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Analytical solutions for acoustic vortex beam radiation from planar and spherically focused circular pistons

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Applied Research Laboratories, The University of Texas at Austin

The University of Texas at Austin
Walker Department of Mechanical Engineering
Cockrell School of Engineering

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Our publications on this work

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Analytical solutions for acoustic vortex beam radiation from planar and spherically focused circular pistons

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Paraxial and ray approximations of acoustic vortex beams

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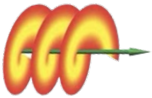
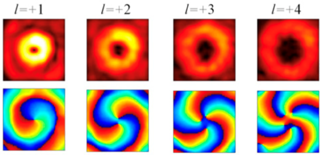
What is an acoustic vortex beam?

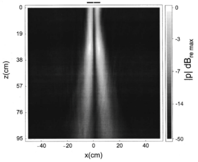
Characterized by...

- helical wavefronts
- orbital number ℓ = number of equiphase wavefronts in $\perp$ plane
- zero acoustic pressure on axis





C. Shi et al. *P. Natl. Acad. Sci. U.S.A.* 114 (2017), pp. 7250–7253

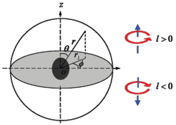


B. T. Hefner and P. L. Marston. *J. Acoust. Soc. Am.* 106 (1999), pp. 3313–3316

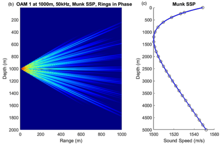
What is an acoustic vortex beam?

Used for…

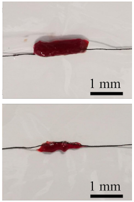
- particle manipulation
- underwater communications
- therapeutic biomedical ultrasound
- sound diffusion



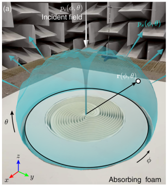
L. Zhang and P. L. Marston. *Phys. Rev. E* 84 (2011), pp. 1–5



M. E. Kelly and C. Shi. *JASA Express Lett.* 3 (2023), pp. 1–5



S. Guo et al. *Ultrasound Med. Biol.* 48 (2022), pp. 1907–1917

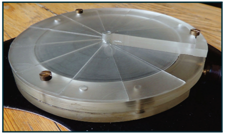


N. Jiménez, J. -P. Groby, and V. Romero-García. *Sci. Rep.* 11 (2021), pp. 1–13

What is an acoustic vortex beam?

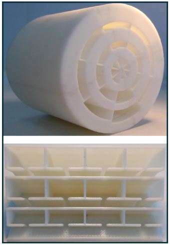
Generated by. . .

- phase plates
- transducer arrays
- metasurfaces



M. E. Terzi et al. *Moscow University Physics Bulletin* 72 (2017), pp. 61–67

A. Marzo, M. Caleap, and B. W. Drinkwater. *Phys. Rev. Lett.* 120 (2018), pp. 1–6



X. Jiang et al. *Phys. Rev. Lett.* 117 (2016), pp. 1–5

Previous analytical descriptions of vortex beams

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- Bessel vortex beams are modes of the cylindrical wave equation:[1]

$$p(r,\theta,z,t) = p_0 J_\ell(k_r r) e^{i(\ell\theta + k_z z - \omega t)}, \quad k = \frac{\omega}{c_0} = \sqrt{k_r^2 + k_z^2}. \tag{1}$$

- Gaussian vortex beams are generalizations of Gaussian beams:[2]

$$\begin{aligned} p(r,\theta,z,t) &= \sqrt{8\pi}\left(\frac{p_0 z}{kr^2}\right)\chi^{3/2}e^{-\chi}\left[I_{(\ell-1)/2}(\chi) - I_{(\ell+1)/2}(\chi)\right] \\ &\quad \times e^{i[\ell\theta - (\ell+1)\pi/2 + kr^2/2z + k_z z - \omega t]}, \\ \chi(r,z) &= \frac{\frac{1}{8}(kar/z)^2}{1 - i(ka^2/2z)(1 - z/d)}. \end{aligned} \tag{2}$$

- Fields described by Eqs. (1) and (2) require infinite source conditions.
- Equation (1) implies infinite energy,[3] because $\int_0^\infty |J_\ell(k_r r)|^2 r\,dr \to \infty$.
- **Objective: derive solutions for vortex fields radiated by circular pistons**

[1] N. Jiménez et al. *Phys. Rev. E* 94 (2016), pp. 1–9.

