# Growth rates of harmonics in nonlinear vortex beams

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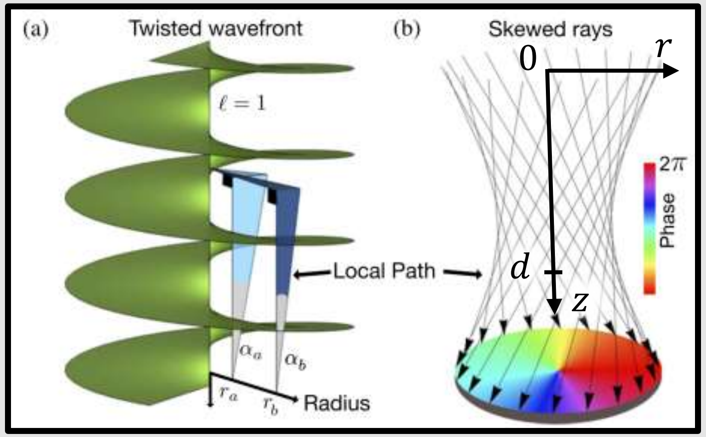
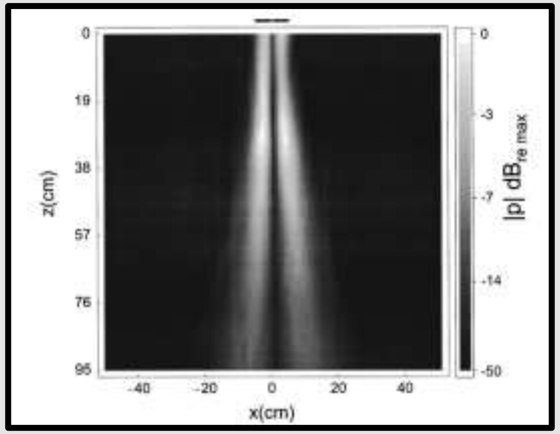
# Outline

1. What is an acoustic vortex beam?
2. Linear solutions
3. Motivation to study nonlinear solution
4. Geometry of Bessel vortex beam
5. Numerical solution of Westervelt equation
6. Comparison to Fubini coefficients
7. Summary

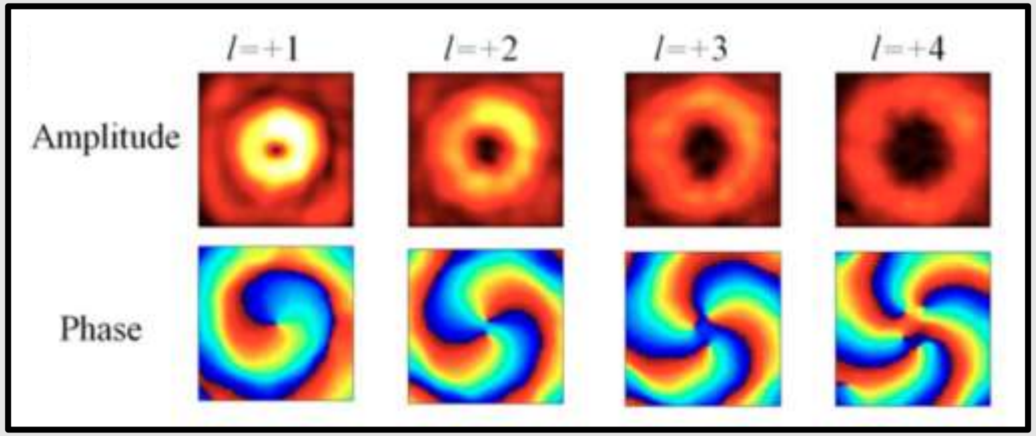
# What is an acoustic vortex beam?




Richard et al., New J. Phys. (2020)

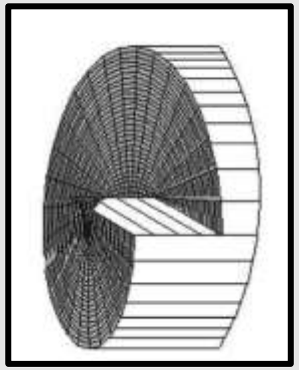



Hefner and Marston, JASA (1999)




Shi et al., Proc. NAS. (2017)

- *Orbital number* $\ell$ = number of wavefronts in transverse plane
- Characterized by null on axis
- Unfocused beams used for communication
- Focused beams used for radial and axial particle manipulation
- Generated using phase plates, transducer arrays, metasurfaces



Gspan et al., JASA (2003)

Terzi et al., MUPB (2017)

Marzo et al., Phys. Rev. Lett. (2018)

Li et al., Wiley (2022)

# Linear solutions



**Helmholtz equation**

$$\nabla^2 p + k^2 p = 0$$

**Fourier acoustics solution of Helmholtz equation**

$$p(x,y,z) = \mathcal{F}_{2D}^{-1}\left\{e^{ik_z z}\mathcal{F}_{2D}[p(x,y,0)]\right\}, \quad k_z = (k^2 - k_x^2 - k_y^2)^{1/2}$$

**Rayleigh integral solution of Helmholtz equation**

$$p(r,\theta,z) = -\frac{ikz}{2\pi}\int_0^{2\pi}\int_0^{\infty} p(r_0,\theta_0,0)\left(1 - \frac{1}{ikR}\right)\frac{e^{ikR}}{R^2}\, r_0\, dr_0\, d\theta_0, \quad R = |\mathbf{r} - \mathbf{r}_0|$$

**Paraxial equation**

$$\nabla_\perp^2 q + 2ik\frac{\partial q}{\partial z} = 0, \quad p = qe^{ikz}$$

**Fourier acoustics solution of paraxial equation**

$$p(x,y,z) = \mathcal{F}_{2D}^{-1}\left\{e^{ik_z z}\mathcal{F}_{2D}[p(x,y,0)]\right\}, \quad k_z = k - (k_x^2 + k_y^2)/2k$$

**Typical pressure source conditions**

$p(r,\theta,0) = p_0 e^{-r^2/a^2} e^{i\ell\theta}$ Gaussian vortex

$p(r,\theta,0) = p_0 \,\mathrm{circ}(r/a)\, e^{i\ell\theta}$ circular vortex

$(\times e^{-ikr^2/2d}$, focusing)

