Born approximation of acoustic radiation force used for acoustofluidic separation

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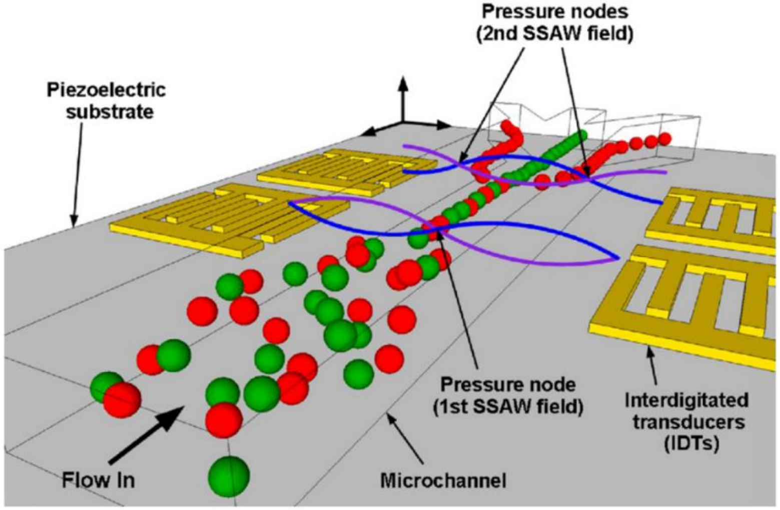
Session 4: Radiation Force

Outline

1. Radiation force on a layered sphere in a standing plane wave
2. Standing surface acoustic wave (SAW) acoustofluidic devices
3. Born analytical solution
4. Born vs. full theory
5. Layered vs. homogenized



Jo and Guldiken, Sens. Actuator A Phys. **187**, 22-28 (2012)

Radiation force on a layered cell in a 1D standing plane wave

Compare Born approximation with full theory for force on a layered cell

- Full theory [Wang et al., J. Appl. Phys. (2017); Ilinskii et al., JASA (2018)]

$$F_z = \frac{i\pi}{\rho_0 c_0^2 k_0^2} \sum_{n=0}^{\infty} \frac{(n+1)}{(2n+1)(2n+3)} a_n^* a_{n+1} (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.}$$

Spherical wave expansions of incident and scattered fields

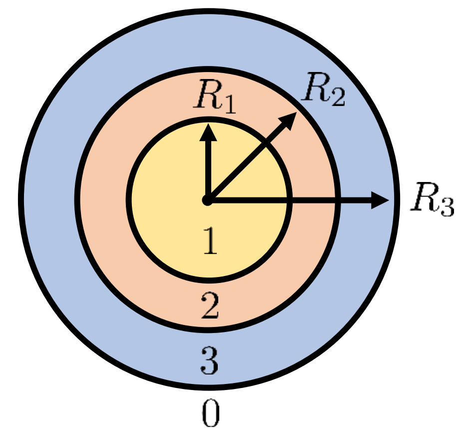
- Born approximation

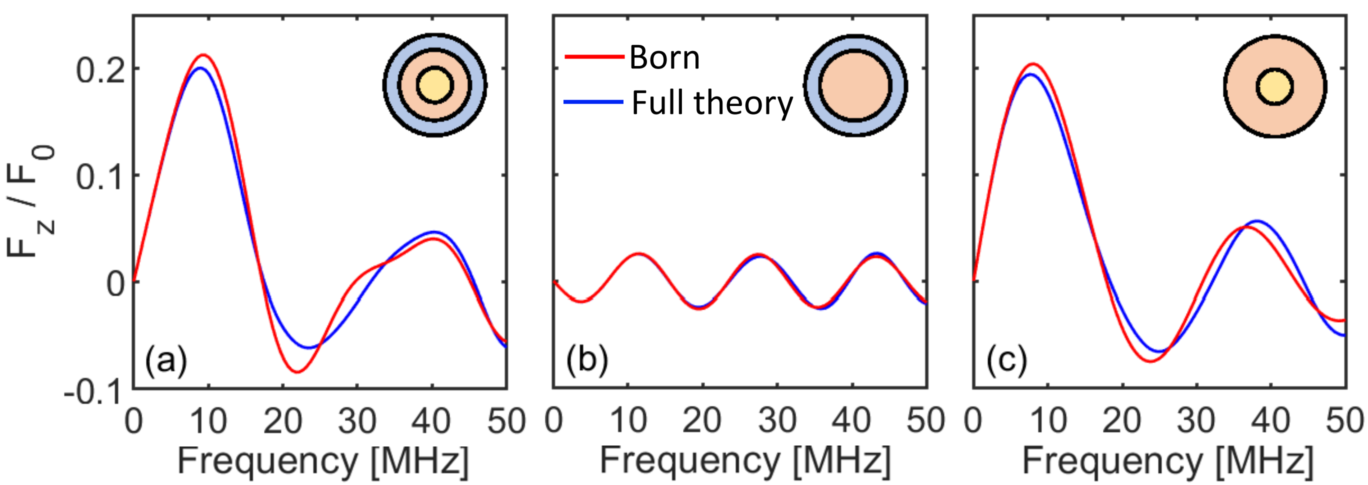
$$F_z = \frac{\pi p_0^2}{2\rho_0 c_0^2} \Phi \sin 2k_0 d,$$

$$\Phi = f_{G,1} R_1^2 j_1(2k_0 R_1) + \sum_{n=1}^{N} f_{G,n+1} \left[R_{n+1}^2 j_1(2k_0 R_{n+1}) - R_n^2 j_1(2k_0 R_n) \right]$$

  - Closed-form analytical result
  - Obtained by summing forces on components of cell

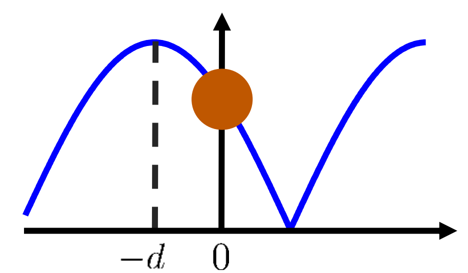
Layered cell:





