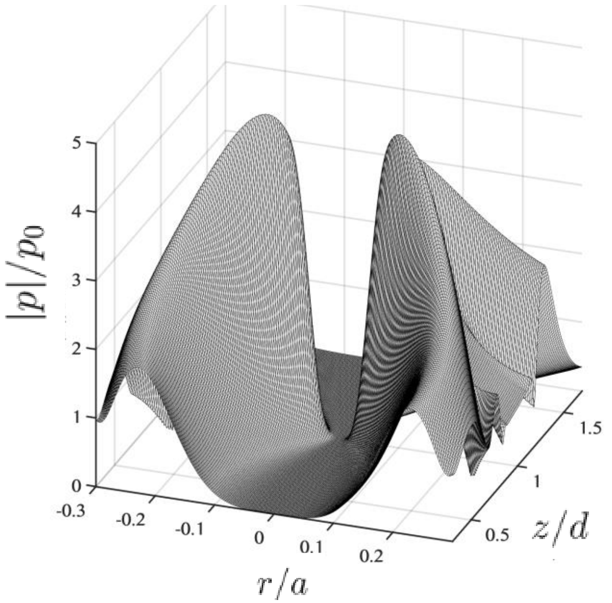
phase



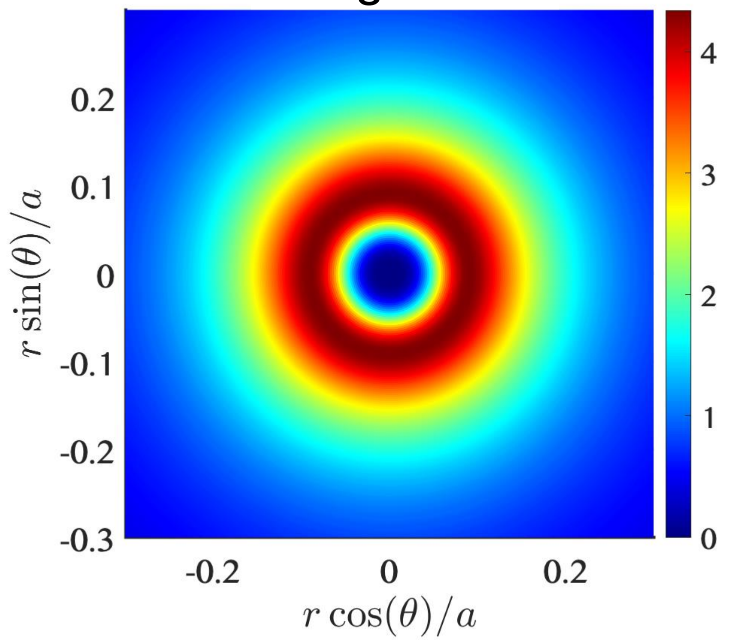
Field plots



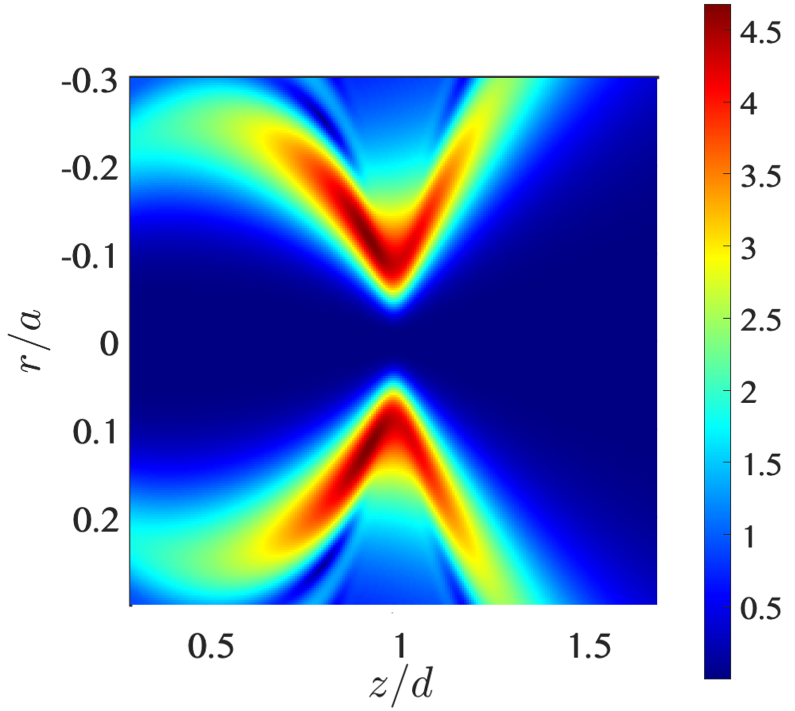
$ka = 500,\ d/a = 10$



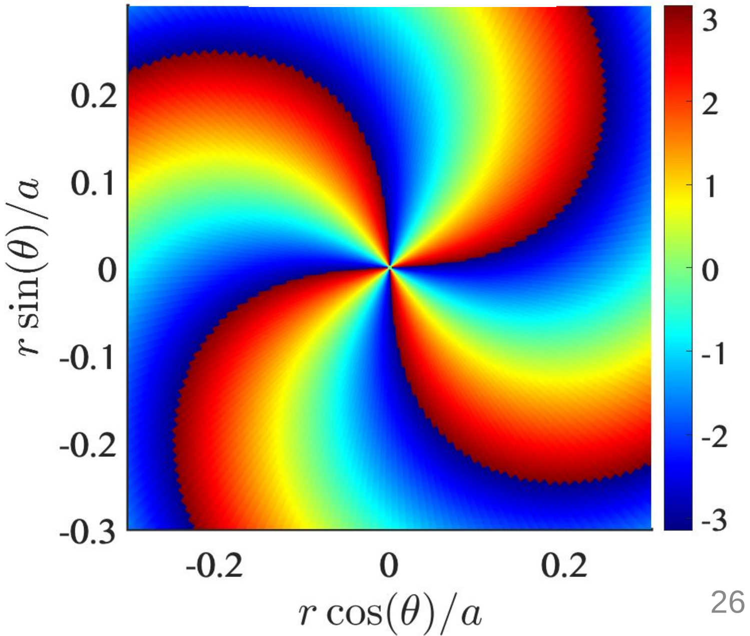
magnitude



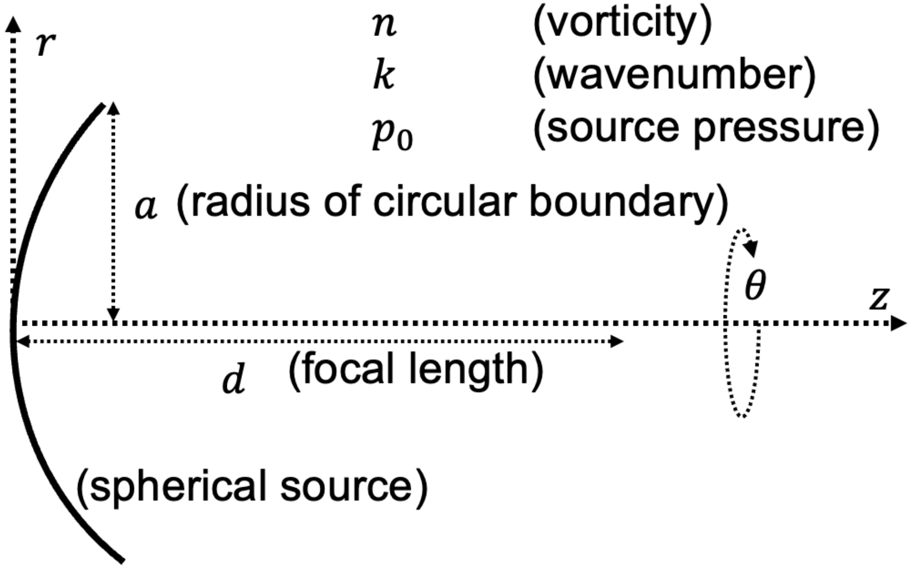
$$n = 4$$



phase

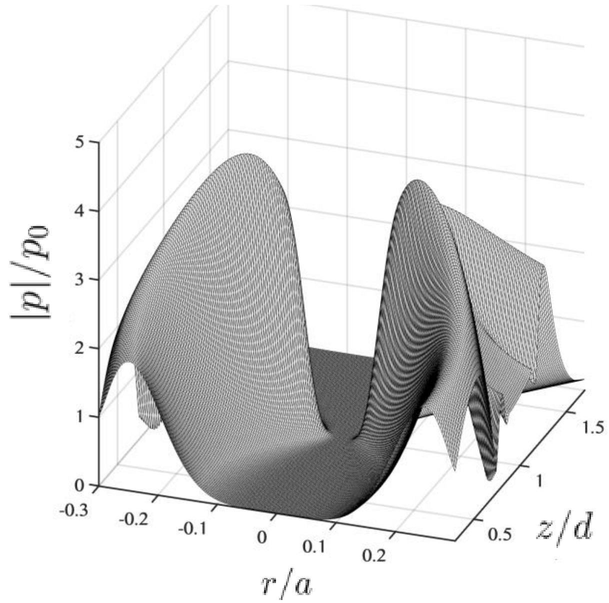


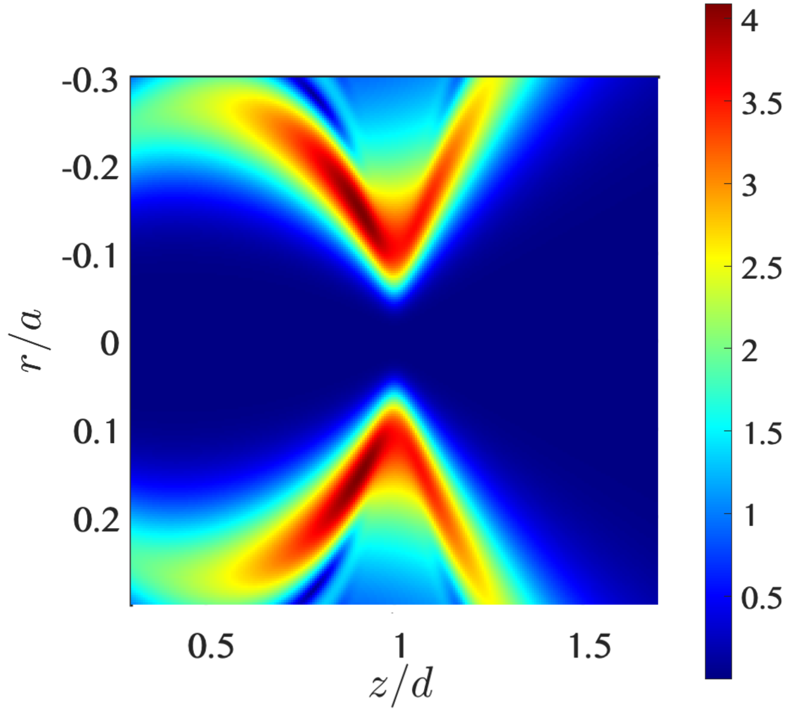