$p(r,\theta,0) = p_0 J_\ell(k_r r)\, e^{i\ell\theta}$ Bessel vortex

**Laguerre-Gauss eigenfunctions of paraxial equation**

$$q(r,\theta,z) = \sum_{n,m} A_n^m \mathrm{LG}_{nm}(r,\theta,z)$$

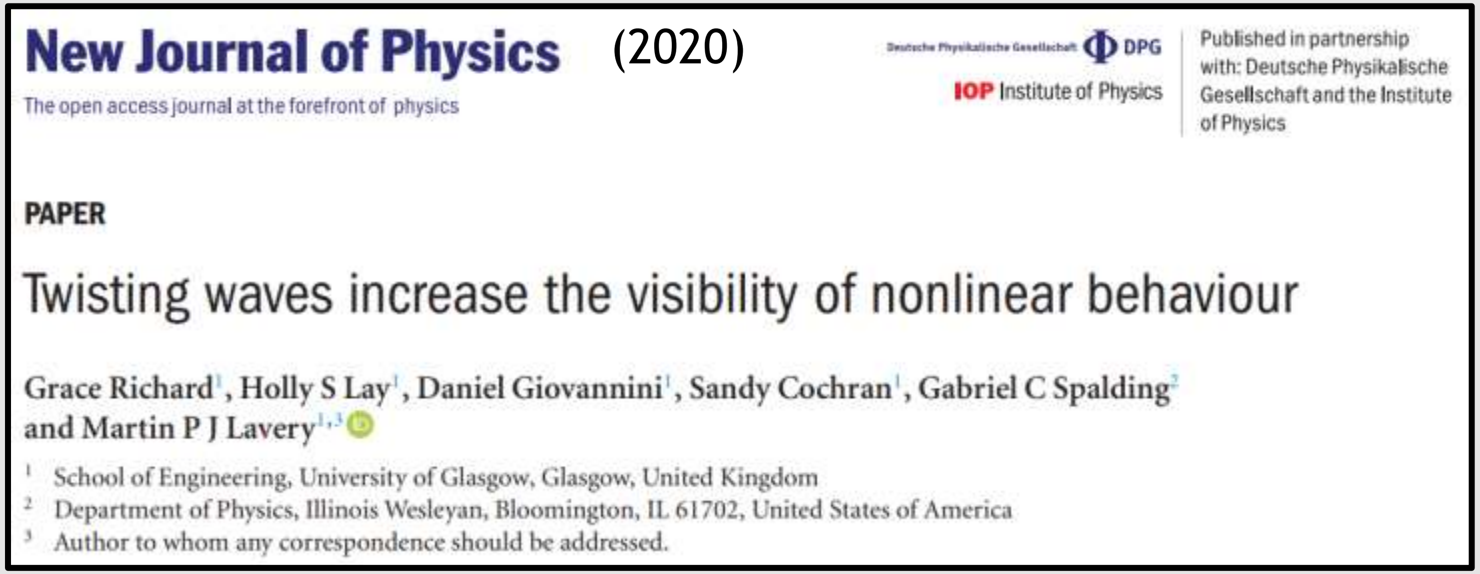
**Integral solution of paraxial equation**

$$q(r,\theta,z) = -\frac{ik}{2\pi z}\int_0^{2\pi}\int_0^{\infty} q(r_0,\theta_0,0)\, e^{i(k/2z)[r^2 + r_0^2 - 2rr_0\cos(\theta_0 - \theta)]}\, r_0\, dr_0\, d\theta_0$$

Gokani et al., “Paraxial and ray theory models of acoustic vortex beams” (in preparation)

# Motivation to study nonlinear solution





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IOP Institute of Physics
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PAPER

Twisting waves increase the visibility of nonlinear behaviour

Grace Richard[1], Holly S Lay[1], Daniel Giovannini[1], Sandy Cochran[1], Gabriel C Spalding[2] and Martin P J Lavery[1,3]

[1] School of Engineering, University of Glasgow, Glasgow, United Kingdom
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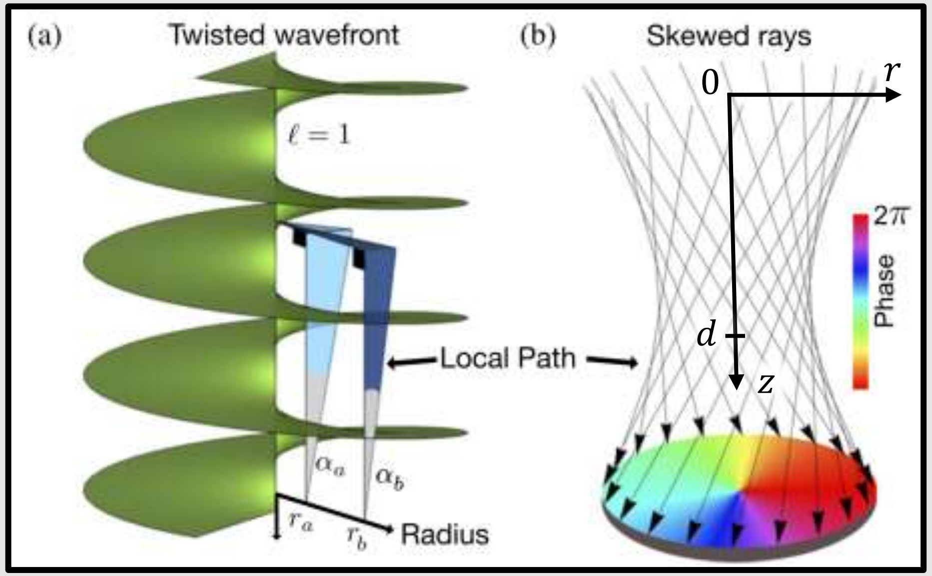
[3] Author to whom any correspondence should be addressed.




- Spiral wavefront carries orbital angular momentum (OAM)
- Wavefront travels farther than it would in the absence of OAM ($\ell = 0$)
- Phase evolves more rapidly due to extra path length in $z$ direction
- **Nonlinear effects in $z$ direction occur over shortened length scale**

# Geometry of Bessel vortex beam



**Bessel vortex beam**

$$p(r,\theta,z) = p_0 J_\ell(k_r r) e^{i(k_z z + \ell\theta)}$$

**Surface of const. phase**

$$f(z,\theta) = k_z z + \ell\theta$$

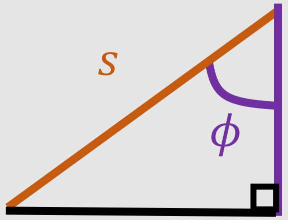
$$k_z = \sqrt{k^2 - k_r^2}$$

**Wave normal[1,2]**

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{k_z r \mathbf{e}_z + \ell \mathbf{e}_\theta}{\sqrt{(k_z r)^2 + \ell^2}}$$

**Angle w.r.t. $z$ axis**

$$\tan\phi = \frac{\ell}{k_z r}$$




$$z = s\cos\phi$$

**Burgers equation in terms of $s$**

$$\frac{\partial p}{\partial s} - \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial \tau^2} = \frac{\beta p}{\rho_0 c_0^3}\frac{\partial p}{\partial \tau}$$

where $s = z/\cos\phi$

**Reduced shock-formation distance**

$$\bar{z} = \frac{\cos\phi}{\beta k \epsilon(r)} = \frac{k_z r}{\sqrt{(k_z r)^2 + \ell^2}} \frac{1}{\beta k \epsilon(r)}$$

$\bar{z} \le \frac{1}{\beta k \epsilon}$ since $\cos\phi \le 1$

➢ Compare to numerical solution of Westervelt equation to test claim

[1]Pierce, *Acoustics: An Introduction to Its Physical Principles and Applications*, Chap. 8 (Springer 2019).