Standing plane wave:

$$|p_0 \cos[k_0(z+d)]|$$



$$F_0 = \frac{p_0^2 S_0}{4\rho_0 c_0^2}, \quad k_0 d = \pi/4$$

Scattering coefficients required in full solution (Wang et al., 2017)

$$A_{n.s} = -\frac{\rho_3 c_3 j_n'(z_0)[Q_2 j_n(z_3) - Y_n(z_3)] - \rho_4 c_4 j_n(z_0)\left[Q_2 j_n'(z_3) - Y_n'(z_3)\right]}{\rho_3 c_3 h_n'(z_0)[Q_2 j_n(z_3) - Y_n(z_3)] - \rho_4 c_4 h_n(z_0)\left[Q_2 j_n'(z_3) - Y_n'(z_3)\right]},$$

$$Q_1 = \frac{\rho_1 c_1 Y_n'(x_2) j_n(x_1) - \rho_2 c_2 Y_n(x_2) j_n'(x_1)}{\rho_1 c_1 j_n'(x_2) j_n(x_1) - \rho_2 c_2 j_n(x_2) j_n'(x_1)},$$

$$Q_2 = \frac{\rho_2 c_2 [Q_1 j_n(y_2) - Y_n(y_2)] Y_n'(y_3) - \rho_3 c_3 \left[Q_1 j_n'(y_2) - Y_n'(y_2)\right] Y_n(y_3)}{\rho_2 c_2 [Q_1 j_n(y_2) - Y_n(y_2)] j_n'(y_3) - \rho_3 c_3 \left[Q_1 j_n'(y_2) - Y_n'(y_2)\right] j_n(y_3)},$$

$$x_1 = k_1 r_1,\ x_2 = k_2 r_1,\ y_2 = k_2 r_2,\ y_3 = k_3 r_2,\ z_0 = k_4 r_3,\ \text{and } z_3 = k_3 r_3.$$

$$Y_p = -\frac{4}{(k_4 r_3)^2} \sum_{n=0}^{\infty} (n+1) \times \left[\mathrm{Re}(\alpha_n + \alpha_{n+1} + 2\alpha_n \alpha_{n+1} + 2\beta_n \beta_{n+1}) + \mathrm{Im}(\beta_{n+1}(1 + 2\alpha_n) - \beta_n(1 + 2\alpha_{n+1}))\right],$$

Wang et al., J. Appl. Phys. **122**, 094902 (2017)

Surface acoustic wave acoustofluidic device (Peng et al., 2020)

2 *X. Peng, W. He and F. Xin et al./Journal of the Mechanics and Physics of Solids 145 (2020) 104134*

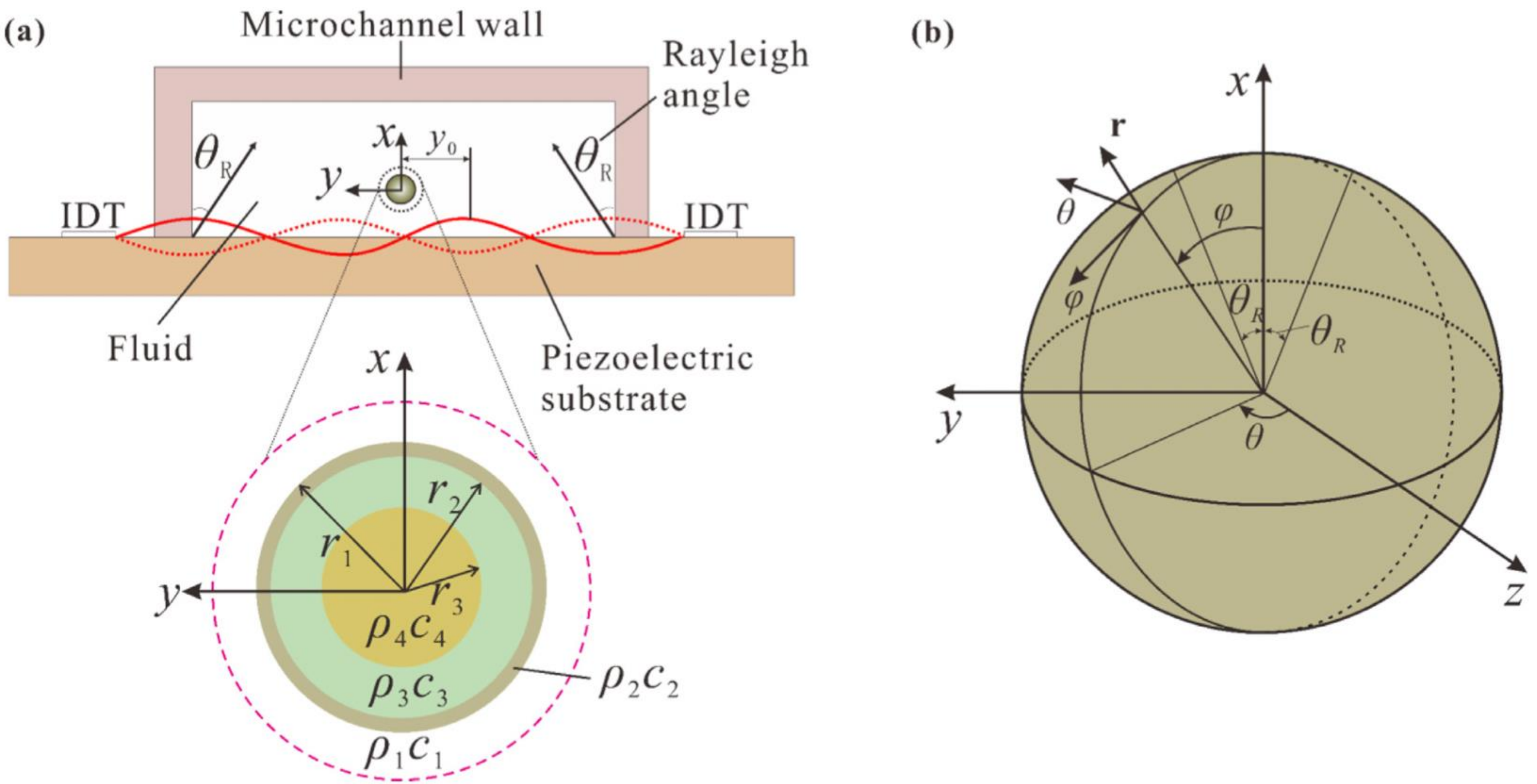
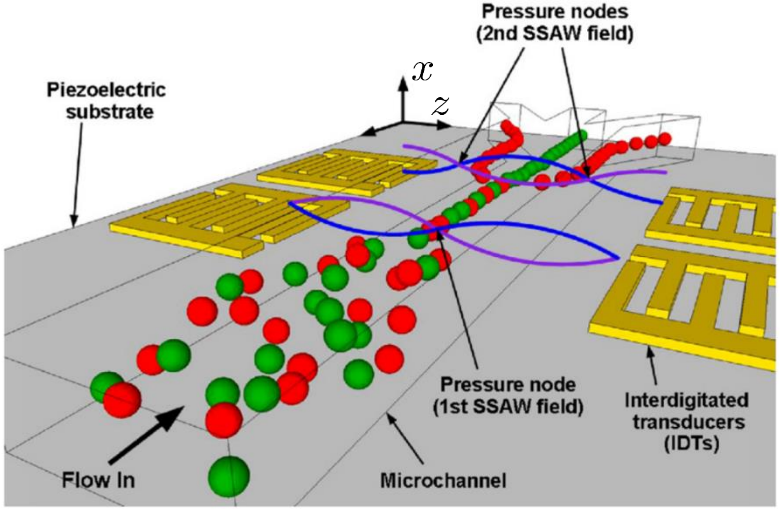


Fig. 1. (a) Schematic of a SSAW incident upon a three-layered model of a eukaryotic cell. (b) The origin of the local spherical coordinate system (r, θ, φ) resides at the instantaneous center of the eukaryotic cell.