Field plots



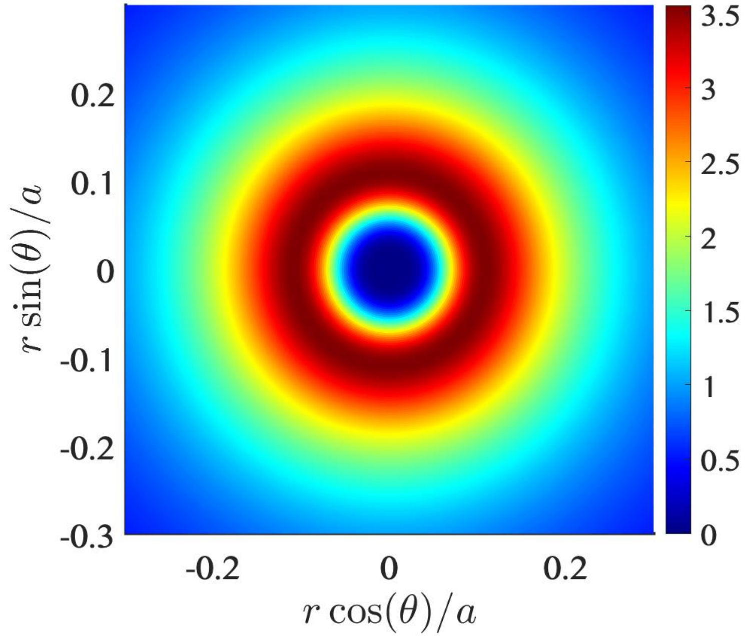
$$n = 5$$

$ka = 500,\ d/a = 10$

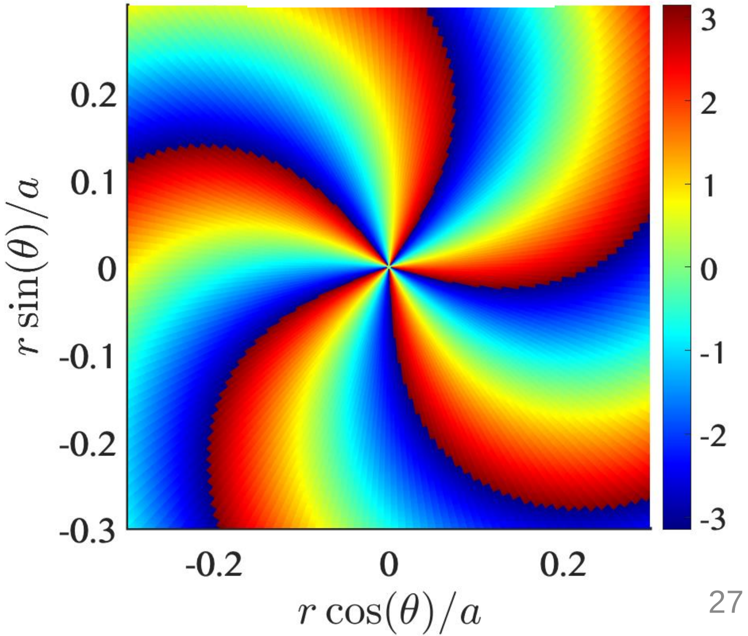




magnitude



phase

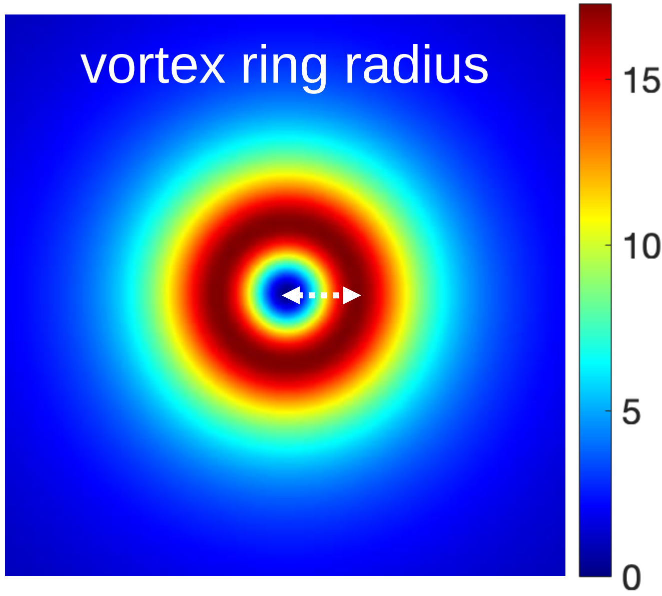


Scaling law for focal beamwidth

- In the focal plane $z = d$,

$$|p(\chi)| = (\pi/2)^{1/2} p_0 \frac{ka^2}{2d} \chi^{1/2} e^{-\chi} \left| I_{(n-1)/2}(\chi) - I_{(n+1)/2}(\chi) \right|$$

