Radiation force on inhomogeneous subwavelength scatterers due to progressive waves

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Linear scattering problem

Examples

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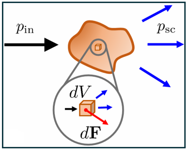
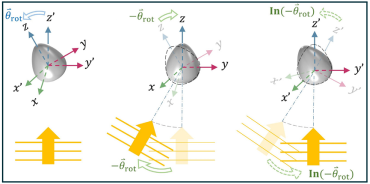
Motivation

Particle manipulation of asymmetric objects

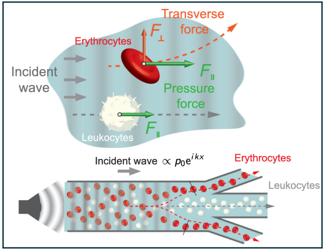
- Forces and torques due to standing waves have been investigated.[1]
- Jerome et al.[2] used the Born approximation to calculate radiation force and torque on subwavelength objects in primarily standing wave fields.
- Forces due to progressive waves have only recently garnered interest.[3]



T. S. Jerome. PhD thesis. University of Texas at Austin, 2022



T. Tang, Y. Zhang, B. Dong, and L. Huang. *J. Acoust. Soc. Am.* 156 (2024), pp. 2767–2782



M. Smagin, I. Toftul, K. Y. Bliokh, and M. Petrov. *Phys. Rev. Appl.* 22 (2024), pp. 1–20

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[1] E. B. Lima and G. T. Silva. *J. Acoust. Soc. Am.* 150 (2021), pp. 376–384.
[2] T. S. Jerome and M. F. Hamilton. *J. Acoust. Soc. Am.* 150 (2021), pp. 3417–3427.
[3] T. Tang and L. Huang. *J. Sound Vib.* 532 (2022), pp. 1–19.

COMET Neowise[4]

[4] J. M. Izen. 2020

Conservation of momentum at quadratic order

- Acoustic radiation force is a consequence of $O(\epsilon^2)$ momentum conservation:

$$\boxed{\frac{\partial \mathbf{g}}{\partial t} = \boldsymbol{\nabla} \cdot \underline{\mathbf{T}}} \tag{1}$$

- The momentum density is

$$\mathbf{g} = p\mathbf{v}/c_0^2 \tag{2}$$

- In free space, the instantaneous acoustic radiation stress tensor is

$$\underline{\mathbf{T}} \equiv L\underline{\mathbf{I}} - \rho_0 \mathbf{v} \otimes \mathbf{v} \,. \tag{3}$$

- The Lagrangian density is

$$L = \tfrac{1}{2}\rho_0 v^2 - \frac{p^2}{2\rho_0 c_0^2} \,, \tag{4}$$

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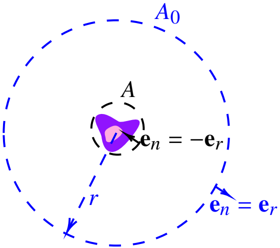
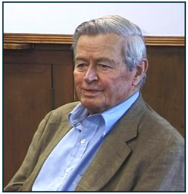
Westervelt's far-field formulation

- The volume integral of $\boldsymbol{\nabla}\cdot\langle\underline{\mathbf{T}}\rangle$ enclosing an object is the radiation force[5]



$$\mathbf{F} = \int_V \boldsymbol{\nabla}\cdot\langle\underline{\mathbf{T}}\rangle dV$$
$$= \oint_A \langle\underline{\mathbf{T}}\rangle\cdot d\mathbf{A}$$



P. J. Westervelt

- Invoking the far-field approximation and energy conservation yields

$$F_{\parallel} = \frac{p_0^2}{2\rho_0 c_0^2}\oint |\Phi_s(\mathbf{k}_s)|^2(1-\cos\psi_0)d\Omega_0\,, \tag{5a}$$

$$F_{\perp} = -\frac{p_0^2}{2\rho_0 c_0^2}\oint |\Phi_s(\mathbf{k}_s)|^2\,\mathbf{e}_m\cdot\mathbf{e}_r\,d\Omega_0\,,\quad d\Omega_0 = r_0^{-2}\,dA_0\,. \tag{5b}$$

where Φ_s = scattered directivity, $\cos\psi = \mathbf{k}_s\cdot\mathbf{k}_i/k^2$, and Ω = solid angle.

[5] P. J. Westervelt. *J. Acoust. Soc. Am.* 29 (1957), pp. 26–29.