[2] C. A. Gokani, M. R. Haberman, and M. F. Hamilton. *J. Acoust. Soc. Am.* 155 (2024), pp. 2707–2723.

[3] M. R. Lapointe. *Opt. Laser Technol.* 24 (1992), pp. 315–321.

Outline

Paraxial equation and its integral solution

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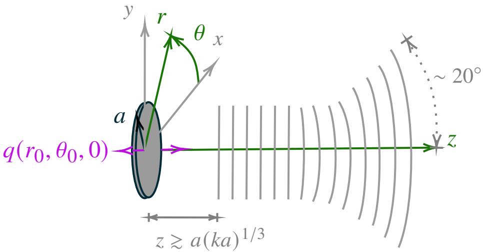
- For $p = qe^{i(kz-\omega t)}$ and $|\partial^2 q/\partial z^2| \ll 2k|\partial q/\partial z|$, $\nabla^2 p - c_0^{-2}\ddot{p} = 0$ reduces to

$$i2k\frac{\partial q}{\partial z} + \nabla_\perp^2 q = 0\,. \tag{3}$$

- $\nabla_\perp^2$ is the Laplacian in the plane perpendicular to the z axis.
- Equation (3) is solved by the Fresnel diffraction integral:

$$q(r,\theta,z) = -\frac{ik}{2\pi z}\int_0^{2\pi}\int_0^{\infty} q(r_0,\theta_0,0)e^{i(k/2z)[r^2+r_0^2-2rr_0\cos(\theta_0-\theta)]}\, r_0 dr_0 d\theta_0\,,$$

where $q(r_0,\theta_0,0)$ is the prescribed pressure field in the plane $z=0$.



Vortex radiation from unfocused circular piston

- A circular vortex source condition is first considered:

$$q(r,\theta,0) = p_0\,\mathrm{circ}(r/a)e^{i\ell\theta}, \tag{4}$$

where $\mathrm{circ}(x) = 1$ for $0 \le x \le 1$ and 0 for $x > 1$.

- Insertion of Eq. (4) in the Fresnel diffraction integral leads to

$$q = -ikp_0\frac{e^{i(ka^2/2z)r^2/a^2}}{z}e^{i\ell(\theta-\pi/2)}\int_0^a e^{i(ka^2/2z)r_0^2/a^2}J_\ell(krr_0/z)\,r_0dr_0\,. \tag{5}$$

- Using Watson's relation,[4] Eq. (5) reduces to

$$q = -\frac{ikp_0}{z}e^{i\ell(\theta-\pi/2)}\int_0^a J_\ell(krr_0/z)\,r_0dr_0 \tag{6}$$

for $z \gg z_R$, where $z_R = ka^2/2$ is the Rayleigh distance.

[4] G. N. Watson. Cambridge, UK: Cambridge University Press, 1944, Sec. 2.2, Eq. (5).