# Numerical solution of Westervelt equation

## Westervelt equation in retarded time[1]

$$\frac{\partial^2 p}{\partial \tau \partial z} = \frac{c_0}{2}\nabla^2 p + \frac{\beta}{2\rho_0 c_0^3}\frac{\partial^2 p^2}{\partial \tau^2} + \frac{\delta}{2c_0^3}\frac{\partial^3 p}{\partial \tau^3}$$

$\tau = t - z/c_0$ diffraction nonlinearity absorption

## Inputs parameters

| | | |
|---|---|---|
| $\ell$ | orbital number | + source condition |
| $ka$ | ratio of source size to wavelength | |
| $B$ | ratio of diffraction to nonlinearity | |
| $A$ | ratio of attenuation to diffraction | |

## Numerical solution based on operator splitting[1,2]

$$\frac{\partial^2 p}{\partial \tau \partial z} = \frac{c_0}{2}\nabla^2 p \dashrightarrow p_n(x,y,z+\Delta z) = \mathcal{F}_{2D}^{-1}\left\{\exp\left(i\sqrt{k_n^2 - k_x^2 - k_y^2}\Delta z - ik_n\Delta z\right)\mathcal{F}_{2D}[p_n(x,y,z)]\right\}$$

Diffraction → Fourier acoustics

$$\frac{\partial p}{\partial z} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial \tau} \dashrightarrow \frac{\partial p_n}{\partial z} = \frac{in\beta\omega}{\rho_0 c_0^3}\left(\sum_{k=1}^{N_{\max}-n} p_k p_{n+k}^* + \frac{1}{2}\sum_{k=1}^{n-1} p_k p_{n-k}\right)$$

Nonlinearity → Runge-Kutta 4th order method

$$\frac{\partial p}{\partial z} = \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial \tau^2} \dashrightarrow p_n(x,y,z+\Delta z) = p_n(x,y,z)\exp[-(\omega_n^2\delta/2c_0^3)\Delta z]$$

Absorption → Exponential decay

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[1] Yuldashev and Khokhlova, Acoustical Physics (2011)
[2] Lee and Hamilton, JASA (1995)

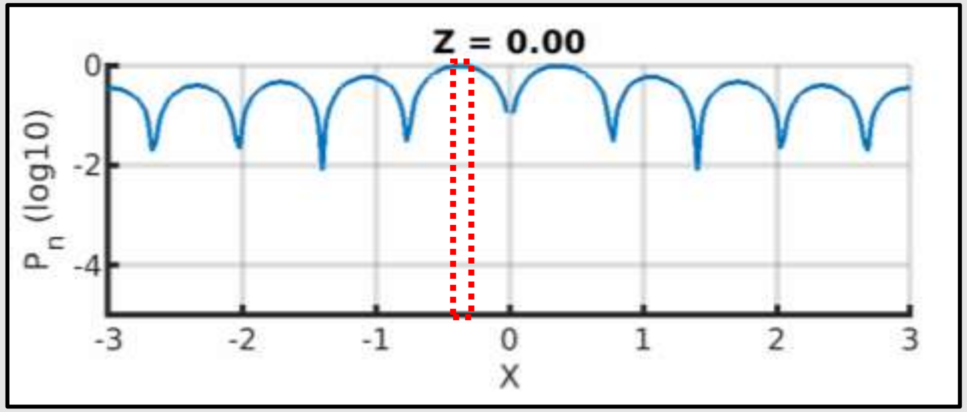
# Comparison to Fubini coefficients

**Westervelt equation in retarded time**

$$\frac{\partial^2 p}{\partial \tau \partial z} = \frac{c_0}{2}\nabla^2 p + \frac{\beta}{2\rho_0 c_0^3}\frac{\partial^2 p^2}{\partial \tau^2} + \frac{\delta}{2c_0^3}\frac{\partial^3 p}{\partial \tau^3}$$

$\tau = t - z/c_0$ diffraction nonlinearity absorption

↓ solved numerically



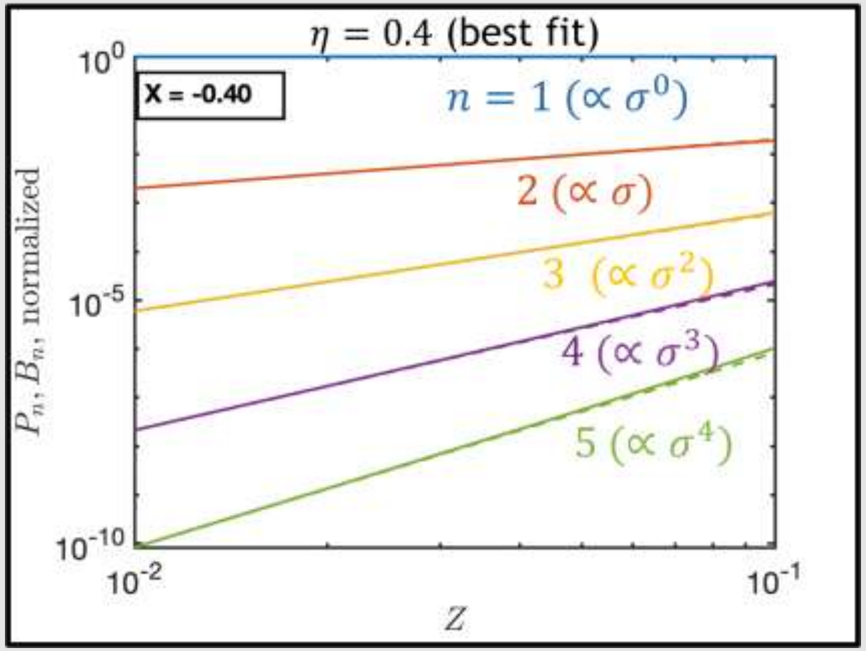

**Mapping $\sigma$ to $Z = z/z_0$**

$$\sigma = \frac{s}{\bar{z}} = \frac{z\beta k \epsilon(r)}{\cos\phi} = \eta Z$$

$$\Longrightarrow \eta = \frac{z_0 \beta \epsilon(r) k}{\cos\phi} = 0.6$$

**Inputs parameters**

| | |
|---|---|
| $\ell = 1$ | orbital number |
| $ka = 1$ | ratio of source size to wavelength |
| $B = 1$ | ratio of diffraction to nonlinearity |
| $A = 0.01$ | ratio of attenuation to diffraction |




Westervelt (—) Fubini (--)

**Bessel vortex source**

$$p(r,\theta,0) = p_0 J_\ell(\mu r) e^{i\ell\theta}$$

**Fubini solution**

$$p(\sigma,\theta) = p_0 \sum_{n=1}^{\infty} \frac{2}{n\sigma} J_n(n\sigma) \sin n\omega\tau$$

$$= p_0 \sum_{n=1}^{\infty} B_n(\sigma) \sin n\omega\tau$$

$$B_1 = 1 + \mathcal{O}(\sigma^2)$$

$$B_2 = \frac{1}{2}\sigma + \mathcal{O}(\sigma^3)$$

$$B_3 = \frac{3}{8}\sigma^2 + \mathcal{O}(\sigma^4)$$

$$B_4 = \frac{1}{3}\sigma^3 + \mathcal{O}(\sigma^5)$$

$$B_5 = \frac{125}{384}\sigma^4 + \mathcal{O}(\sigma^6)$$

# Thanks for listening!

## Summary

- Developed geometric interpretation of Bessel vortex beam
- Numerically solved Westervelt equation for Bessel vortex source condition
- Mapped Fubini solution to nonlinear evolution along ray of path length $s = z/\cos\phi$
- Compared Fubini solution in terms of $s$ to numerical solution to Westervelt equation

