$19^{th}$ *International Congress on Artificial Materials for Novel Wave Phenomena - Metamaterials 2025*
Amsterdam, The Netherlands, Sep. $1^{\text{st}}$ - $6^{\text{th}}$ 2025

# Acousto-electromagnetic media: Homogenization and constraints

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March 19, 2025

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***Abstract*** **– Ensembles of asymmetric piezoelectric scatterers embedded in a background medium have been predicted to couple acceleration to electric displacement. Previous models of this so-called electromomentum coupling are based on electrostatics. However, energy conservation involving time-varying electric fields requires considering the magnetic field. This work employs an acousto-electromagnetic polarizability matrix to calculate the fields scattered by a one-dimensional lattice of asymmetric piezoelectric scatterers. The effective constitutive relations couple acoustics and electromagnetism and satisfy passivity and reciprocity.**

## I. Introduction

Electromomentum coupling is the dependence of the electric field $\boldsymbol{E}$ and displacement $\boldsymbol{D}$ on mechanical momentum density $\boldsymbol{\mu}_v$ and acceleration $\dot{\boldsymbol{v}}$ [1]. If $\boldsymbol{E} = \boldsymbol{E}(t)$, the effect of the magnetic field $\boldsymbol{H}$ on 1D piezoelectric scatterers responsible for electromomentum coupling can be considered using the polarizability relation [2–5]

$$\begin{bmatrix} m_0 \\ d_0 \\ P_0 \\ M_0 \end{bmatrix} = \begin{bmatrix} -\beta_0\alpha_{pp} & -i(\rho_0\beta_0)^{1/2}\alpha_{pv} & (\beta_0\epsilon_0)^{1/2}\alpha_{pE} & 0 \\ -i(\rho_0\beta_0)^{1/2}\alpha_{pv} & \rho_0\alpha_{vv} & -i(\rho_0\epsilon_0)^{1/2}\alpha_{vE} & 0 \\ -(\beta_0\epsilon_0)^{1/2}\alpha_{pE} & i(\rho_0\epsilon_0)^{1/2}\alpha_{vE} & \epsilon_0\alpha_{EE} & 0 \\ 0 & 0 & 0 & \mu_0\alpha_{HH} \end{bmatrix} \begin{bmatrix} p_{\text{loc}} \\ v_{\text{loc}} \\ E_{\text{loc}} \\ H_{\text{loc}} \end{bmatrix}, \tag{1}$$

where $m_0$ is the acoustic monopole strength and $d_0$, $P_0$, and $M_0$ are the acoustic, electric, and magnetic dipole moments, respectively. The background mass density, compressibility, permittivity, and permeability are respectively denoted $\rho_0$, $\beta_0$, $\epsilon_0$, and $\mu_0$, and minus signs in Eq. (1) are due to the local pressure field $p_{\text{loc}}$ being positive for compressions. In Sec. II, Eq. (1) is used to describe a lattice of scatterers, leading to effective constitutive relations that couple acoustics and electromagnetism. Constraints due to passivity and reciprocity are derived in Sec. III.

## II. Source-driven homogenization

A harmonic force $f_{\text{ext}}$, volume velocity $q_{\text{ext}}$, and magnetic and electric current density $K_{\text{ext}}$ and $J_{\text{ext}}$ satisfy

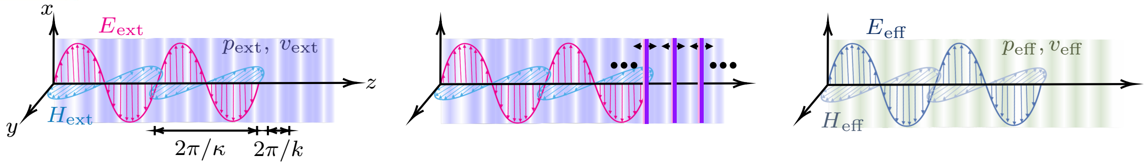
$$ikp_{\text{ext}} = i\omega\rho_0 v_{\text{ext}} + f_{\text{ext}}\,, \qquad ikv_{\text{ext}} = i\omega\beta_0 p_{\text{ext}} + q_{\text{ext}}\,, \qquad c_0 = (\beta_0\rho_0)^{-1/2} = \omega/k_0\,, \tag{2a}$$

$$i\kappa E_{\text{ext}} = i\omega\mu_0 H_{\text{ext}} - K_{\text{ext}}\,, \quad i\kappa H_{\text{ext}} = i\omega\epsilon_0 E_{\text{ext}} - J_{\text{ext}}\,, \quad C_0 = (\epsilon_0\mu_0)^{-1/2} = \omega/\kappa_0\,, \tag{2b}$$