Vortex radiation from unfocused circular piston

- Taking the integral in Eq. (6) leads to an analytical solution:

$$q_\ell(r,\theta,z) = -ip_0 \frac{z}{kr^2} e^{i\ell(\theta-\pi/2)} F_\ell(kar/z)\,, \quad z \gg z_R\,, \tag{7}$$

where[5]

$$F_\ell(\xi) = \int_0^\xi J_\ell(t)t\,dt = \xi\,\frac{\Gamma(\ell/2+1)}{\Gamma(\ell/2)} \sum_{k=0}^{\infty} \frac{(\ell+2k+1)\Gamma(\ell/2+k)}{\Gamma(\ell/2+2+k)} J_{\ell+2k+1}(\xi). \tag{8}$$

- Equation (8) equals the following closed-form expressions for $1 \le \ell \le 4$:

$$F_0(\xi) = \xi J_1(\xi)\,, \qquad \ell = 0 \tag{9a}$$

$$F_1(\xi) = \tfrac{\pi}{2}\xi\left[\mathbf{H}_0(\xi)J_1(\xi) - \mathbf{H}_1(\xi)J_0(\xi)\right]\,, \qquad \ell = 1 \tag{9b}$$

$$F_2(\xi) = 2 - 2J_0(\xi) - \xi J_1(\xi)\,, \qquad \ell = 2 \tag{9c}$$

$$F_3(\xi) = \left[\tfrac{3\pi}{2}\xi\,\mathbf{H}_0(\xi) - 8\right] J_1(\xi) + \left[4\xi - \tfrac{3\pi}{2}\xi\,\mathbf{H}_1(\xi)\right] J_0(\xi)\,, \qquad \ell = 3 \tag{9d}$$

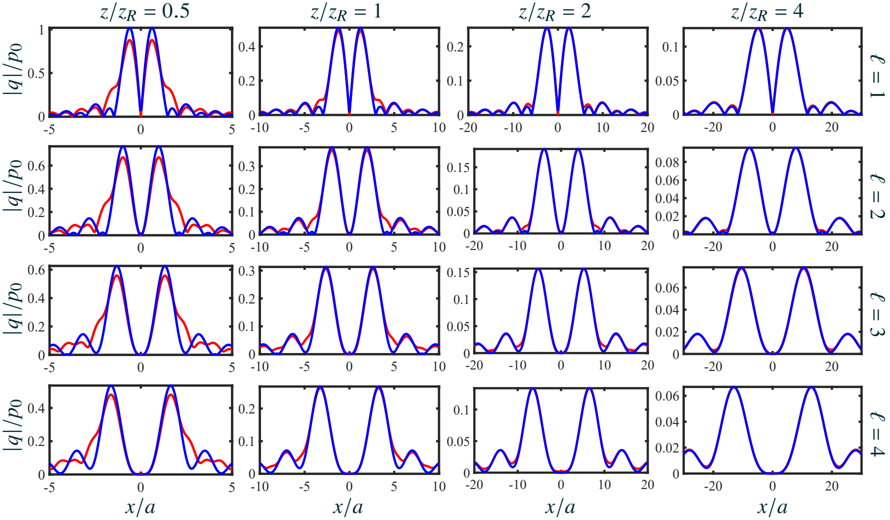
$$F_4(\xi) = 4 - 8J_1(\xi)/\xi - 4J_2(\xi) - \xi J_3(\xi)\,. \qquad \ell = 4 \tag{9e}$$

[5] M. Abramowitz and I. A. Stegun, editors. New York: Dover Publications, 1972, Item 11.1.1.

Verification of Eq. (7)

- The validity of Eq. (7) is assessed by comparison to

$$q(x,y,z) = \mathcal{F}_{xy}^{-1}\left\{e^{ik_z z}\mathcal{F}_{xy}[q(x,y,0)]\right\}\,, \quad k_z = k - (k_x^2 + k_y^2)/2k\,. \qquad (10)$$



Vortex radiation from focused circular piston

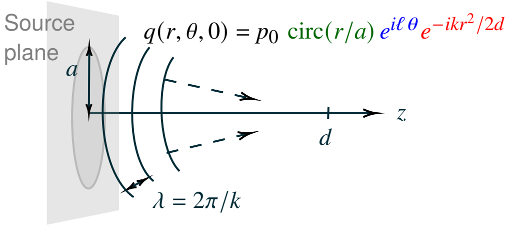
- To describe spherical focusing at a geometric focal length d, source condition (4) is multiplied by $\exp(-ikr^2/2d)$:

$$q(r,\theta,0) = p_0 \, \mathrm{circ}(r/a) \, e^{i\ell\theta} e^{-ikr^2/2d} \,, \tag{11}$$