Linear scattering problem

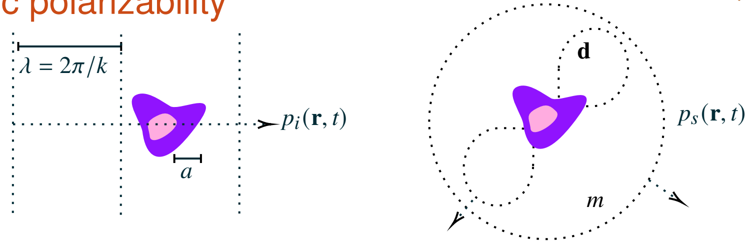
[F]or the calculation of the average force correct up to terms of the second order in the velocity, it is sufficient to find the solution of the linear scattering problem.[6]

G. Boebinger, S. Iordansky, D. Pines, and L. Pitaevskii. *Physics Today* 70 (2017), pp. 68–69

[6] L. P. Gor'kov. *Sov. Phys. Dokl.* 6 (1962), pp. 773–775.

Outline

Acoustic polarizability



(a) incident acoustic waves, $ka \ll 1$

(b) scattered acoustic waves

- *Polarizabilities* describe the heterogeneity's response to local fields:[7]
  - α_{m} relates $p_i \leftrightarrow m$
  - $\underline{\boldsymbol{\alpha}}_{\mathrm{d}}$ relates $\mathbf{v}_i \leftrightarrow \mathbf{d}$
  - $\boldsymbol{\alpha}_{\mathrm{c}}$ relates $\mathbf{v}_i \leftrightarrow m$ and $p_i \leftrightarrow \mathbf{d}$
- The scattered monopole strength m and dipole moment $\mathbf{d}$ are given by

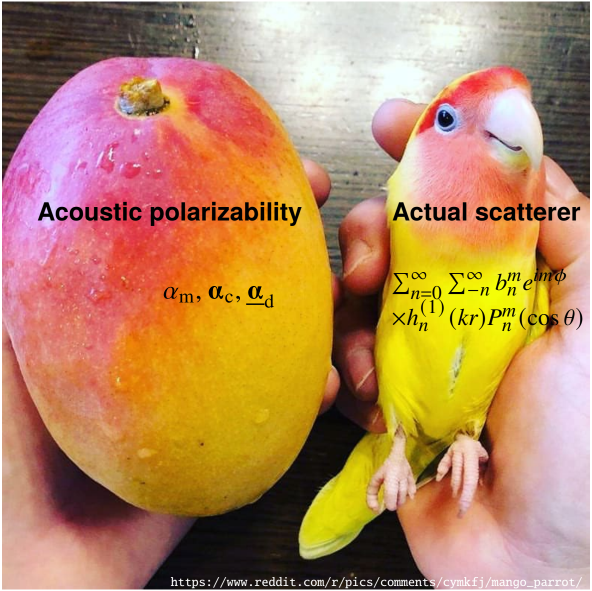
$$m = -\beta_0 \alpha_{\mathrm{m}} p_i - i c_0^{-1} \boldsymbol{\alpha}_{\mathrm{c}} \cdot \mathbf{v}_i \,, \tag{6a}$$

$$\mathbf{d} = -i c_0^{-1} \boldsymbol{\alpha}_{\mathrm{c}} p_i + \rho_0 \underline{\boldsymbol{\alpha}}_{\mathrm{d}} \cdot \mathbf{v}_i \,. \tag{6b}$$

- **Objective: Obtain a general approach to calculate α_{m}, $\boldsymbol{\alpha}_{\mathrm{c}}$, and $\underline{\boldsymbol{\alpha}}_{\mathrm{d}}$.**

[7] C. F. Sieck, A. Alù, and M. R. Haberman. *Phys. Rev. B.* 96 (2017), pp. 1–20

Multipole expansion to dipole order



Scattering of sound from heterogeneities

- The equation of state in heterogeneous media is[8]

$$\frac{\partial \rho'}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} - \mathbf{v} \cdot \boldsymbol{\nabla} \rho(\mathbf{r}) \,. \tag{7}$$

- The linearized mass conservation equation combined with Eq. (7) yields

$$\boldsymbol{\nabla} \cdot \mathbf{v} = -\beta(\mathbf{r}) \frac{\partial p}{\partial t} \,. \tag{8}$$

- Linearizing the momentum equation yields

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho(\mathbf{r})} \boldsymbol{\nabla} p \,. \tag{9}$$

- Combining Eqs. (8) and (9) and assuming time-harmonic solutions yields

$$\nabla^2 \tilde{p}_\omega + k^2 \tilde{p}_\omega = k^2 f_1 \tilde{p}_\omega + \boldsymbol{\nabla} \cdot \left(\tfrac{3f_2}{2+f_2} \boldsymbol{\nabla} \tilde{p}_\omega \right) \,. \tag{10}$$

where the contrast factors are[9]

$$f_1(\mathbf{r}) = 1 - \frac{\beta(\mathbf{r})}{\beta_0} \,, \quad f_2(\mathbf{r}) = \frac{2[\rho(\mathbf{r}) - \rho_0]}{2\rho(\mathbf{r}) + \rho_0} \,. \tag{11}$$

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[8] A. D. Pierce. 2nd edition. Woodbury, New York: Acoustical Society of America, 1989.