where $\chi = \frac{1}{8}\left(\frac{kar}{d}\right)^2$



Scaling law for focal beamwidth

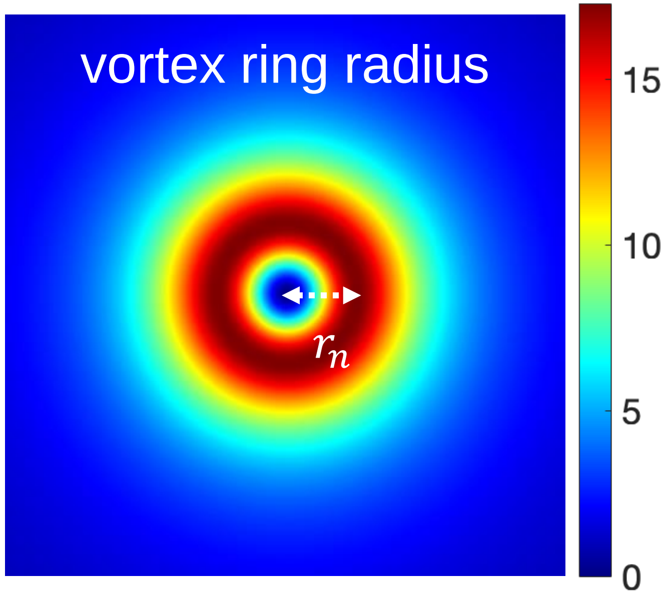
- In the focal plane $z = d$,

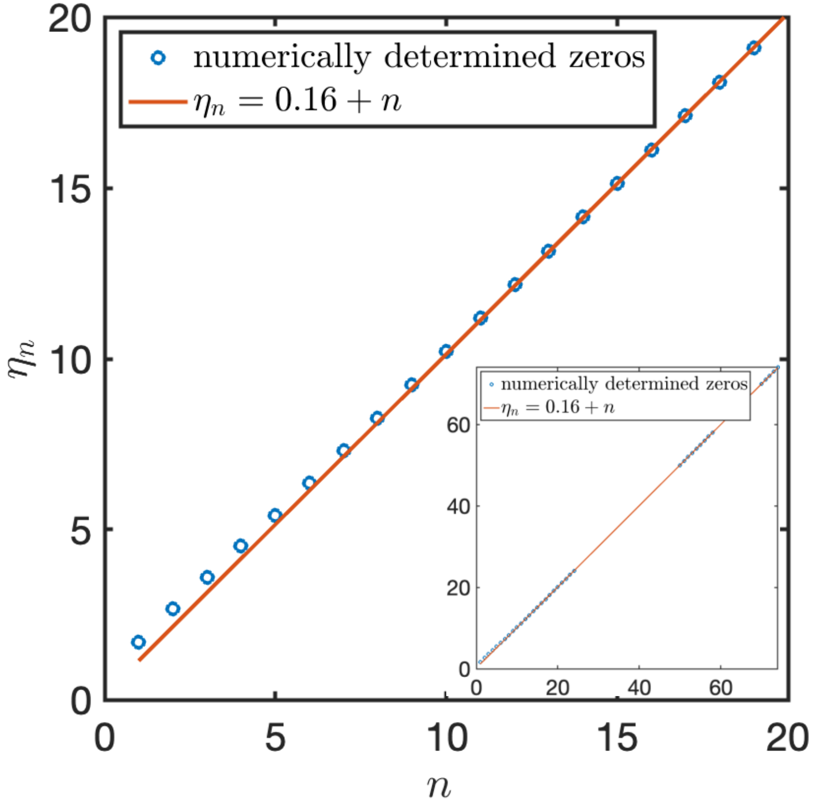
$$\left|p(\chi)\right| = (\pi/2)^{1/2} p_0 \frac{ka^2}{2d} \chi^{1/2} e^{-\chi} \left| I_{(n-1)/2}(\chi) - I_{(n+1)/2}(\chi) \right|$$

where $\chi = \frac{1}{8}\left(\frac{kar}{d}\right)^2$

- Solving $d|p(\chi)|/d\chi = 0$ for r_n leads to a scaling law:

$$r_n = \frac{\eta_n d}{ka} \quad \text{where} \quad \eta_n = 0.16 + n$$





Scaling law for focal beamwidth

- In the focal plane $z = d$,

$$|p(\chi)| = (\pi/2)^{1/2} p_0 \frac{ka^2}{2d} \chi^{1/2} e^{-\chi} \left| I_{(n-1)/2}(\chi) - I_{(n+1)/2}(\chi) \right|$$