Surface acoustic wave acoustofluidic device: Radiation force due to noncollinear plane waves

- Configuration considered by Peng et al. [J. Mech. Phys. Solids **145**, 104134 (2020)]



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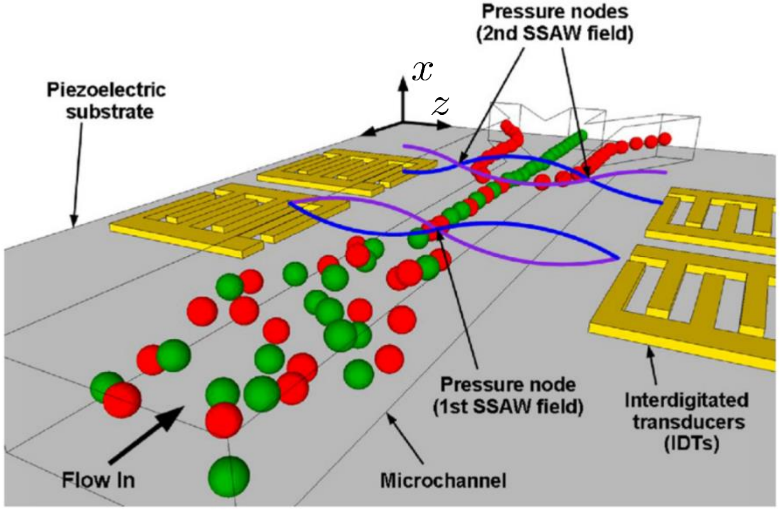

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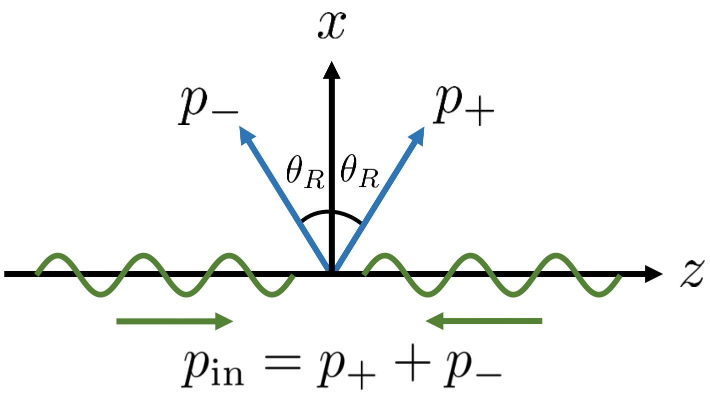

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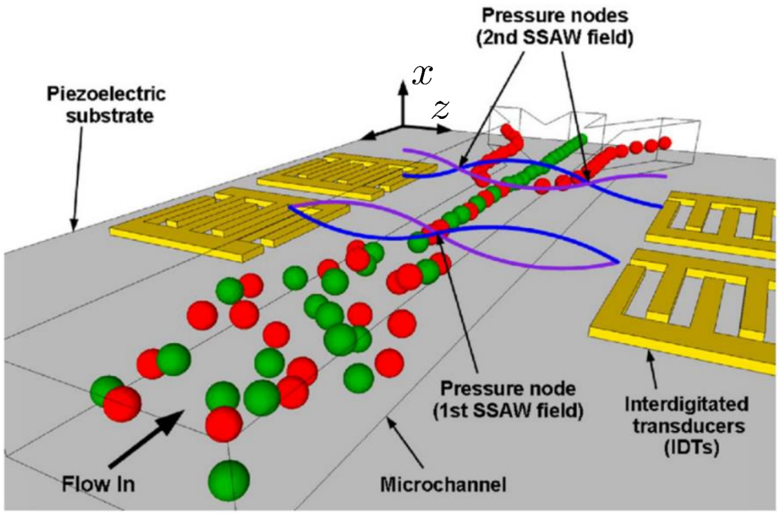

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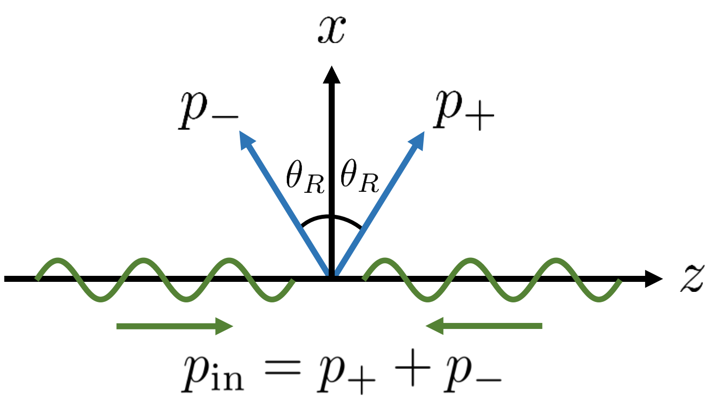

$$p_{\text{in}} = 2p_0 \underline{\cos[k_0(z+d)\sin\theta_R]} e^{\underline{i(k_0 x \cos\theta_R - \omega t)}}$$

- Standing wave in z direction (Horizontal)
- Traveling in x direction (Vertical)

Surface acoustic wave acoustofluidic device: Radiation force due to noncollinear plane waves