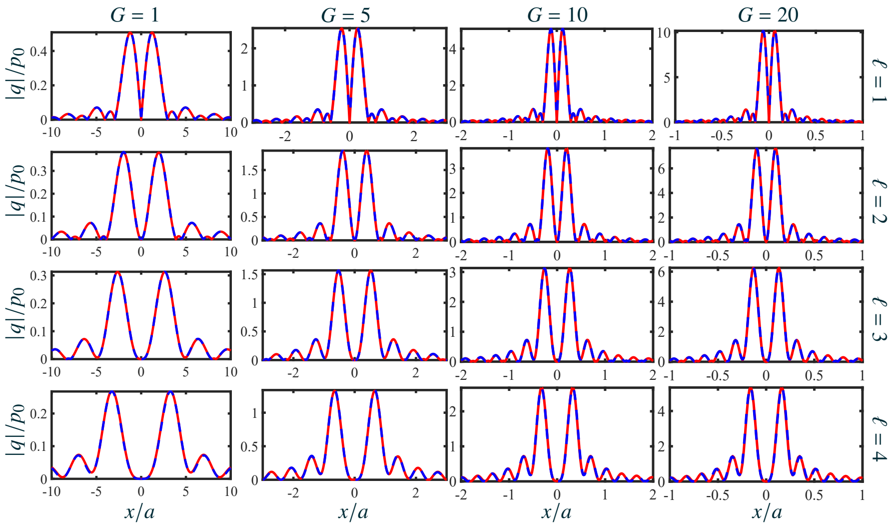
- An analytical solution of Eq. (3) is available at $z = d$:

$$q_\ell(r,\theta,d) = -ip_0 \frac{d}{kr^2} e^{ikr^2/2d} e^{i\ell(\theta-\pi/2)} F_\ell(kar/d) \,, \tag{12}$$

where $F_\ell(\xi)$ is given by Eq. (8).



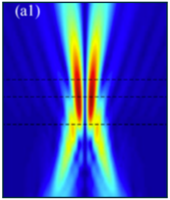
Verification of Eq. (12): analytical, Fourier



Outline

Vortex ring radius

- The magnitudes of vortex beam fields are axisymmetric.
- In the far field of Eq. (7), the field is conical.
- In the geometric focal plane $z = d$ of Eq. (12), the field forms toroidal ring.
- Equations (7) and (12) can be used to find the radius of these features.



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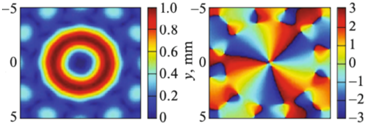

M. E. Terzi et al. *Moscow University Physics Bulletin* 72 (2017), pp. 61–67

Vortex ring radius

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- Maximizing Eqs. (7) and (12) in r amounts to solving

$$\frac{d|\xi^{-1}F_\ell(\xi)|}{d\xi} = 0\,,$$

where $\xi = kar/z$ (unfocused) and $\xi = kar/d$ (focused).