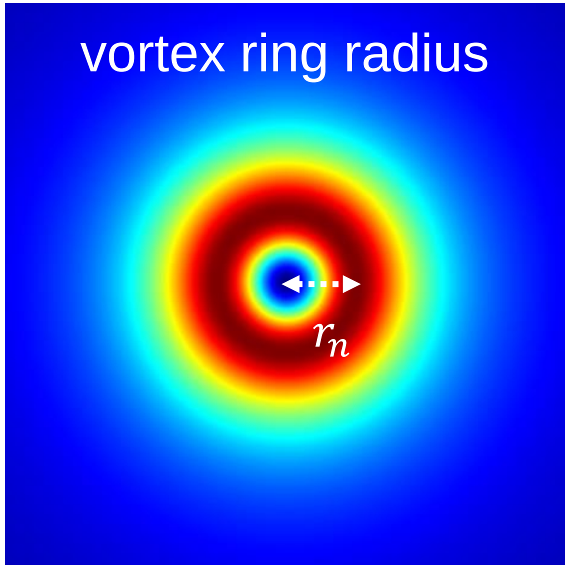
where $\chi = \frac{1}{8}\left(\frac{kar}{d}\right)^2$

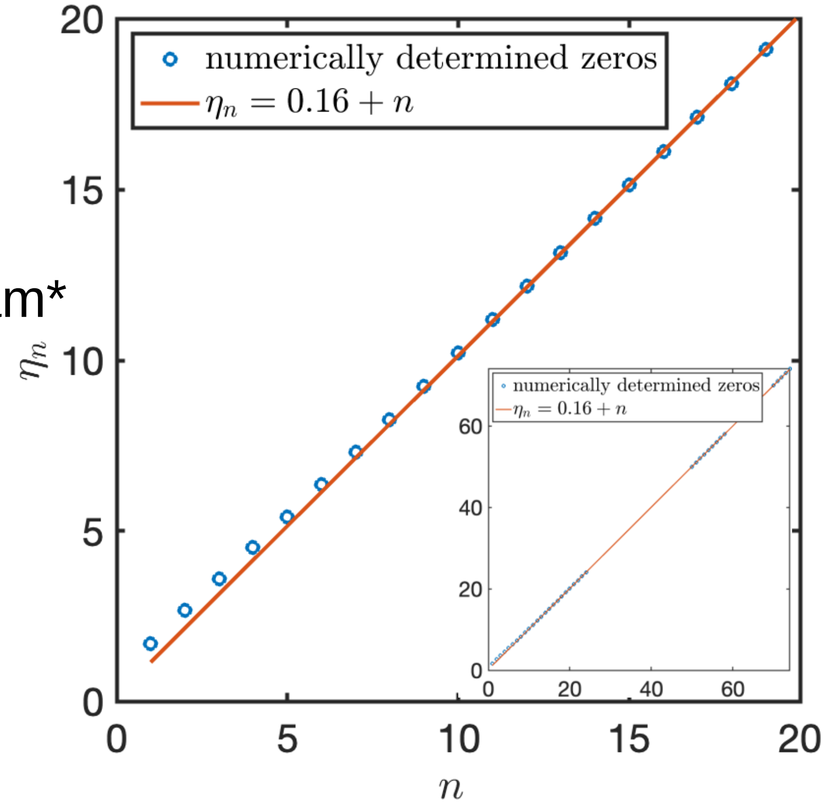
- Solving $d|p(\chi)|/d\chi = 0$ for r_n leads to a scaling law:

$$r_n = \frac{\eta_n d}{ka} \quad \text{where} \quad \eta_n = 0.16 + n$$

- Same functional dependence as for optical uniform vortex beam*

$$r_n = \frac{\eta_n d}{ka} \quad \text{where} \quad \eta_n = 1.29 + 0.13n$$





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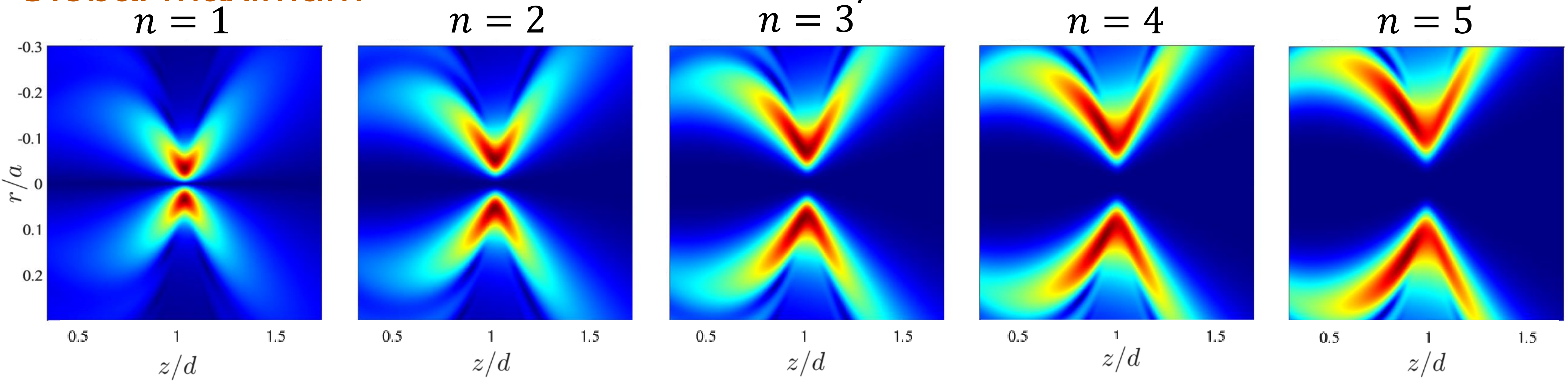
* Curtis et al., "Structure of Optical Vortices." PRL **90**, 13 (2003).