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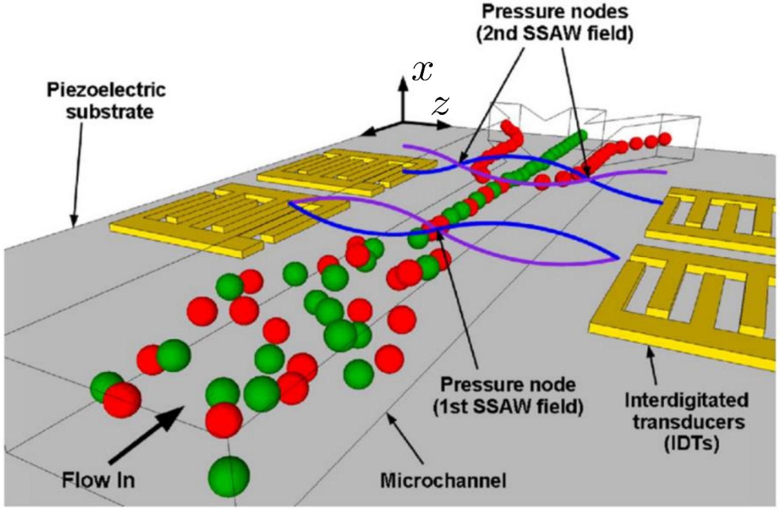

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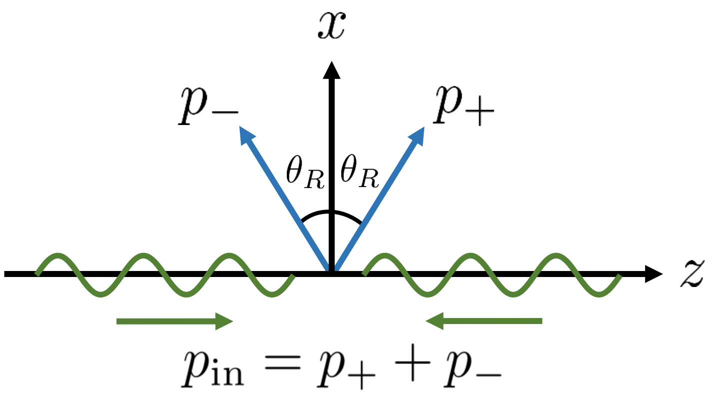

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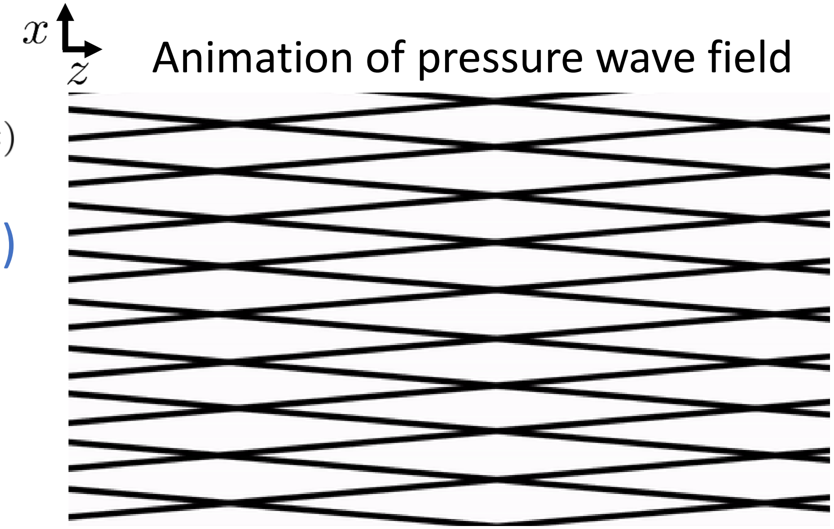
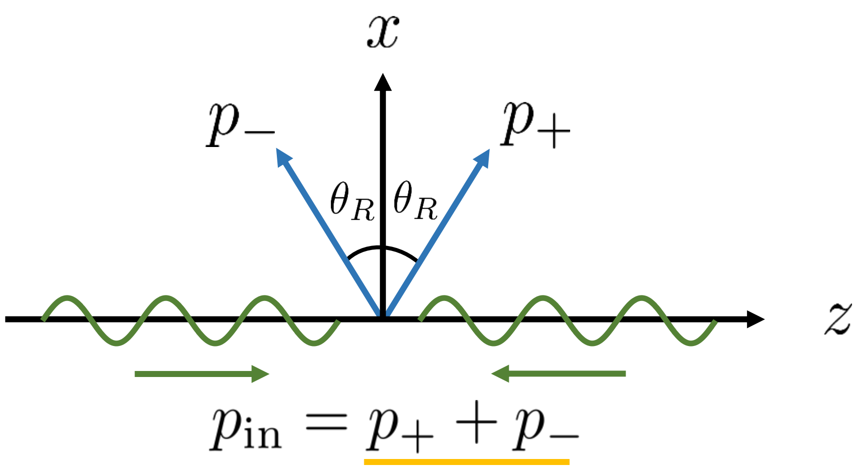


Surface acoustic wave acoustofluidic device: Full solution

- Plane wave radiated into fluid by traveling surface acoustic wave



- Horizonal radiation force (full theory):

$$F_z = \frac{i\pi}{\rho_0 c_0^2 k_0^2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(n+m+1)(n-m)!}{(2n+1)(2n+3)(n-m)!} a_n^{m*} a_{n+1}^m (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.}$$

Given by Wang et al.

$$p_{\rm in} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (a_{n+}^m + a_{n-}^m) j_n(k_0 r) P_n^m(\cos\theta) e^{im\phi}$$

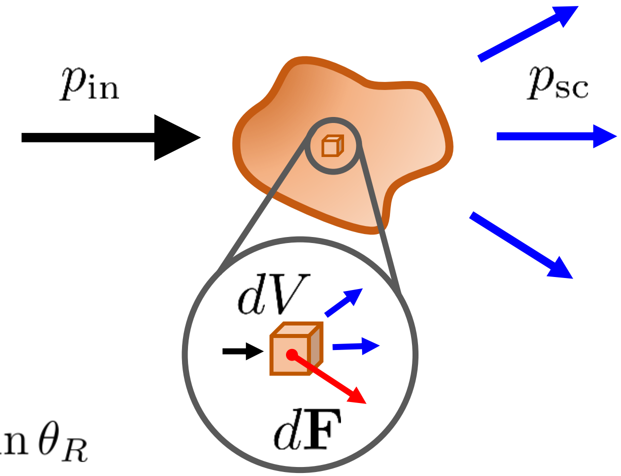
$$a_{n\pm}^m = p_0 (2n+1) i^n \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta_\pm) e^{im\phi_\pm}$$