- Using Eq. (8) for F_ℓ and taking the derivative yields[6]

$$\sum_{k=0}^{\infty} \frac{(\ell+2k+1)\Gamma(\ell/2+k)}{\Gamma(\ell/2+2+k)} \left[\frac{J_{\ell+2k}(\xi) - J_{\ell+2k+2}(\xi)}{2\xi} - \frac{J_{\ell+2k+1}(\xi)}{\xi^2} \right] = 0\,. \tag{13}$$

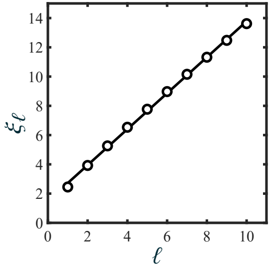
- The roots ξ_ℓ of Eq. (13) are fit to a line: $\xi_\ell = 1.23\ell + 1.49$.
- Solving $\xi_\ell = kar_\ell/z$ and $\xi_\ell = kar_\ell/d$ for r_ℓ yields

$$r_\ell = \frac{\xi_\ell z}{ka}\,, \quad z \gg z_R\,, \tag{14a}$$

$$= \frac{\xi_\ell d}{ka}\,, \quad z = d\,. \tag{14b}$$

[6] I. S. Gradshteyn and I. M. Ryzhik. New York: Academic Press, 1980, Item 8.471-2.

Vortex ring radius



Comparison of roots ξ_ℓ (circles) with least-squares fit $\xi_\ell = 1.23\ell + 1.49$ (line)

Outline

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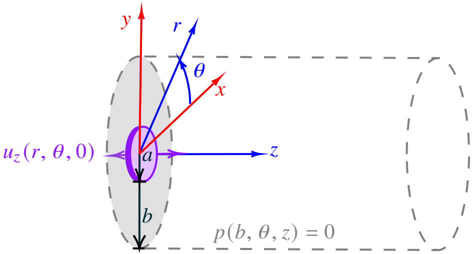
Alternative approach to diffraction theory

- The Rayleigh integral

$$p(\mathbf{r}) = -\frac{ik\rho_0 c_0}{2\pi}\int_{A_0} u_z(\mathbf{r}_0)\frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\,dA_0 \tag{15}$$

  is the starting-point for the study of diffraction from circular pistons.
- Equation (15) is traditionally derived from the Helmholtz-Kirchhoff integral.[7]
- Consider a piston placed concentrically within a tube of radius b.



[7] A. D. Pierce. Cham, Switzerland: Springer, 2019, Eqs. (5.2.6) and (5.7.3).

Alternative approach to diffraction theory

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- The solution of the Helmholtz equation for that scenario is

$$p(r,\theta,z) = \sum_{n=1}^{\infty} A_{\ell n} J_\ell(\alpha_{\ell n} r/b) e^{i(\ell\theta + \beta_{\ell n} z)}\,, \tag{16a}$$

$$A_{\ell n} = \frac{2k\rho_0 c_0 u_0}{\alpha_{\ell n}^2 \beta_{\ell n}} \frac{F_\ell(\alpha_{\ell n} a/b)}{J_{\ell+1}^2(\alpha_{\ell n})}\,, \quad \beta_{\ell n} = \sqrt{k^2 - k_r^2}\,, \quad k_r = \alpha_{\ell n}/b\,, \tag{16b}$$

where $\alpha_{\ell n}$ is the n^{th} root of J_ℓ.

- The ratio $\alpha_{\ell n}/b$ is vanishingly small except for large n, for which[8]

$$\alpha_{\ell n} \approx \pi(n - 1/4 + \ell/2), \quad n \gg 1\,. \tag{17}$$

- Defining $\zeta = \alpha_{\ell n} a/b \approx \frac{a}{b}\pi(n - 1/4 + \ell/2)$ for $n \gg 1$ sets $\Delta\zeta = \pi a/b$:

$$p = \rho_0 c_0 u_0 e^{i\ell\theta} \sum_{n=1}^{\infty} \frac{\Delta\zeta}{\zeta} \frac{F_\ell(\zeta) J_\ell(\zeta r/a)}{\sqrt{1 - (\zeta/ka)^2}} e^{ikz\sqrt{1-(\zeta/ka)^2}}\,. \tag{18}$$

- As $b \longrightarrow \infty$, Eq. (18) tends to

$$p = \rho_0 c_0 u_0 e^{i\ell\theta} \int_0^\infty \frac{F_\ell(\zeta) J_\ell(\zeta r/a)}{\zeta\sqrt{1 - (\zeta/ka)^2}} e^{ikz\sqrt{1-(\zeta/ka)^2}} d\zeta\,. \tag{19}$$

[8] I. S. Gradshteyn and I. M. Ryzhik. New York: Academic Press, 1980, Eq. (8.547).