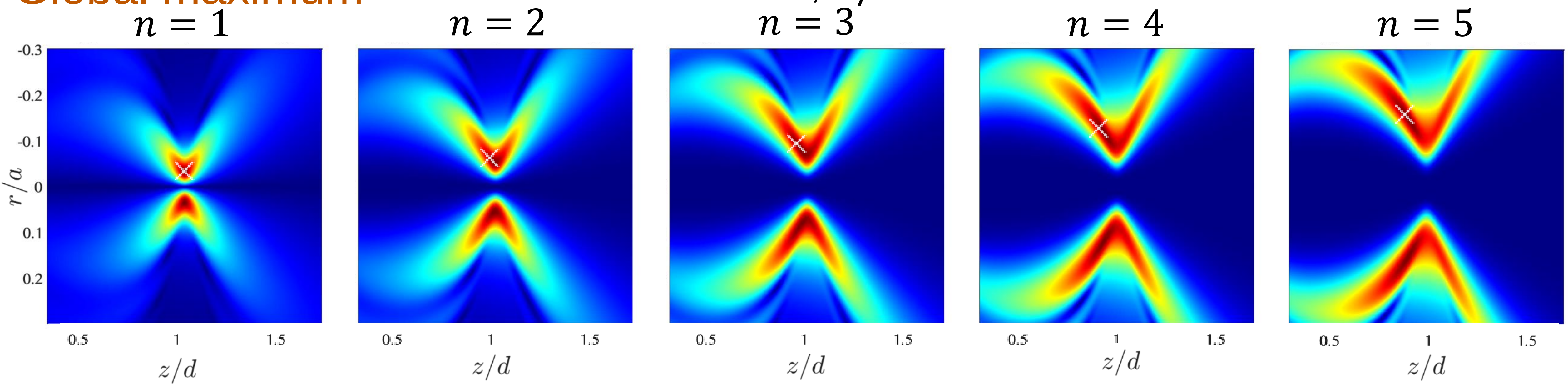
Global maximum

$ka = 500,\ d/a = 10$



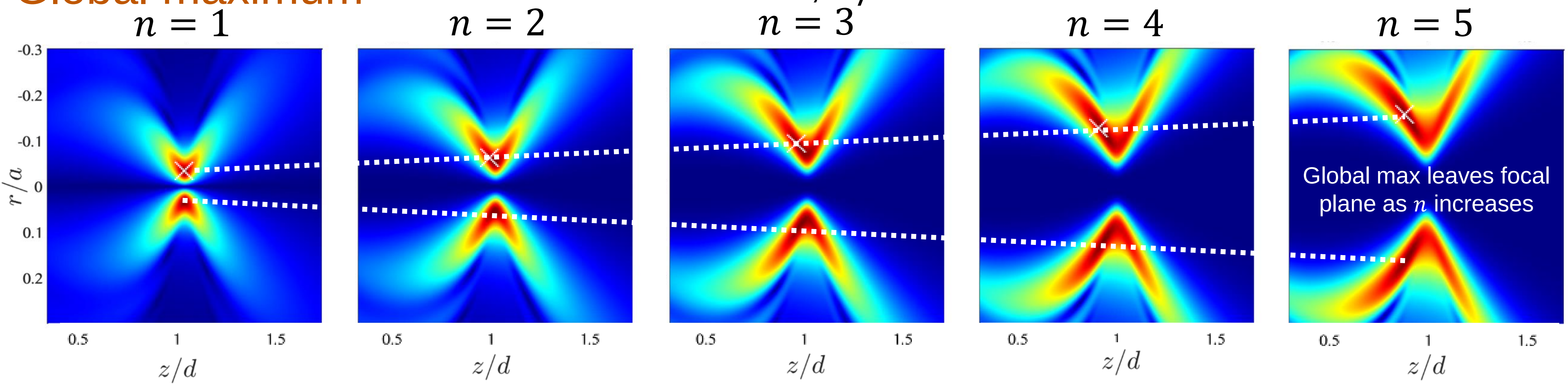
Global maximum

$ka = 500,\ d/a = 10$



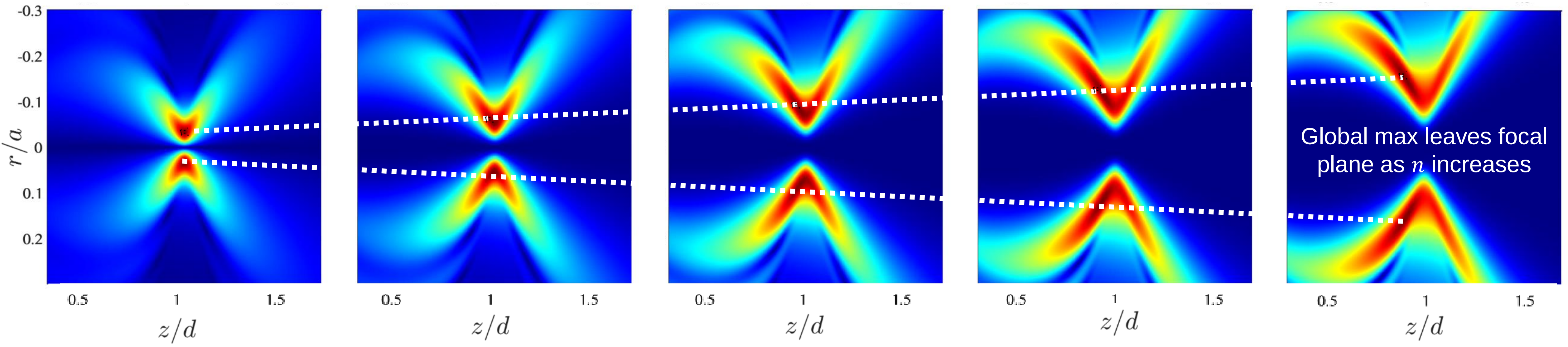
Global maximum