- Vertical radiation force not described by Born approximation

Born approximation of horizontal radiation force

- Incident field: $p_{\mathrm{in}} = 2p_0 \cos[k_0(z+d)\sin\theta_R]e^{i(k_0 x\cos\theta_R - \omega t)}$
- Radiation force on a volume element

$$d\mathbf{F} = -\left[f_K \boldsymbol{\nabla}\langle E_p\rangle - \tfrac{3}{2} f_\rho \boldsymbol{\nabla}\langle E_k\rangle\right] dV$$
$$= \frac{p_0^2 \tilde{k}}{2\rho_0 c_0^2} \tilde{f}_G \sin[2\tilde{k}(z+d)]\mathbf{e}_z$$



- Same form as result for 1D standing wave with

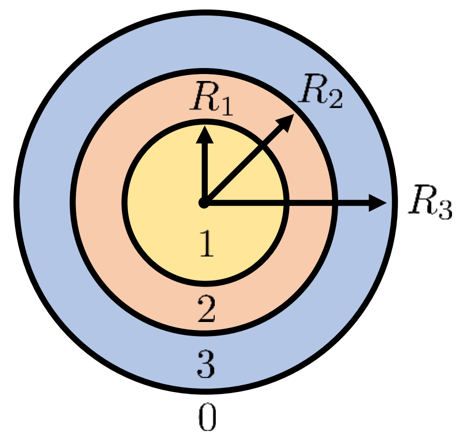
$$f_G \to \tilde{f}_G = f_K - \frac{3}{2} f_\rho \sin 2\theta_R, \quad k_0 \to \tilde{k} = k_0 \sin\theta_R$$

- Layered cell

$$F_z = \frac{\pi p_0^2}{2\rho_0 c_0^2}\Phi \sin 2\tilde{k}d,$$

$$\Phi = \tilde{f}_{G,1} R_1^2 j_1(2\tilde{k}R_1) + \sum_{n=1}^{N} \tilde{f}_{G,n+1}\left[R_{n+1}^2 j_1(2\tilde{k}R_{n+1}) - R_n^2 j_1(2\tilde{k}R_n)\right]$$

n	Material	c_n [m/s]	ρ_n [kg/m^3]	R_n [μm]
1	Nucleus	1508.5	1430	6
2	Cytoplasm	1508	1139	14
3	Cell wall	1450	970	15
0	Water (ref)	1500	1000	--



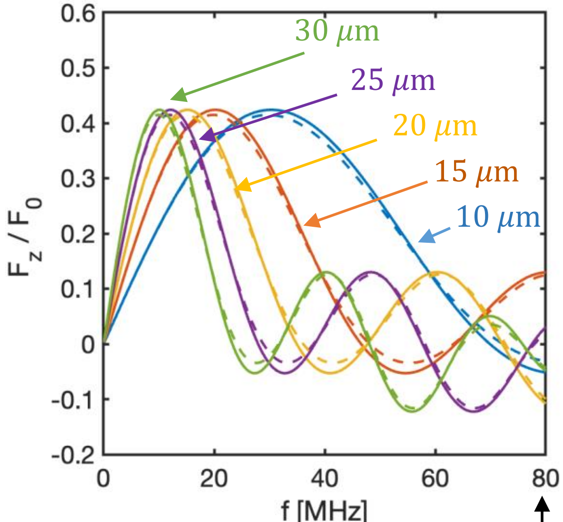
Peng et al. [J. Mech. Phys. Solids **145**, 104134 (2020)]

Born approximation vs. full theory for horizontal force

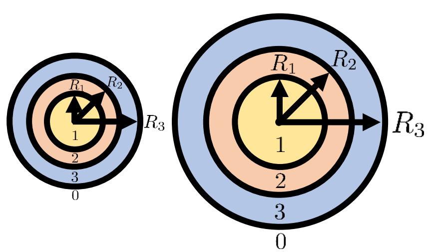
——— Born approximation

- - - - - - - - Full theory

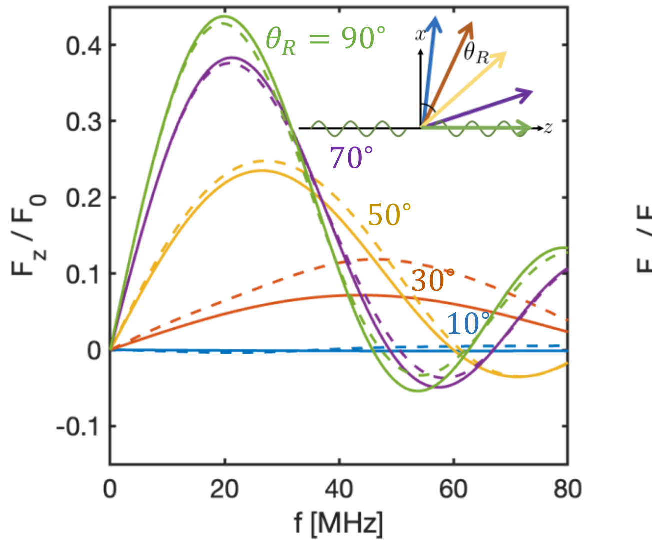
Dependence on outer radius R_3



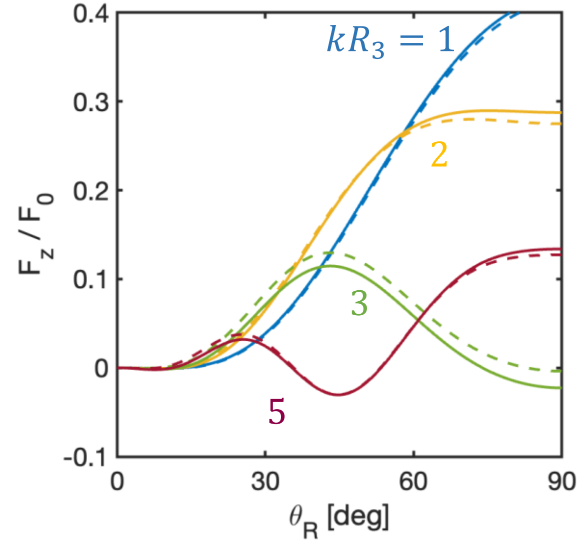
corresponds to $kR_3 \simeq 5$ for $R_3 = 30\ \mu$m ($2R_3 \simeq 2\lambda$)



Dependence on Rayleigh (intersection) angle θ_R



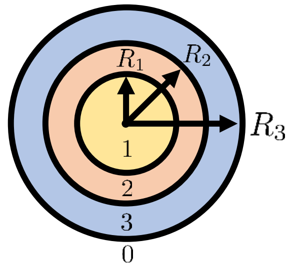
Dependence on kR_3



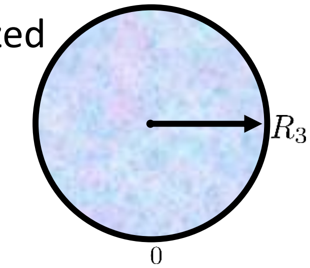
Comparison of Born approximation for layered vs. homogenized cell