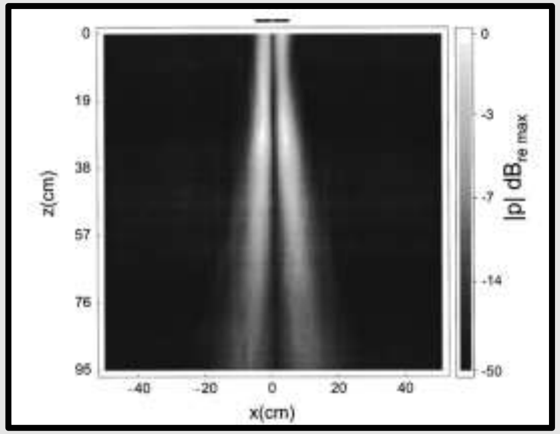

# Acknowledgments

ARL:UT Chester M. McKinney Graduate Fellowship in Acoustics

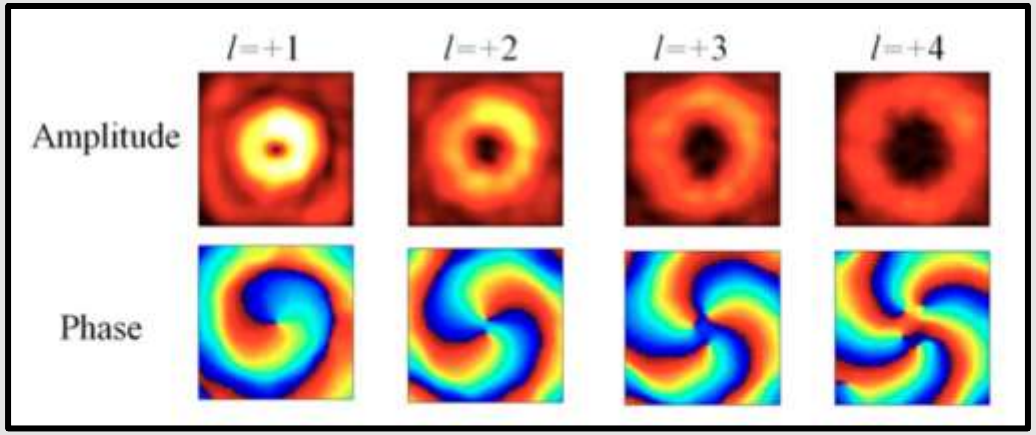
# Extra slides

# Unfocused linear vortex beams






Hefner and Marston, JASA (1999)




Shi et al., Proc. NAS. (2017)

**Paraxial equation**

$$\nabla_\perp^2 q + 2ik\frac{\partial q}{\partial z} = 0$$

**Gaussian vortex source condition**

$$q(r,\theta,0) = p_0 e^{-r^2/a^2} e^{i\ell\theta}$$

$\ell$ = number of surfaces of equal phase

**Analytical solution of paraxial equation for Gaussian vortex source**

$$q(r,\theta,z) = \sqrt{8\pi}\,\frac{p_0 z}{kr^2}\chi^{3/2}e^{-\chi}\left[I_{(\ell-1)/2}(\chi) - I_{(\ell+1)/2}(\chi)\right]e^{i[\ell\theta-(\ell+1)\pi/2+kr^2/2z]}$$

$$\chi(r,z) = \frac{(kar/z)^2}{8[1-i(ka^2/2z)]}$$

**Power**

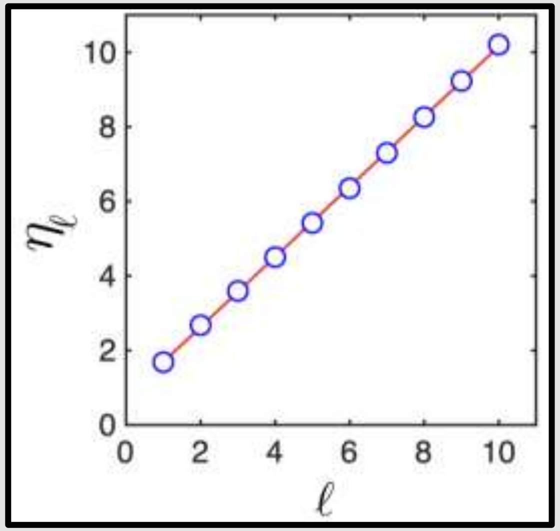
$$\mathcal{P}_0 = \frac{\pi a^2 p_0^2}{4\rho_0 c_0}$$

independent of orbital number

**Laguerre-Gauss eigenfunctions of paraxial equation**

$$q(r,\theta,z) = \sum_{n,m}^{\infty} A_n^m \,\mathrm{LG}_{nm}(r,\theta,z),$$

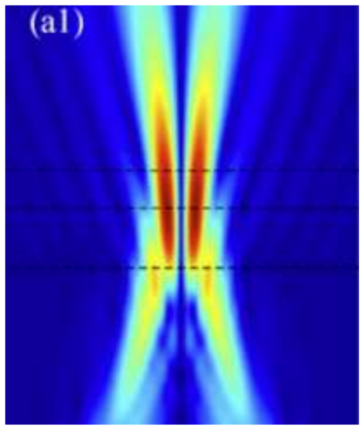
$$\mathrm{LG}_{nm}(r,\theta,z) = N_n^m \frac{1}{w(z)}\left(\frac{\sqrt{2}r}{w(z)}\right)^{|m|} L_n^m\left(\frac{2r^2}{w^2(z)}\right) \times \exp\left\{-\frac{r^2}{w^2(z)} + i\left[m\theta + \frac{kr^2}{2R(z)} - (2n+|m|+1)\phi(z)\right]\right\}$$



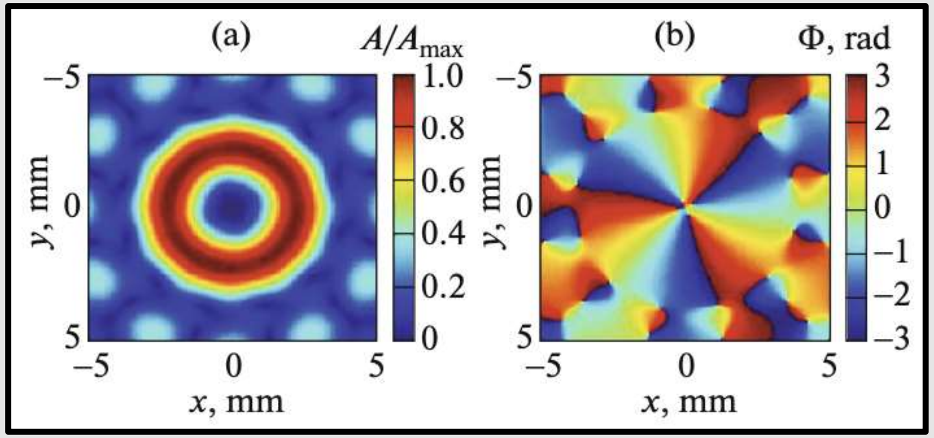
# Focused linear vortex beams

TEXAS
The University of Texas at Austin

Baresch et al., PRL (2016)




Zhou et al., JAP (2020)




Terzi et al., Moscow Univ. Phys. Btn. (2017)