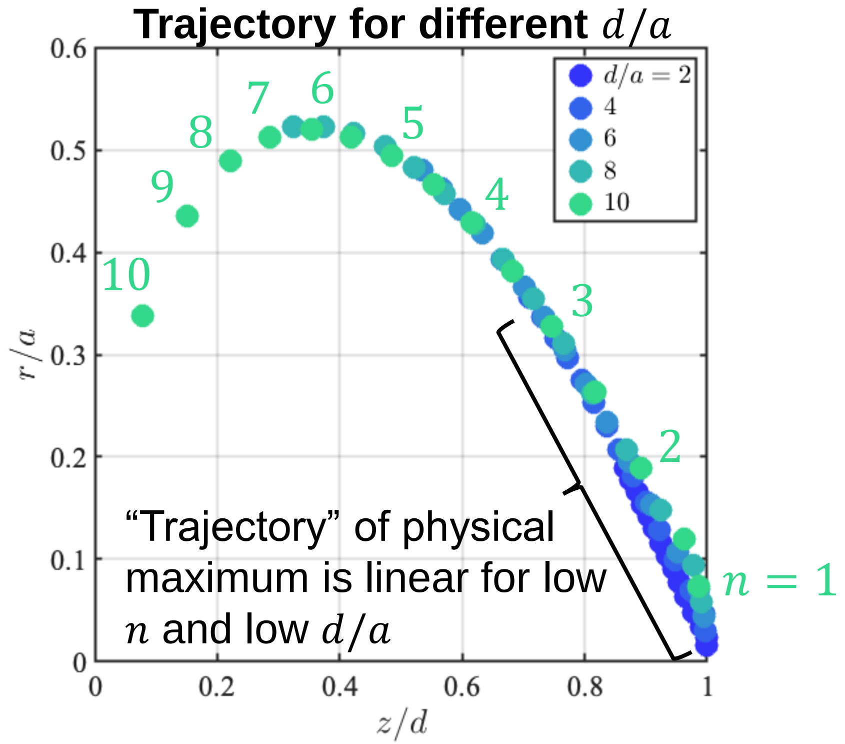
$ka = 500,\ d/a = 10$



Global maximum

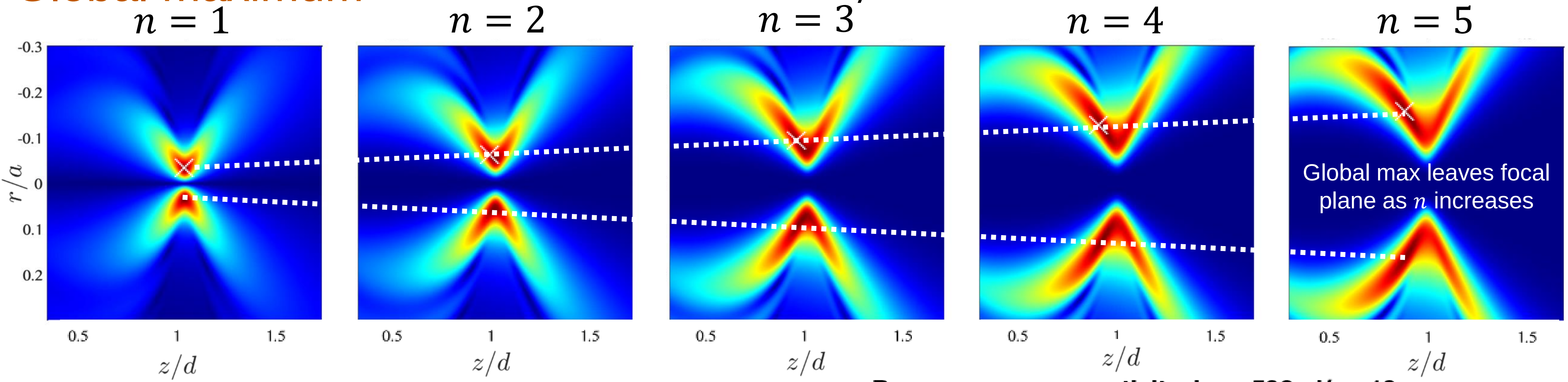
$ka = 500,\ d/a = 10$

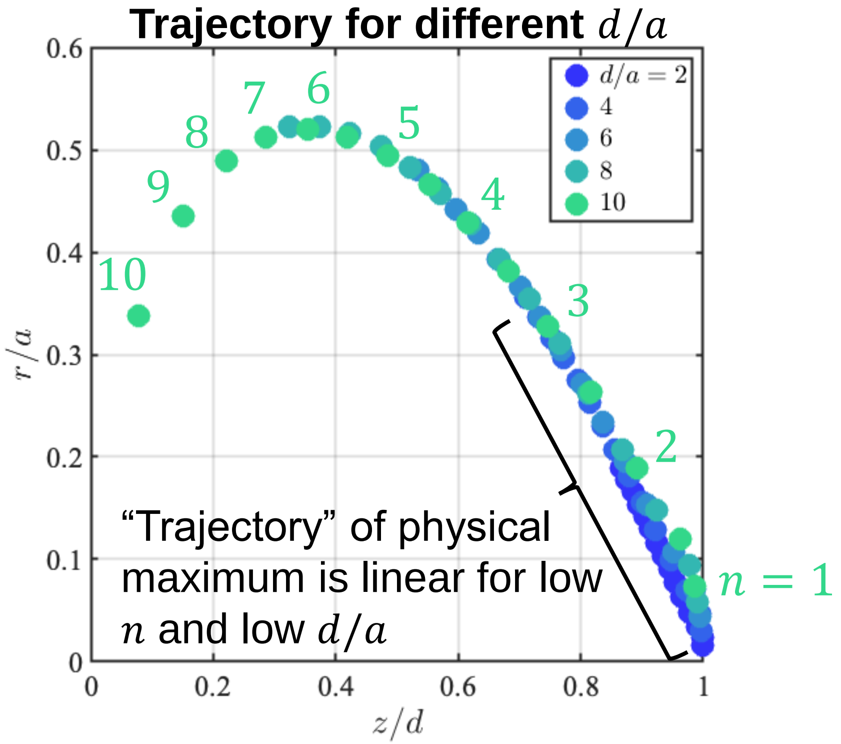