- Homogenization is based on volume average of compressibilities and densities of the layers:

——— Layered



- - - - - - Homogenized



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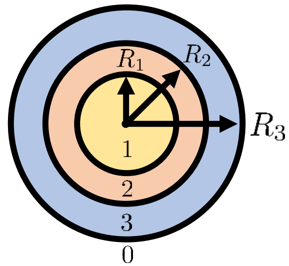
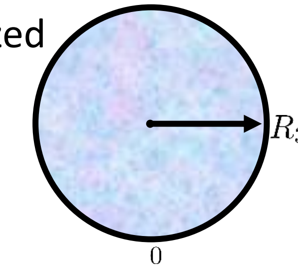
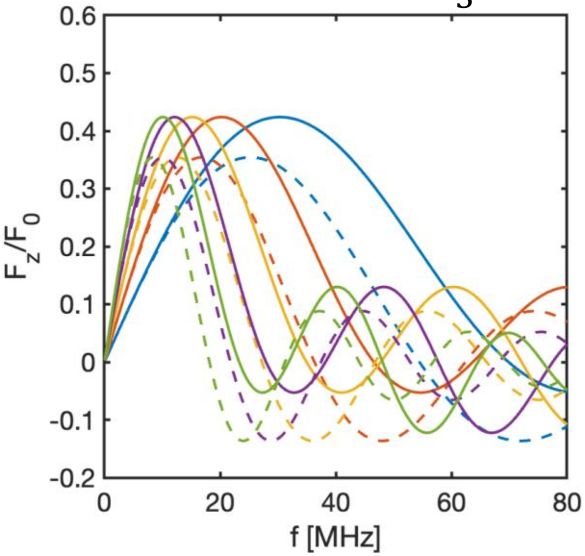


- - - - - - Homogenized

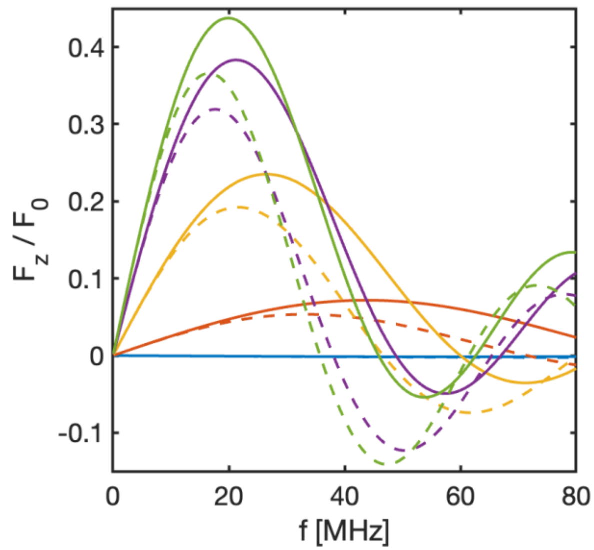


Dependence on outer radius R_3



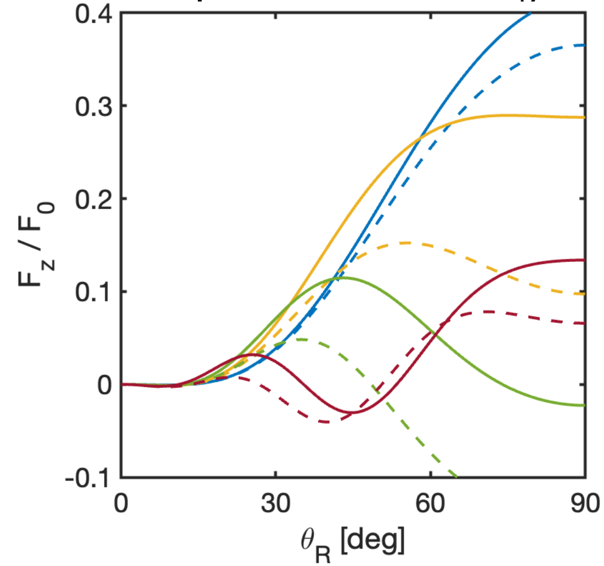
$10\ \mu m \leq R_3 \leq 30\ \mu m$

Dependence on Rayleigh (intersection) angle θ_R



$10^\circ \leq \theta_R \leq 90^\circ$

Dependence on kR_3

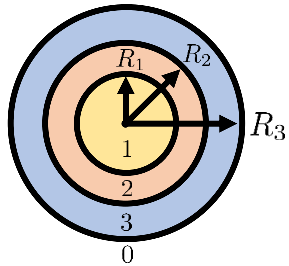


$1 \leq kR_3 \leq 5$

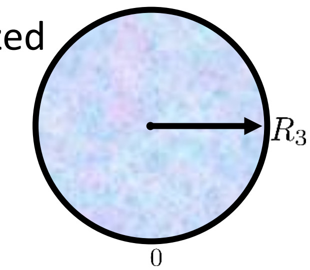
Comparison of Born approximation for layered vs. homogenized cell

- Homogenization is based on volume average of compressibilities and densities of the layers:

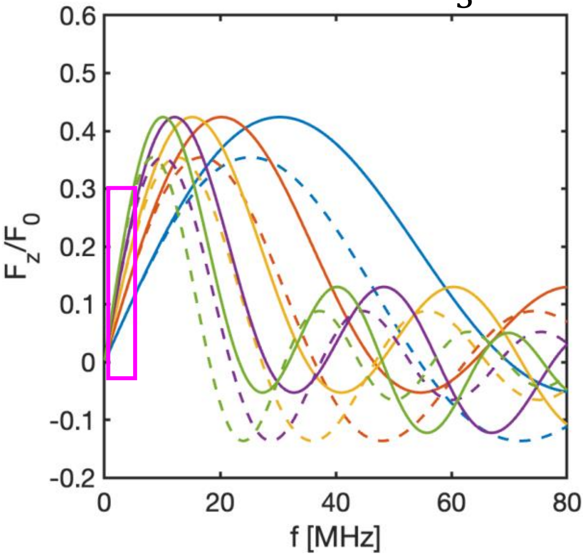
——— Layered



- - - - - - Homogenized

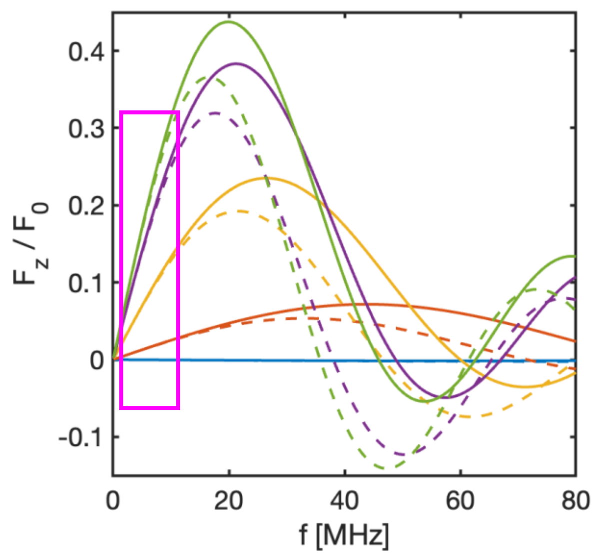


Dependence on outer radius R_3



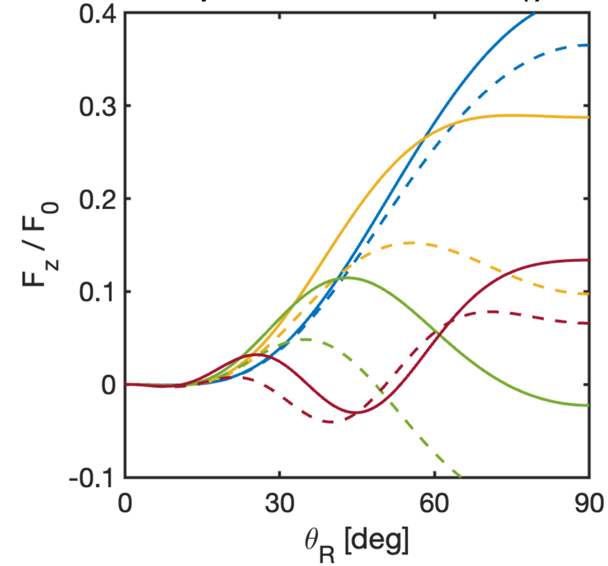
$10\ \mu m \le R_3 \le 30\ \mu m$

Dependence on Rayleigh (intersection) angle θ_R



$10^\circ \le \theta_R \le 90^\circ$

Dependence on kR_3



$1 \le kR_3 \le 5$

Summary

- Closed form solutions often available in Born approximation
- Accurately describes horizontal forces in acoustofluidic devices
- Accurate for inhomogeneous objects on the order of one wavelength
- Homogenization inaccurate unless $kR << 1$
- Normal restriction applies—acoustic contrast of objects must not differ substantially from that of host fluid

This work supported by the Applied Research Laboratories Chester M. McKinney Graduate Fellowship in Acoustics.

Extra slides

EXTRA: Radiation force on a nucleated cell

Compare Born approximation with theory for force on a nucleated cell*

- Based on theory for radiation force on a sphere

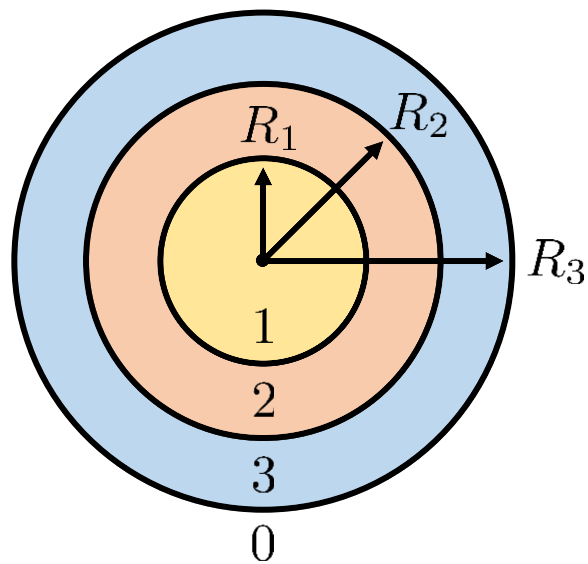
$$F_z = \frac{i\pi}{\rho_0 c_0^2 k_0^2} \sum_{n=0}^{\infty} \frac{(n+1)}{(2n+1)(2n+3)} a_n^* a_{n+1} (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) + \text{c.c.}$$

- Spherical wave expansion coefficients a_n for axisymmetric incident field
- Scattering coefficients A_n for spherically symmetric scatterer

*Wang et al., J. Appl. Phys. **122**, 094902 (2017)

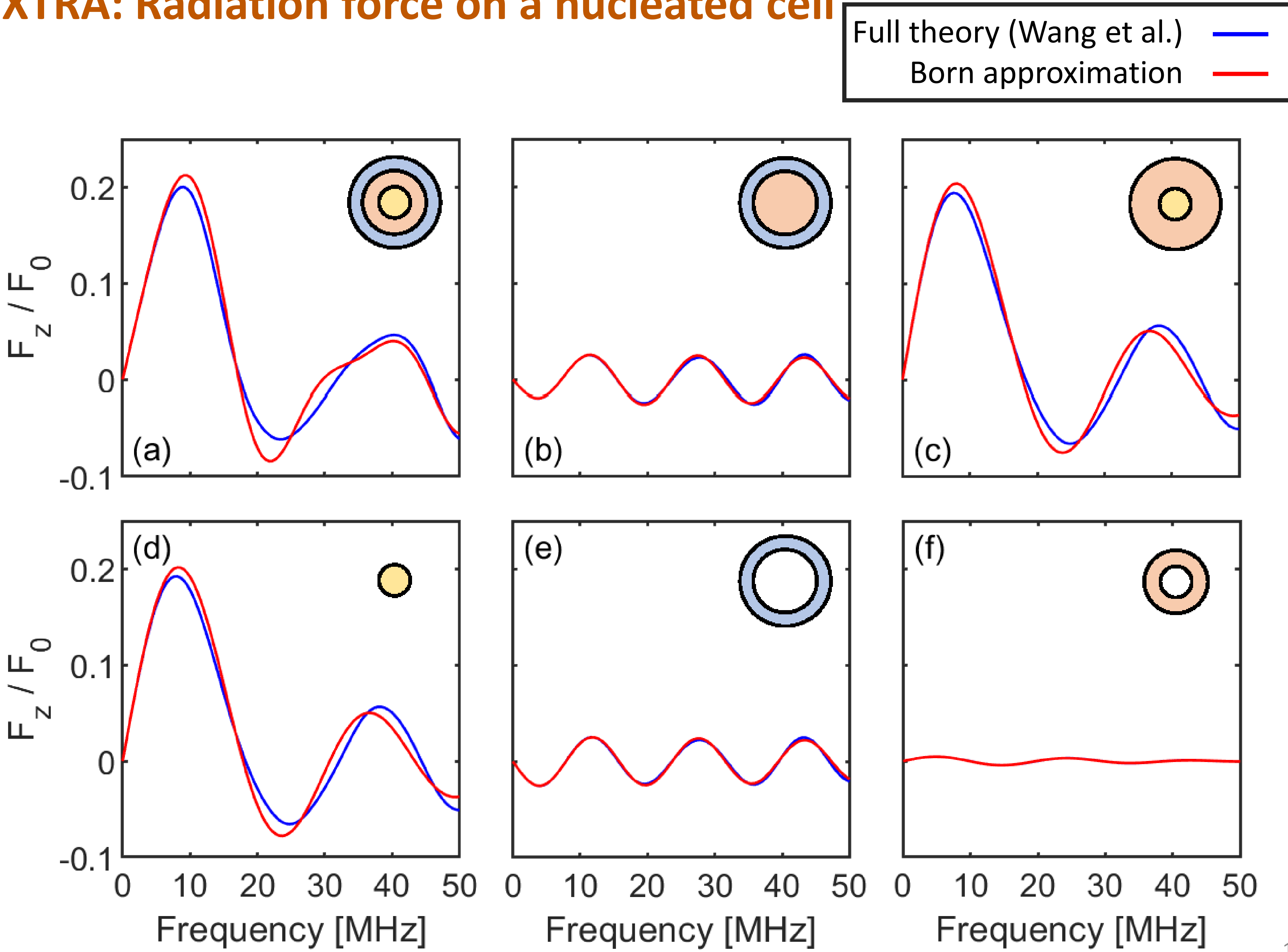
- Cell modeled as a sphere surrounded by $N = 2$ layers:

	Material	$f_{G,n}$
1	Nucleus	0.6428
2	Cytoplasm	0.0106
3	Cell membrane	-0.1339
0	Water (reference)	--



- Neglect shear and absorption in cell

EXTRA: Radiation force on a nucleated cell



Extra: Wang et al. full theory

$$p_i = p_0 e^{-i\omega t} \sum_{n=0}^{\infty} (2n+1) i^n j_n(k_4 r) P_n(\cos\theta),$$

$$p_s = p_0 e^{-i\omega t} \sum_{n=0}^{\infty} (2n+1) i^n A_{n.s} h_n^{(1)}(k_4 r) P_n(\cos\theta),$$

$$p_1 = p_0 e^{-i\omega t} \sum_{n=0}^{\infty} (2n+1) i^n P_n(\cos\theta)[F_n j_n(k_1 r)], \qquad (3)$$

$$p_2 = p_0 e^{-i\omega t} \sum_{n=0}^{\infty} (2n+1) i^n P_n(\cos\theta)[D_n j_n(k_2 r) + E_n Y_n(k_2 r)], \qquad (4)$$

$$p_3 = p_0 e^{-i\omega t} \sum_{n=0}^{\infty} (2n+1) i^n P_n(\cos\theta)[B_n j_n(k_3 r) + C_n Y_n(k_3 r)]. \qquad (5)$$

$$A_{n.s} = -\frac{\rho_3 c_3 j_n'(z_0)[Q_2 j_n(z_3) - Y_n(z_3)] - \rho_4 c_4 j_n(z_0)\left[Q_2 j_n'(z_3) - Y_n'(z_3)\right]}{\rho_3 c_3 h_n'(z_0)[Q_2 j_n(z_3) - Y_n(z_3)] - \rho_4 c_4 h_n(z_0)\left[Q_2 j_n'(z_3) - Y_n'(z_3)\right]},$$

$$Q_1 = \frac{\rho_1 c_1 Y_n'(x_2) j_n(x_1) - \rho_2 c_2 Y_n(x_2) j_n'(x_1)}{\rho_1 c_1 j_n'(x_2) j_n(x_1) - \rho_2 c_2 j_n(x_2) j_n'(x_1)},$$

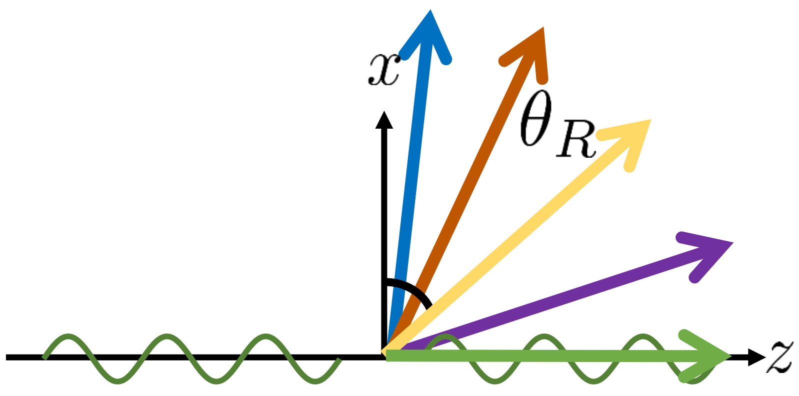
$$Q_2 = \frac{\rho_2 c_2 [Q_1 j_n(y_2) - Y_n(y_2)] Y_n'(y_3) - \rho_3 c_3 \left[Q_1 j_n'(y_2) - Y_n'(y_2)\right] Y_n(y_3)}{\rho_2 c_2 [Q_1 j_n(y_2) - Y_n(y_2)] j_n'(y_3) - \rho_3 c_3 \left[Q_1 j_n'(y_2) - Y_n'(y_2)\right] j_n(y_3)},$$

$$x_1 = k_1 r_1,\ x_2 = k_2 r_1,\ y_2 = k_2 r_2,\ y_3 = k_3 r_2,\ z_0 = k_4 r_3,\ \text{and}\ z_3 = k_3 r_3.$$

$$Y_p = -\frac{4}{(k_4 r_3)^2} \sum_{n=0}^{\infty} (n+1) \times \left[\mathrm{Re}(\alpha_n + \alpha_{n+1} + 2\alpha_n \alpha_{n+1} + 2\beta_n \beta_{n+1}) + \mathrm{Im}(\beta_{n+1}(1 + 2\alpha_n) - \beta_n(1 + 2\alpha_{n+1}))\right],$$

Wang et al., J. Appl. Phys. **122**, 094902 (2017)

Extra: Rayleigh angle



Peng et al., J. Mech. Phys. Solids. **145**, 104134 (2020)