where $k$ and $\kappa$ are the acoustic and electromagnetic wavenumbers, respectively, as depicted in Fig. 1(a) [2, 3]. A periodic array of passive, reciprocal, and identically oriented asymmetric subwavelength piezoelectric scatterers is introduced, as shown in Fig. 1(b). The local fields are the sum of external and scattered fields [3],

$$p_{\text{loc}} = p_{\text{ext}} - \beta_0^{-1} s m_0 + c_0 s d_0\,, \qquad v_{\text{loc}} = v_{\text{ext}} - c_0 s m_0 + \rho_0^{-1} s d_0\,, \qquad s \equiv \frac{ik_0}{2A}\sum_{n\neq 0} e^{ikz_n + ik_0|z_n|}, \tag{3a}$$

$$E_{\text{loc}} = E_{\text{ext}} + \epsilon_0^{-1} S P_0 + C_0 S M_0\,, \quad H_{\text{loc}} = H_{\text{ext}} + C_0 S P_0 + \mu_0^{-1} S M_0\,, \quad S \equiv \frac{i\kappa_0}{2A}\sum_{n\neq 0} e^{i\kappa z_n + i\kappa_0|z_n|}, \tag{3b}$$




(a) Free space (b) Multiple scattering (c) Effective medium

Fig. 1: (a) External fields obeying Eqs. (2). (b) Local fields given by Eqs. (3). (c) Effective fields satisfying Eq. (5).

where $A$ denotes the cross-sectional area of the unit cell. Equations (1)–(3) are combined to eliminate the local and external fields. Identifying $m_{\text{eff}} \equiv m_0/V$, $d_{\text{eff}} \equiv d_0/V$, $P_{\text{eff}} \equiv P_0/V$, and $M_{\text{eff}} \equiv M_0/V$, where $V$ is the unit cell volume, the resulting equations are combined with the effective displacements

$$\varepsilon_{\text{eff}} = -\beta_0 p_{\text{eff}} + m_{\text{eff}}\,, \quad \mu_{v,\text{eff}} = \rho_0 v_{\text{eff}} + d_{\text{eff}}\,, \quad D_{\text{eff}} = \epsilon_0 E_{\text{eff}} + P_{\text{eff}}\,, \quad B_{\text{eff}} = \mu_0 H_{\text{eff}} + M_{\text{eff}}\,, \tag{4}$$

yielding a system of linear equations, which, when inverted, results in fully coupled effective constitutive relations

$$\begin{bmatrix} \varepsilon_{\text{eff}} \\ \mu_{v,\text{eff}} \\ D_{\text{eff}} \\ B_{\text{eff}} \end{bmatrix} = \begin{bmatrix} \beta_{\text{eff}} & \chi^{\text{o}}_{pv} - i\chi^{\text{e}}_{pv} & i\chi^{\text{o}}_{pE} - \chi^{\text{e}}_{pE} & -\chi^{\text{o}}_{pH} + i\chi^{\text{e}}_{pH} \\ \chi^{\text{o}}_{pv} + i\chi^{\text{e}}_{pu} & \rho_{\text{eff}} & \chi^{\text{o}}_{vE} - i\chi^{\text{e}}_{vE} & -i\chi^{\text{o}}_{vH} + \chi^{\text{e}}_{vH} \\ -i\chi^{\text{o}}_{pE} - \chi^{\text{e}}_{pE} & \chi^{\text{o}}_{vE} + i\chi^{\text{e}}_{vE} & \epsilon_{\text{eff}} & \chi^{\text{o}}_{EH} \\ -\chi^{\text{o}}_{pH} - i\chi^{\text{e}}_{pH} & i\chi^{\text{o}}_{vH} + \chi^{\text{e}}_{vH} & \chi^{\text{o}}_{EH} & \mu_{\text{eff}} \end{bmatrix} \begin{bmatrix} -p_{\text{eff}} \\ v_{\text{eff}} \\ E_{\text{eff}} \\ H_{\text{eff}} \end{bmatrix}. \tag{5}$$