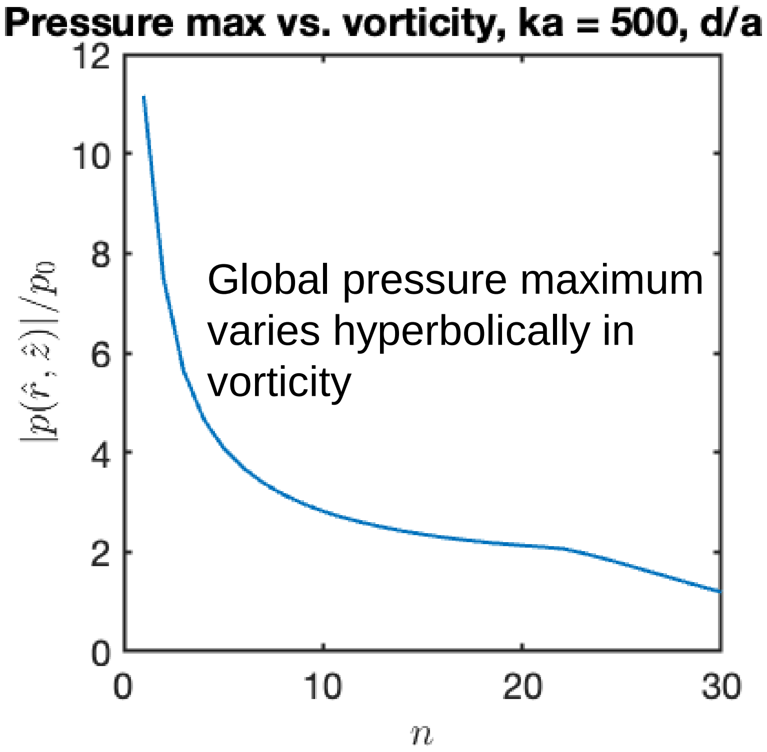


Global maximum

$ka = 500,\ d/a = 10$







Acknowledgments

- ARL:UT Chester M. McKinney Graduate Fellowship in Acoustics

Questions/Comments

- Thank you for your attention!

Extra slides