[9] L. P. Gor'kov. *Sov. Phys. Dokl.* 6 (1962), pp. 773–775.

Scattering of sound from heterogeneities

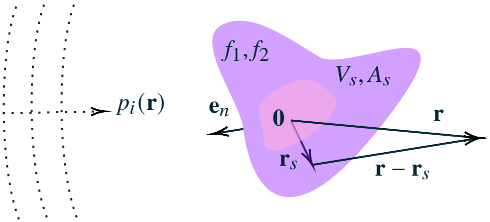
- The Helmholtz-Kirchhoff integral theorem solves Eq. (10),[10]

$$\tilde{p}_\omega(\mathbf{r}) = \tilde{p}_{i,\omega}(\mathbf{r}) + \tilde{p}_{s,\omega}(\mathbf{r}) \,,$$

$$\tilde{p}_{s,\omega}(\mathbf{r}) = \int_{V_s} \left[k^2 f_1(\mathbf{r}_s) \tilde{p}_\omega(\mathbf{r}_s) g(\mathbf{r}|\mathbf{r}_s) - \tfrac{3 f_2(\mathbf{r}_s)}{2 + f_2(\mathbf{r}_s)} \boldsymbol{\nabla}_s \tilde{p}_\omega(\mathbf{r}_s) \cdot \boldsymbol{\nabla}_s g(\mathbf{r}|\mathbf{r}_s) \right] dV_s \,.$$

- The origin is defined by the *centroid*,

$$\mathbf{0} \equiv \frac{\int_{V_s} \mathbf{r}_s \, dV_s}{\int_{V_s} dV_s} \,. \tag{12}$$



[10] P. M. Morse and K. U. Ingard. McGraw-Hill, 1968.

Three approximations

1. If $r \gg a$, then $g(\mathbf{r}|\mathbf{r}_s)$ and $\boldsymbol{\nabla}_s g(\mathbf{r}|\mathbf{r}_s)$ can be written in terms of $\mathbf{k}_s = k\mathbf{e}_r$:

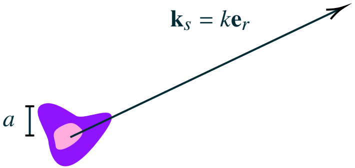
$$g(\mathbf{r}|\mathbf{r}_s) \simeq \frac{e^{ikr}}{4\pi r} e^{-i\mathbf{k}_s \cdot \mathbf{r}_s} , \quad \boldsymbol{\nabla}_s g \simeq -i\mathbf{k}_s g \, . \tag{13}$$

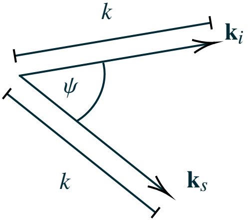
2. In the long-wavelength limit $ka \sim |\mathbf{k}_s \cdot \mathbf{r}_s| \ll 1$, so

$$e^{-i\mathbf{k}_s \cdot \mathbf{r}_s} \simeq 1 - i\mathbf{k}_s \cdot \mathbf{r}_s \, , \tag{14}$$

   and therefore $p_i(\mathbf{r}) = p_0 e^{i\mathbf{k}_i \cdot \mathbf{r}} \simeq p_0(1 + i\mathbf{k}_i \cdot \mathbf{r})$.

3. If $f_1, f_2 \ll 1$, then $|p_s| \ll |p_i|$, and the problem becomes explicit.





Solution of scattering problem to dipole order

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- Identifying

$$\alpha_\mathrm{m} = -\int_{V_s} f_1(\mathbf{r}_s)\, dV_s\,, \tag{15a}$$

$$\underline{\boldsymbol{\alpha}}_\mathrm{d} = \underline{\mathbf{I}} \int_{V_s} \tfrac{3f_2(\mathbf{r}_s)}{2+f_2(\mathbf{r}_s)} dV_s \equiv \alpha_d \underline{\mathbf{I}}\,, \tag{15b}$$