**Paraxial equation**

$$\nabla_\perp^2 q + 2ik\frac{\partial q}{\partial z} = 0$$

**Gaussian focused vortex source**

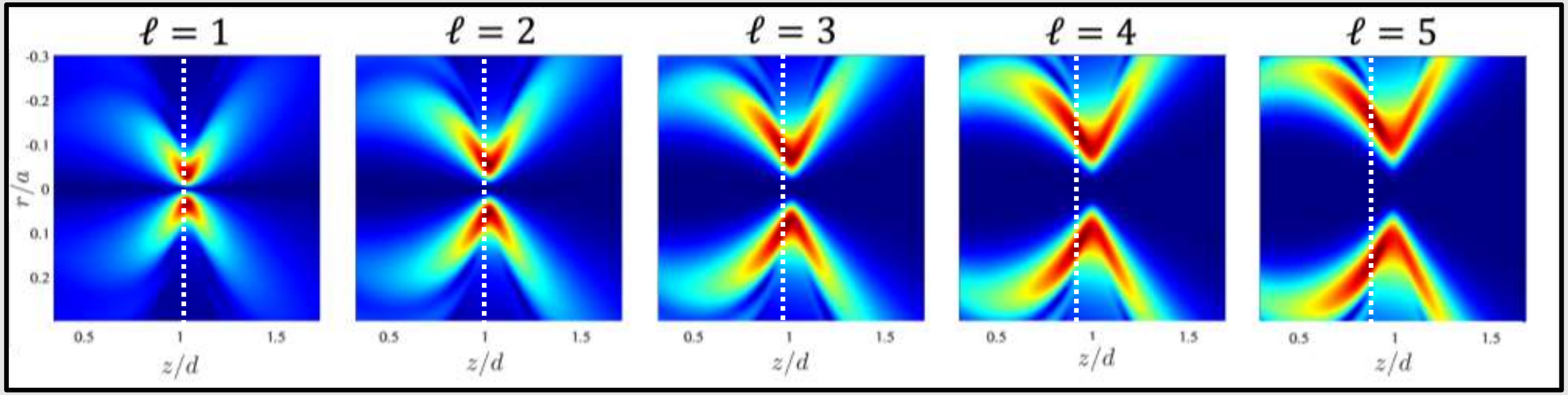
$$q(r,\theta,0) = p_0 e^{-r^2/a^2} e^{-ikr^2/2d} e^{i\ell\theta}$$

$\ell$ = number of surfaces of equal phase

**Analytical solution of paraxial equation for Gaussian vortex source**

$$q(r,\theta,z) = \sqrt{8\pi}\frac{p_0 z}{kr^2}\chi^{3/2}e^{-\chi}\left[I_{(\ell-1)/2}(\chi) - I_{(\ell+1)/2}(\chi)\right]e^{i[\ell\theta-(\ell+1)\pi/2+kr^2/2z]}$$

$$\chi(r,z) = \frac{(kar/z)^2}{8[1-i(ka^2/2z)(1-z/d)]}$$

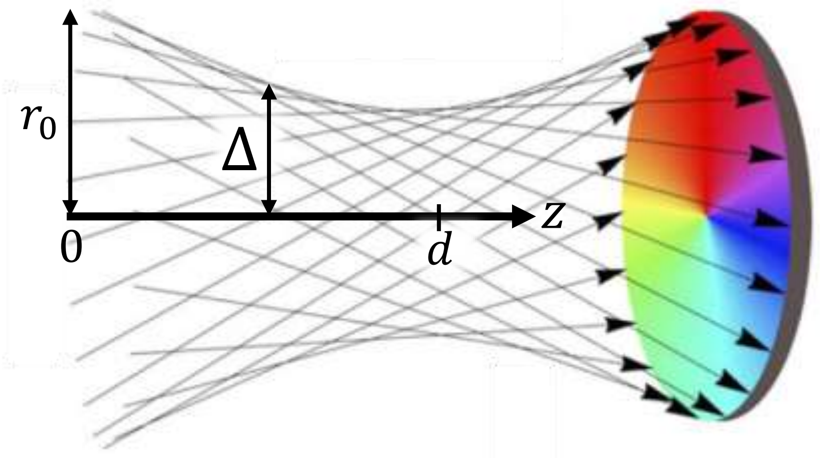


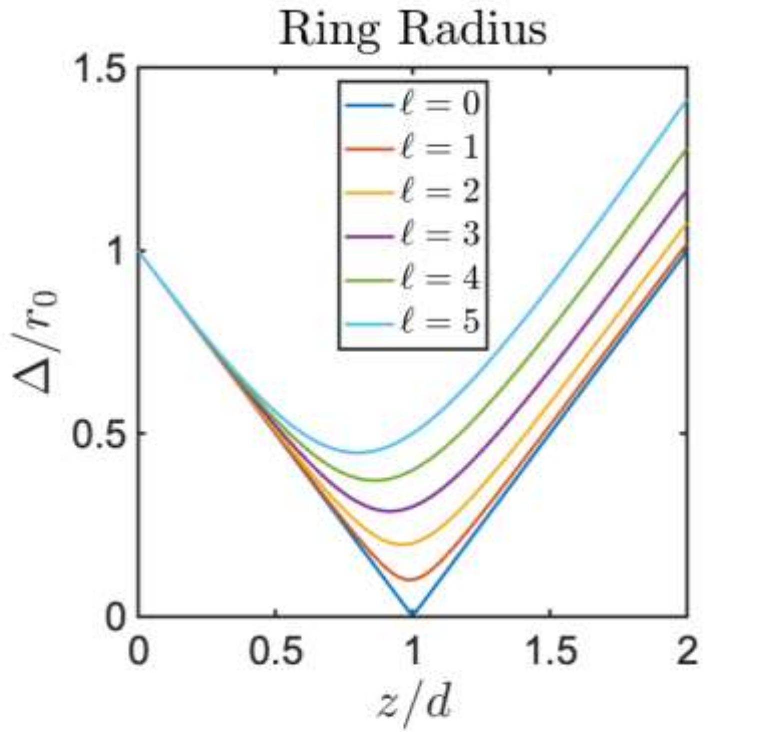

Global max moves out of focal plane toward source as $\ell$ increases

$ka = 500,\ d/a = 10$

# Focused linear vortex beams

Adapted from Richard et al., *New J. Phys.* (2020)