Alternative approach to diffraction theory

- Using Watson's relation[9] and the 2D Fourier transforms

$$\mathcal{F}_{\rm 2D}\{f(r,\theta)\} = g(k_r,\psi) = \int_0^{2\pi}\int_0^{\infty} f(r,\theta)e^{-ik_r r\cos(\theta-\psi)}\, r\, dr\, d\theta\,, \tag{20a}$$

$$\mathcal{F}_{\rm 2D}^{-1}\{g(k_r,\psi)\} = f(r,\theta) = \int_0^{2\pi}\int_0^{\infty} g(k_r,\psi)e^{ik_r r\cos(\theta-\psi)}\, k_r\, dk_r\, d\psi\,, \tag{20b}$$

Eq. (19) recovers the angular spectrum method,

$$p(r,\theta,z) = \rho_0 c_0 k\, \mathcal{F}_{\rm 2D}^{-1}\{\mathcal{F}_{\rm 2D}\{u_z(r,\theta)\}e^{ik_z z}/k_z\}\,. \tag{21}$$

- Noting that $\mathcal{F}_{\rm 2D}\{e^{ikr}/r\} = i2\pi e^{ik_z|z|}/k_z$,[10] where $r = \sqrt{x^2+y^2+z^2}$, Eq. (21) reduces to

$$\boxed{p = -\frac{ik\rho_0 c_0}{2\pi}\, u(x,y) ** \frac{e^{ikr}}{r}\,.} \tag{22}$$

- By the definition of the convolution operation, Eq. (22) recovers Eq. (15).

[9] G. N. Watson. Cambridge, UK: Cambridge University Press, 1944, Sec. 2.2, Eq. (5).

[10] L. T. Brekhovskikh, translated by R. T. Beyer. Academic Press, 1980, pp. 227–234.

Outline

Introduction

Analytical solution of paraxial equation

Vortex ring radius

Alternative approach to diffraction theory

Summary

Thank you for listening!

Summary

- Solved paraxial equation for planar and focused circular vortex sources
- Calculated the ring radius for both solutions
- Derived alternative theory of diffraction from circular pistons

Further reading

- “Paraxial and ray approximations of acoustic vortex beams,” J. Acoust. Soc. Am. **155**, 2707–2723 (2024).
- “Analytical solutions for acoustic vortex beam radiation from planar and spherically focused circular pistons,” JASA Express Lett. **4**, 124001 (2024).

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Notation I

Symbol	Description	Dimensions
a	source radius	m
c_0	speed of sound	m s^{-1}
d	focal length	m
G	focusing gain $ka^2/2d$	1
i	complex unit	1
$\mathbf{k}$	wavenumber	m^{-1}
ℓ	orbital number	1
ρ_0	ambient mass density	kg m^{-3}
p	acoustic pressure	kg m^{-1} s^{-2}
q	paraxial pressure	kg m^{-1} s^{-2}
$\mathbf{R}$	separation vector $\mathbf{R} = \mathbf{r} - \mathbf{r}'$	m
$\mathbf{r}$	position vector	m
$\mathbf{v}$	particle velocity	m s^{-1}
z_R	Rayleigh distance, $ka^2/2$	m
ω	angular frequency, $\omega = 2\pi f$	s^{-1}

Dimensionless form of Eq. (19)