The matrix elements of Eq. (5) are functions of $s$, $S$, and the matrix elements of the inverse of Eq. (1), where the superscripts "e" and "o" denote whether each term is even or odd in $k$ and $\kappa$. The top-left $2 \times 2$ block of Eq. (5) recovers Eq. (35) of Ref. [3], while the bottom-right $2 \times 2$ block recovers Eq. (22) of Ref. [2], where the fact that $\chi^{\text{e}}_{EH} = 0$ is consistent with the assumption in Eq. (1) that the scatterers are magnetically symmetric. The fact that $\chi^{\text{o}}_{EH} \neq 0$ in Eq. (5) reflects that "even when inclusions are perfectly centersymmetric and with no inherent bianisotropy, a form of magnetoelectric coupling is still expected…due to lattice effects and the nonzero value of [$\kappa$]" [2]. Equation (5) corroborates the finding of Ref. [1] that asymmetric piezoelectric scatterers lead to electromomentum coupling, subscripted "$vE$." Also predicted by Eq. (5) are emergent piezomagnetic and magnetomomentum effects, subscripted "$pH$" and "$vH$," respectively.

## III. Constraints due to Passivity and Reciprocity

Since the piezoelectric scatterers described by Eq. (1) are passive and reciprocal, Eq. (5) must also obey passivity and reciprocity. According to Poynting's theorem, passive and lossless acousto-electromagnetic media satisfy

$$\tfrac{1}{2}\Re[\boldsymbol{\nabla} \cdot (p\boldsymbol{v}^* + \boldsymbol{E} \times \boldsymbol{H}^*)] = 0\,. \tag{6}$$

Substituting the general form of the acousto-electromagnetic constitutive relations

$$\begin{bmatrix} \varepsilon \\ \boldsymbol{\mu}_v \\ \boldsymbol{D} \\ \boldsymbol{B} \end{bmatrix} = [Y] \begin{bmatrix} -p \\ \boldsymbol{v} \\ \boldsymbol{E} \\ \boldsymbol{H} \end{bmatrix}, \quad [Y] \equiv \begin{bmatrix} \beta & \boldsymbol{\gamma} & \boldsymbol{e} & \boldsymbol{c} \\ \boldsymbol{\eta} & \underline{\boldsymbol{\rho}} & \underline{\boldsymbol{w}} & \underline{\boldsymbol{n}} \\ \boldsymbol{d} & \underline{\boldsymbol{v}} & \underline{\boldsymbol{\epsilon}} & \underline{\boldsymbol{\xi}} \\ \boldsymbol{b} & \underline{\boldsymbol{m}} & \underline{\boldsymbol{\zeta}} & \underline{\boldsymbol{\mu}} \end{bmatrix} \tag{7}$$

into Eq. (6) shows that

$$[Y] = [Y]^{\dagger}, \tag{8}$$

which is consistent Eq. (5). Meanwhile, a reciprocity theorem is formulated between points "1" and "2" [6],

$$\begin{aligned} &\oint_A (p_1\boldsymbol{v}_2 - p_2\boldsymbol{v}_1 + \boldsymbol{E}_1 \times \boldsymbol{H}_2 - \boldsymbol{E}_2 \times \boldsymbol{H}_1) \cdot \boldsymbol{e}_n\, dA \\ &\quad = \int_V (\boldsymbol{v}_2 \cdot \boldsymbol{f}_1 - \boldsymbol{v}_1 \cdot \boldsymbol{f}_2 + p_1 q_2 - p_2 q_1 + \boldsymbol{E}_2 \cdot \boldsymbol{J}_1 - \boldsymbol{E}_1 \cdot \boldsymbol{J}_2 + \boldsymbol{H}_1 \cdot \boldsymbol{K}_2 - \boldsymbol{H}_2 \cdot \boldsymbol{K}_1)\, dV \\ &\qquad - i\omega \int_V (p_1\varepsilon_2 - p_2\varepsilon_1 + \boldsymbol{v}_1 \cdot \boldsymbol{\mu}_2 - \boldsymbol{v}_2 \cdot \boldsymbol{\mu}_1 - \boldsymbol{E}_1 \cdot \boldsymbol{D}_2 + \boldsymbol{E}_2 \cdot \boldsymbol{D}_1 + \boldsymbol{H}_1 \cdot \boldsymbol{B}_2 - \boldsymbol{H}_2 \cdot \boldsymbol{B}_1)\, dV\,, \end{aligned} \tag{9}$$

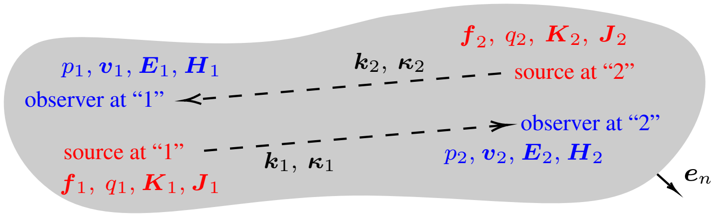



Fig. 2: Reciprocity between points "1" and "2" for acousto-electromagnetic media with surface $A$ and volume $V$.