$d/kr_0^2 = 1/10$

**Gaussian focused vortex source**

$$q(r,\theta,0) = p_0 e^{-r^2/a^2} e^{-ikr^2/2d} e^{i\ell\theta}$$

number of surfaces of equal phase

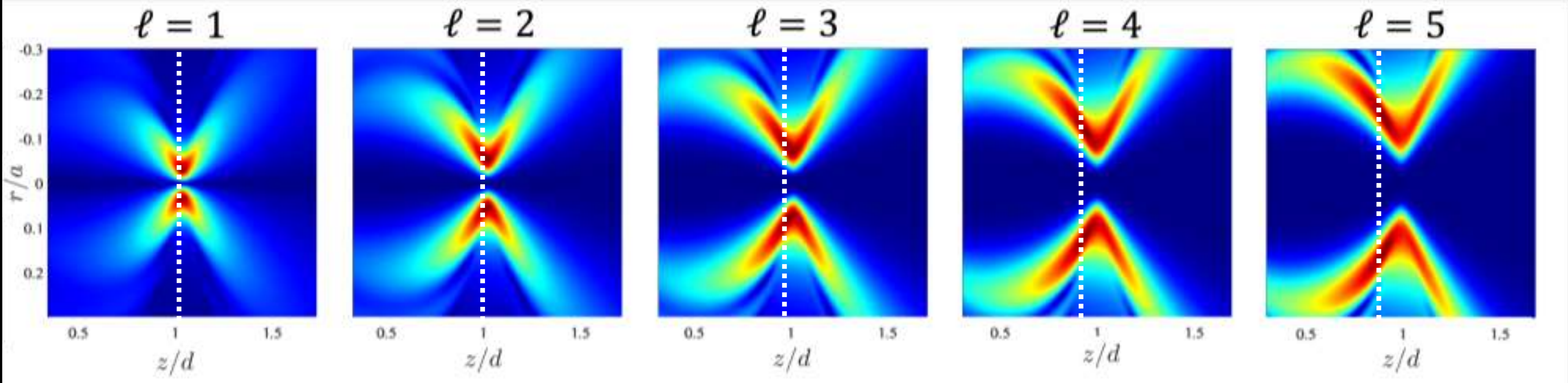
**Ring Radius (ray theory)**

$$\frac{\Delta}{r_0} = \sqrt{(1 - z/d)^2 + (\ell d/kr_0^2)^2 (z/d)^2}$$

**Focal ring radius**

$$r_\ell = \frac{\eta_\ell d}{ka},$$

$$\eta_\ell = 0.94\ell + 0.75$$



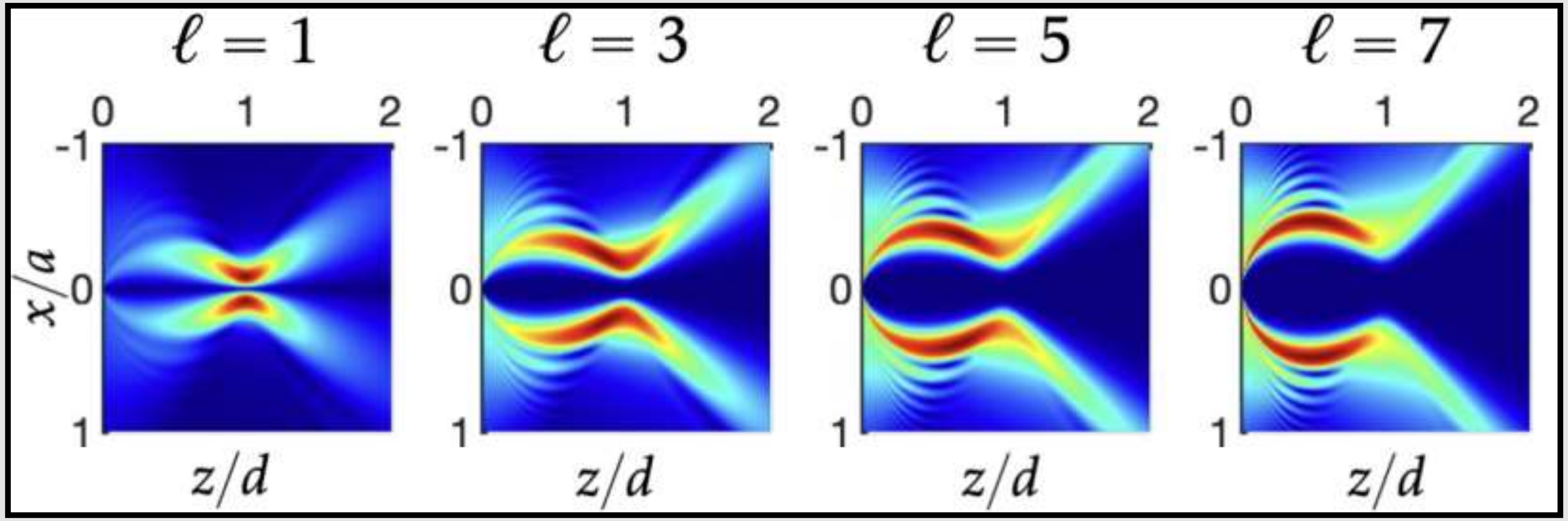

Global max moves out of focal plane toward source as $\ell$ increases

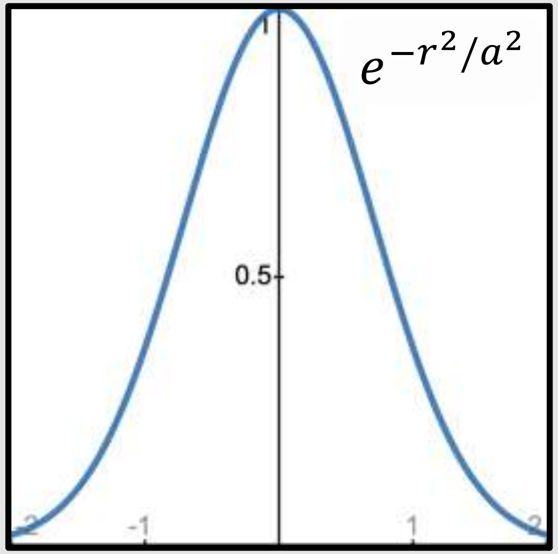
$ka = 500$, $d/a = 10$

# Focused linear vortex beams: "cupped hands" phenomenon

TEXAS
The University of Texas at Austin

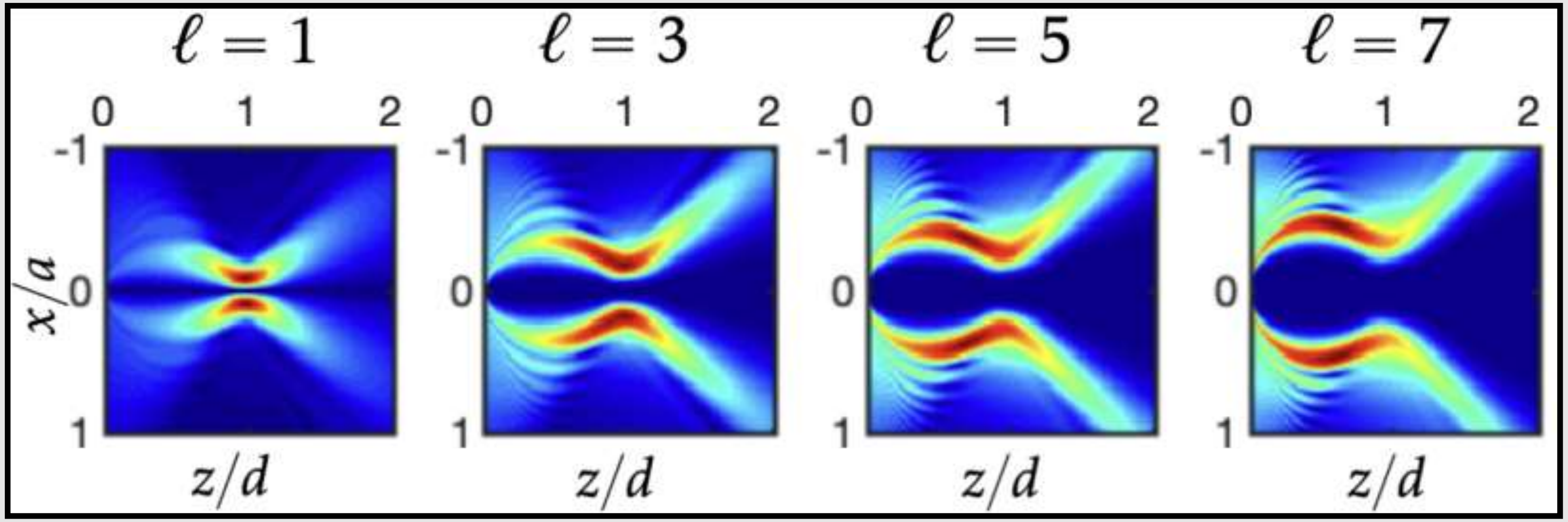
Gaussian amplitude shading

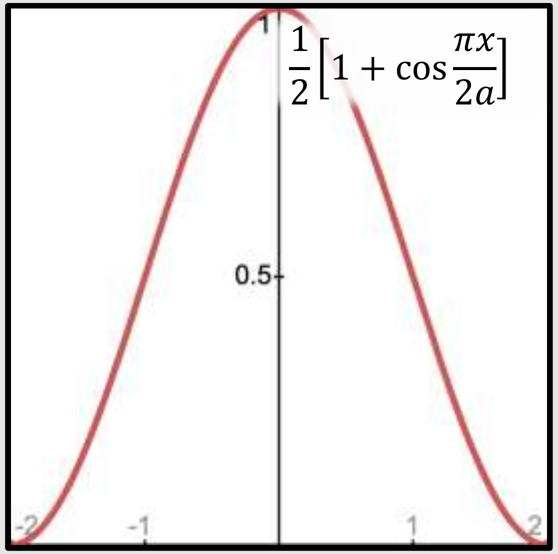



$G = 10$

Raised cosine amplitude shading