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- In terms of the dimensionless parameters

$$P \equiv p/\rho_0 c_0 u_0 \,, \quad R \equiv r/a \,, \quad Z \equiv z/z_R \,, \quad K \equiv ka \,, \tag{23}$$

where z_R is the Rayleigh distance $ka^2/2$, and where

$$k_z/k = \sqrt{1 - (\zeta/K)^2} \,, \quad \zeta \equiv k_r a \,, \tag{24}$$

Eq. (19) becomes

$$P = e^{i\ell\theta} \int_0^\infty \frac{F_\ell(\zeta) J_\ell(\zeta R)}{\zeta\sqrt{1 - (\zeta/K)^2}} e^{iK^2 Z\sqrt{1-(\zeta/K)^2}/2} \, d\zeta \,. \tag{25}$$

- Equation (25) is equivalent to and easier to evaluate than Eq. (21).

On-axis pressure of baffled circular piston

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- Equation (25) can be evaluated analytically for $R = \ell = 0$:

$$P(Z) = \int_0^\infty \frac{J_1(\zeta)}{\sqrt{1-(\zeta/K)^2}}\, e^{iK^2 Z\sqrt{1-(\zeta/K)^2}/2} d\zeta\,. \tag{26}$$

- Equation (26) evaluates to[11]

$$P(Z) = -iK\, I_{1/2}[-i\chi_-(Z)]\, K_{1/2}[-i\chi_+(Z)]\,, \tag{27}$$

where I_ν and K_ν are the modified Bessel functions of order ν, and where

$$\chi_\pm(Z) = \tfrac{K}{2}\left[\sqrt{1+(KZ/2)^2} \pm KZ/2\right]. \tag{28}$$

- Bessel function identities reduce Eq. (27) to

$$P(Z) = -2i\sin[\chi_-(Z)]e^{i\chi_+(Z)}\,, \tag{29}$$

recovering the on-axis pressure radiated by a planar circular piston.[12]

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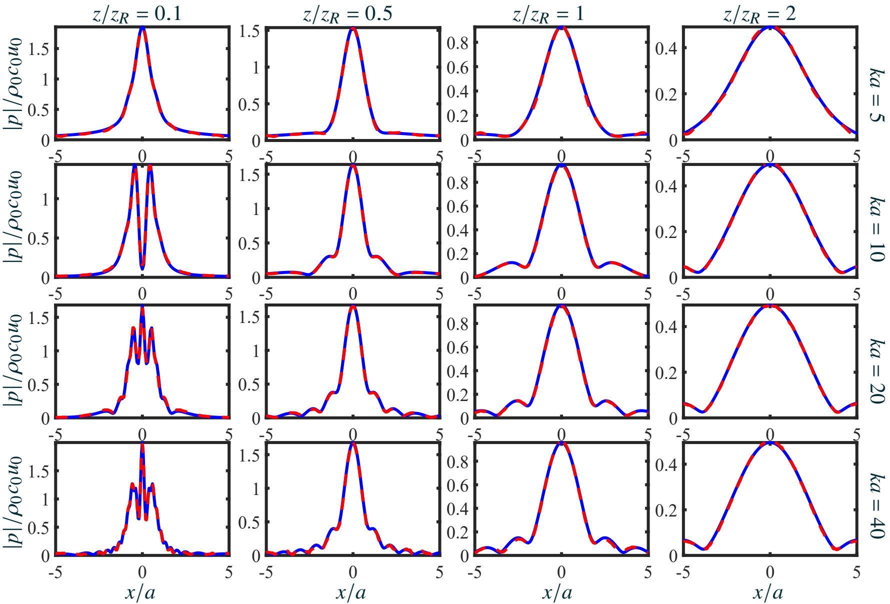
[11] I. S. Gradshteyn and I. M. Ryzhik. New York: Academic Press, 1980, Item 6.637-1.

[12] A. D. Pierce. Cham, Switzerland: Springer, 2019, Eq. (5.7.3).

Equation (25) for $\ell = 0$: semi-analytical, Fourier



Equation (25) for $\ell = 1$: semi-analytical, Fourier