as shown in Fig. 2. The first integral of Eq. (9) vanishes by radiation condition, while the second integral vanishes due to the invariance under the exchange of source and observer [7]. Substitution of Eq. (7) into the integrand of the remaining integral in Eq. (9) shows that $\underline{\boldsymbol{\rho}}$, $\underline{\boldsymbol{\epsilon}}$, and $\underline{\boldsymbol{\mu}}$ are symmetric rank-2 tensors, while

$$\boldsymbol{\gamma} = \boldsymbol{\eta}^{\mathrm{o}} - \boldsymbol{\eta}^{\mathrm{e}}\,, \qquad \boldsymbol{e} = \boldsymbol{d}^{\mathrm{e}} - \boldsymbol{d}^{\mathrm{o}}\,, \qquad \boldsymbol{c} = \boldsymbol{b}^{\mathrm{o}} - \boldsymbol{b}^{\mathrm{e}}\,, \tag{10a}$$

$$\underline{\boldsymbol{w}} = (\underline{\boldsymbol{v}}^{\mathrm{o}} - \underline{\boldsymbol{v}}^{\mathrm{e}})^{\mathrm{T}}\,, \quad \underline{\boldsymbol{n}} = (\underline{\boldsymbol{m}}^{\mathrm{e}} - \underline{\boldsymbol{m}}^{\mathrm{o}})^{\mathrm{T}}\,, \quad \underline{\boldsymbol{\xi}} = (\underline{\boldsymbol{\zeta}}^{\mathrm{o}} - \underline{\boldsymbol{\zeta}}^{\mathrm{e}})^{\mathrm{T}}\,. \tag{10b}$$

Combining Eqs. (10) with the decompositions [2, 3]

$$\boldsymbol{\eta} = \boldsymbol{\chi}^{\mathrm{o}}_{pv} + i\boldsymbol{\chi}^{\mathrm{e}}_{pv}\,, \qquad \boldsymbol{d} = -i\boldsymbol{\chi}^{\mathrm{o}}_{pE} - \boldsymbol{\chi}^{\mathrm{e}}_{pE}\,, \qquad \boldsymbol{b} = -\boldsymbol{\chi}^{\mathrm{o}}_{pH} - i\boldsymbol{\chi}^{\mathrm{e}}_{pH}\,, \tag{11a}$$

$$\underline{\boldsymbol{v}} = \underline{\boldsymbol{\chi}}^{\mathrm{o}}_{uE} + i\underline{\boldsymbol{\chi}}^{\mathrm{e}}_{uE}\,, \quad \underline{\boldsymbol{m}} = i\underline{\boldsymbol{\chi}}^{\mathrm{o}}_{uH} + \underline{\boldsymbol{\chi}}^{\mathrm{e}}_{uH}\,, \qquad \underline{\boldsymbol{\zeta}} = \underline{\boldsymbol{\chi}}^{\mathrm{o}}_{EH} + i\underline{\boldsymbol{\chi}}^{\mathrm{e}}_{EH} \tag{11b}$$

recovers the form of the $4\times4$ matrix in Eq. (5), showing that the homogenized medium in Sec. II obeys reciprocity.

The relations $\boldsymbol{b} = \boldsymbol{c}^*$ and $\underline{\boldsymbol{m}} = \underline{\boldsymbol{n}}^\dagger$ from Eq. (8) due to passivity, along with the third of Eqs. (10a) and the second of Eqs. (10b) due to reciprocity, are novel. The constraints provided by Eqs. (8) and (10) on the top-left $2\times2$ block of Eq. (7) recover Eq. (5) of Ref. [3] and the results of Eqs. (3.14) and (3.16) of Ref. [7]; those on the top-left $3\times3$ block of Eq. (7) recover the acoustic limit of the strain-charge form of Eqs. (18), (20), (33), and (42) of Ref. [8], and those on the bottom-right $2\times2$ block of Eq. (7) recover Eqs. (8) and (23) of Ref. [9].

## IV. Conclusion

A periodic array of asymmetric piezoelectric scatterers was homogenized, and constraints due to passivity and reciprocity were derived. Limiting cases of Eqs. (5), (8), and (10) recover the results of Refs. [2, 3, 7, 8, 9]. The additional couplings to the magnetic field may be relevant to the design of multiphysics metamaterials [10].

## Acknowledgement

Work supported by DARPA and the ARL:UT Chester M. McKinney Graduate Fellowship in Acoustics.

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