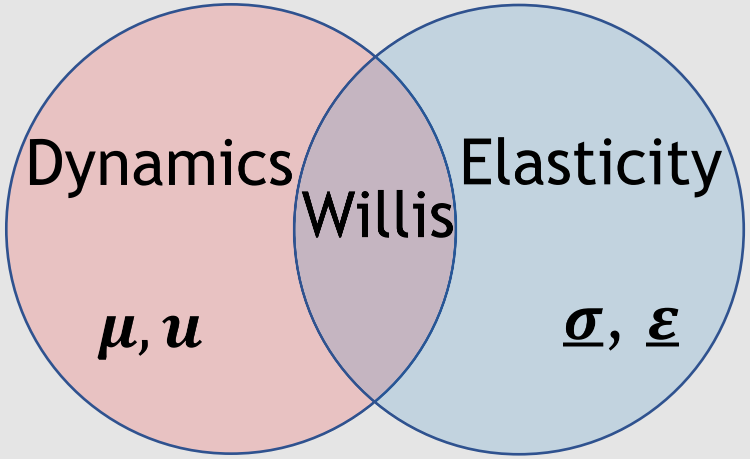



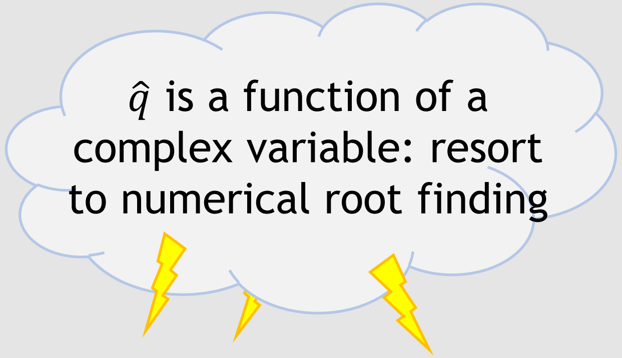

$G = 10$

# Figure board

$\mathcal{F}_{2\text{D}}$ = 2D spatial Fourier transform

$k_n = n\omega/c_0$




$$P(\sigma,\theta) = \sum_{n=1}^{\infty} B_n(\sigma)\sin n\theta$$

$$\sigma = z/\bar{z}, \quad \theta = \omega\tau$$

**Fubini coefficients (--)**

$$B_1 = 1 - \frac{1}{8}\sigma^2 + \mathcal{O}(\sigma^4)$$

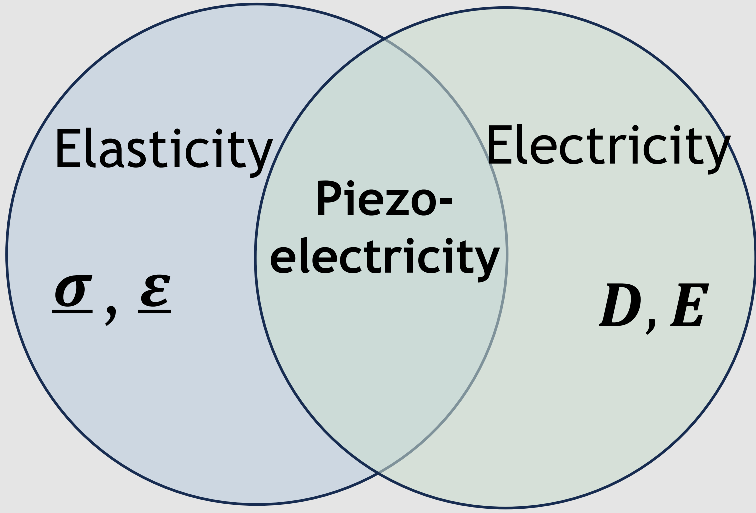
$$B_2 = \frac{1}{2}\sigma + \mathcal{O}(\sigma^3)$$

$$B_3 = \frac{3}{8}\sigma^2 + \mathcal{O}(\sigma^4)$$

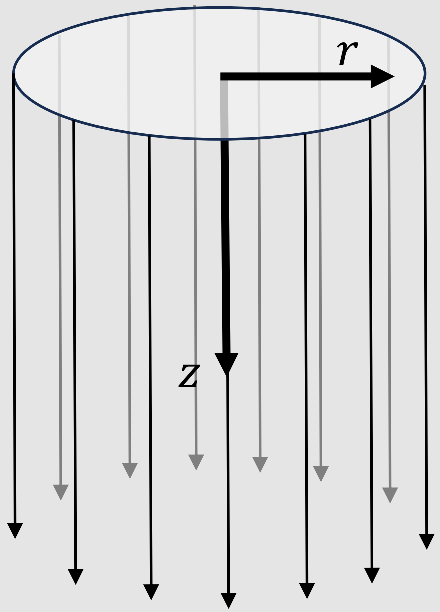
$$B_4 = \frac{1}{3}\sigma^3 + \mathcal{O}(\sigma^5)$$

$$B_5 = 0.3255\,\sigma^4 + \mathcal{O}(\sigma^6)$$

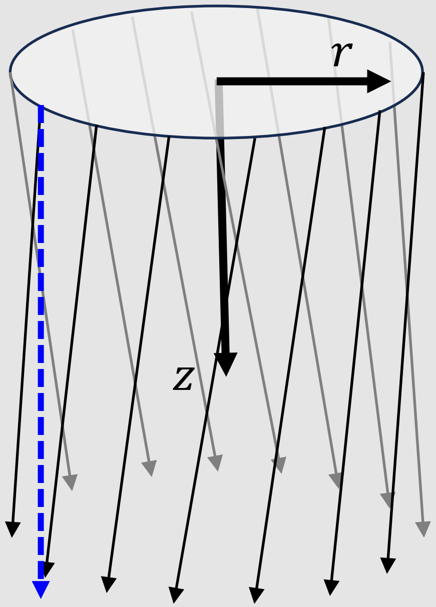
$$B_n = \frac{(n\sigma)^{n-1}}{2^{n-1}n!} + \mathcal{O}(\sigma^{n+1}).$$



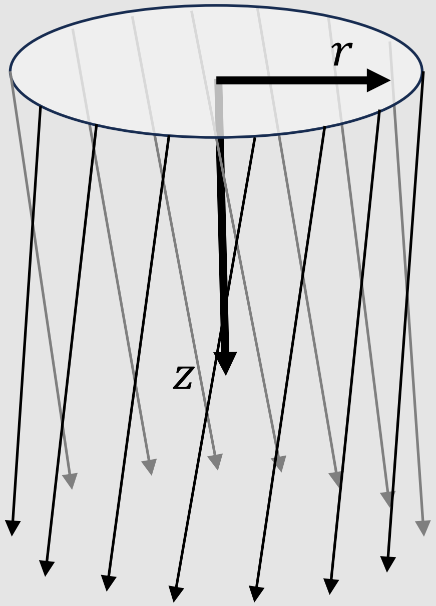
# Figure board

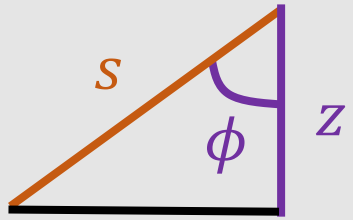



Field parallel to $z$ is described by individual rays




Field parallel to $z$ is contribution of many skewed rays

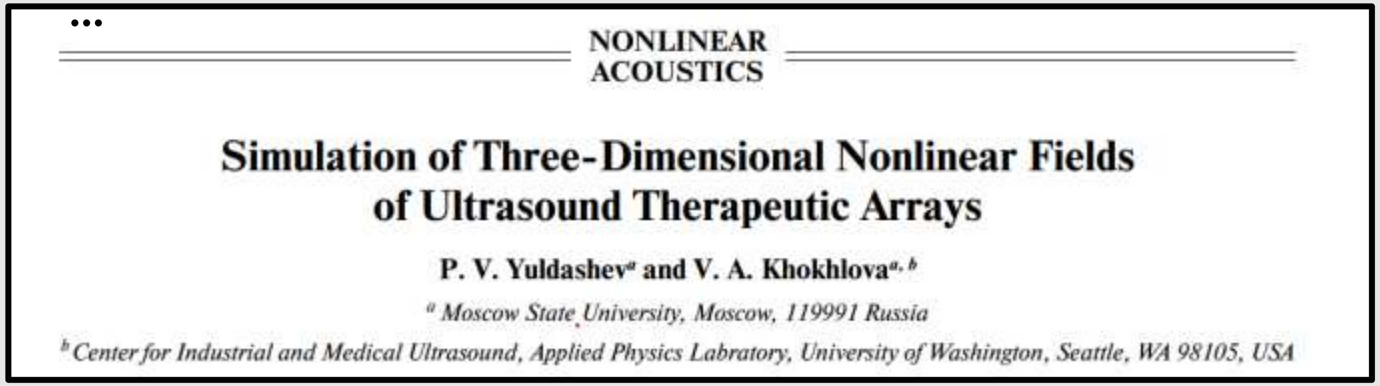



$(\tau, x, y, z)$ → $(\tau, x, y, z + \Delta z)$

$(\tau, x, y, z + \Delta z)$ ←

…



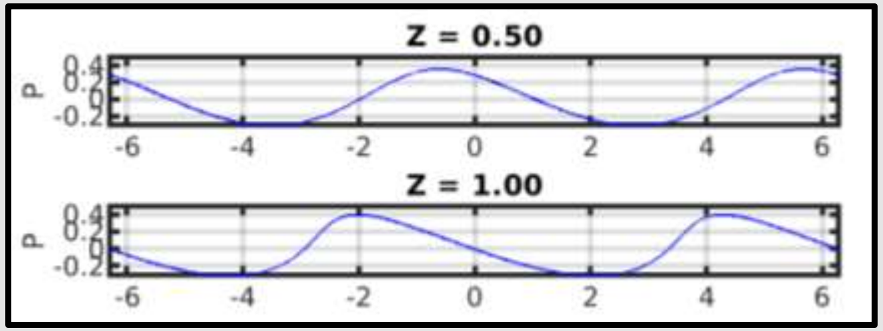
NONLINEAR ACOUSTICS

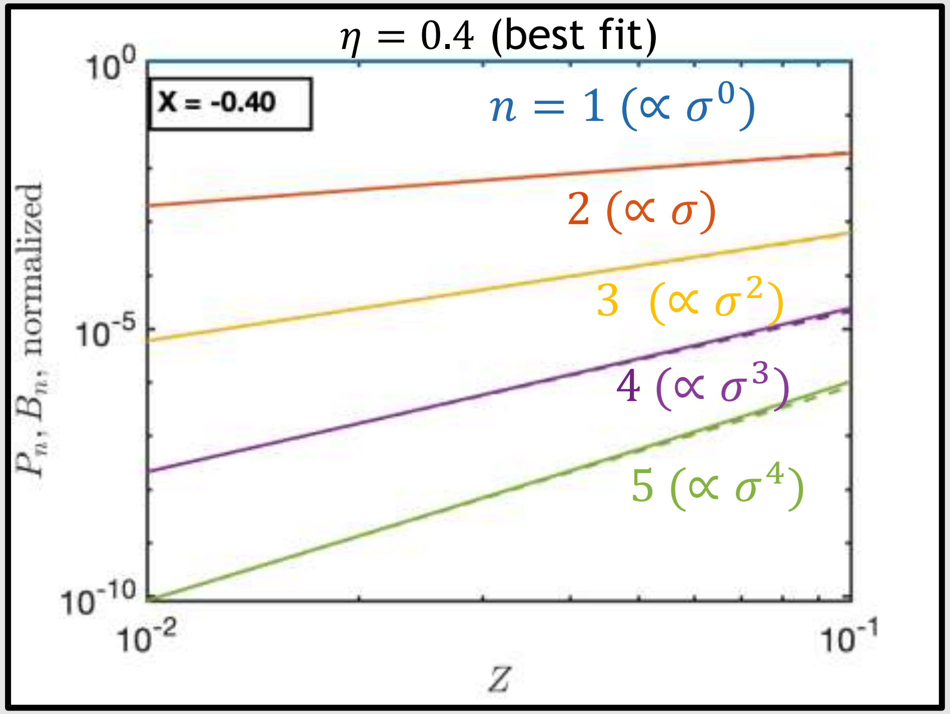
**Simulation of Three-Dimensional Nonlinear Fields of Ultrasound Therapeutic Arrays**

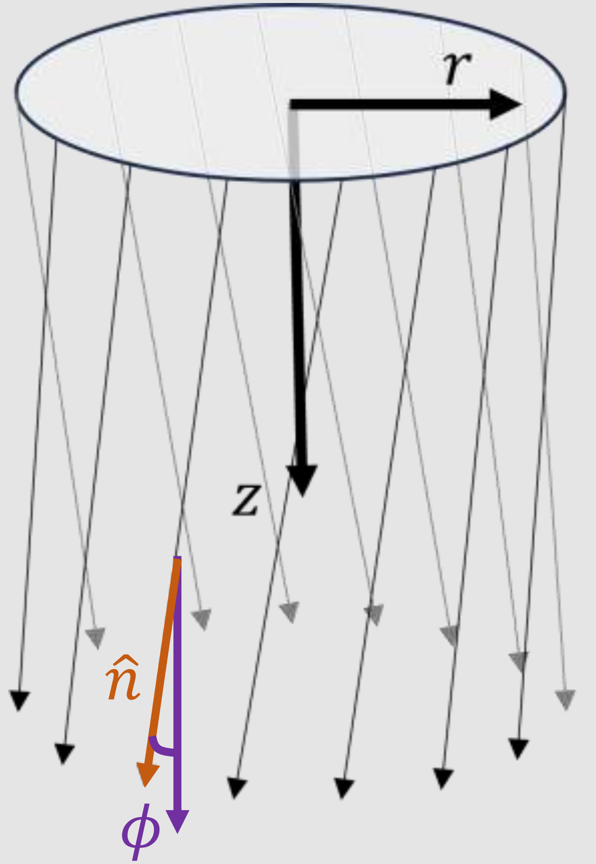
P. V. Yuldashev[a] and V. A. Khokhlova[a, b]

[a] *Moscow State University, Moscow, 119991 Russia*

[b] *Center for Industrial and Medical Ultrasound, Applied Physics Laboratory, University of Washington, Seattle, WA 98105, USA*






$\eta = 0.6$ (analytical)

[2]Morfey and Cotaras, “Propagation in Inhomogeneous Media (Ray Theory),” *Nonlinear Acoustics*, Hamilton and Blackstock, eds. (ASA 2008).

# Outline