$$\boldsymbol{\alpha}_\mathrm{c} = -k\left[\int_{V_s} f_1(\mathbf{r}_s)\mathbf{r}_s\, dV_s - \cos\psi \int_{V_s} \tfrac{3f_2(\mathbf{r}_s)}{2+f_2(\mathbf{r}_s)} \mathbf{r}_s\, dV_s\right] \tag{15c}$$

yields the scattered far field:

$$p_s(\mathbf{r}) = p_0 \frac{e^{ikr}}{r} \Phi_s\,, \quad \Phi_s(\mathbf{k}_s) = \tfrac{k^2}{4\pi}\left[\alpha_\mathrm{m} + i\boldsymbol{\alpha}_\mathrm{c}\cdot(\mathbf{e}_i - \mathbf{e}_r) + \alpha_\mathrm{d}\cos\psi\right]. \tag{16}$$

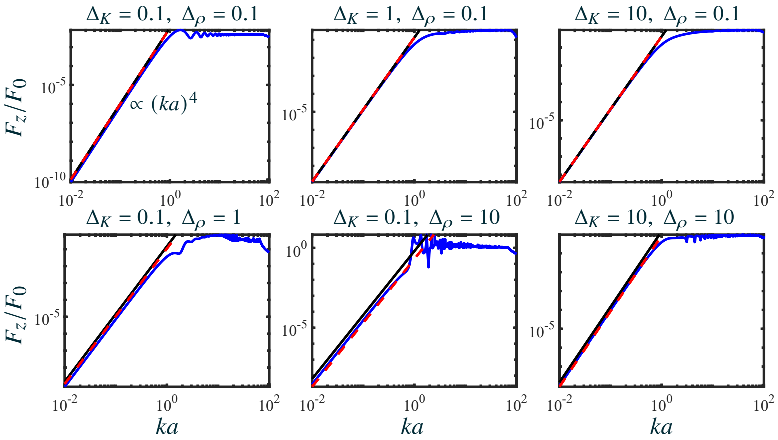
- Thus using

$$|\Phi_s|^2 = \tfrac{k^4}{16\pi^2}\left\{\alpha_\mathrm{m}^2 + 2\alpha_\mathrm{m}\alpha_\mathrm{d}\cos\psi + [\boldsymbol{\alpha}_\mathrm{c}\cdot(\mathbf{e}_i - \mathbf{e}_r)]^2 + \alpha_\mathrm{d}^2\cos^2\psi\right\} \tag{17}$$

with Eq. (5) gives the radiation force in terms of the polarizabilities.

Outline

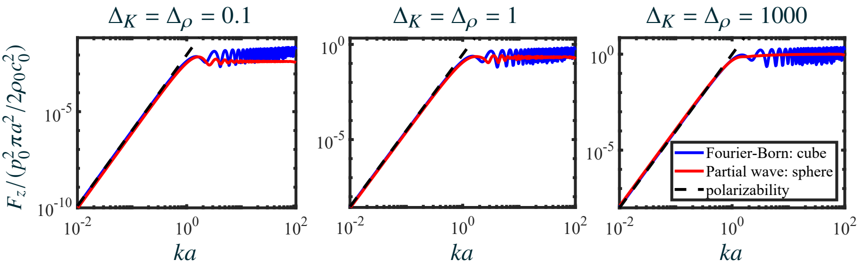
Example 1: Homogeneous sphere[11]



Comparison of the polarizability formulation (42), Gor'kov's result (43), and the exact solution based on spherical wave expansions (44), for values of $\Delta_K = (K_s - K_0)/K_0$ and $\Delta_\rho = (\rho_s - \rho_0)/\rho_0$ ranging two orders of magnitude.

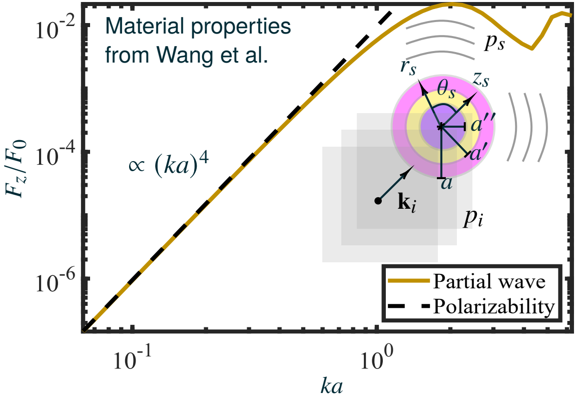
[11] The forces are normalized to $F_0 = p_0^2 \pi a^2 / 2\rho_0 c_0^2$.

Example 2: Homogeneous cube



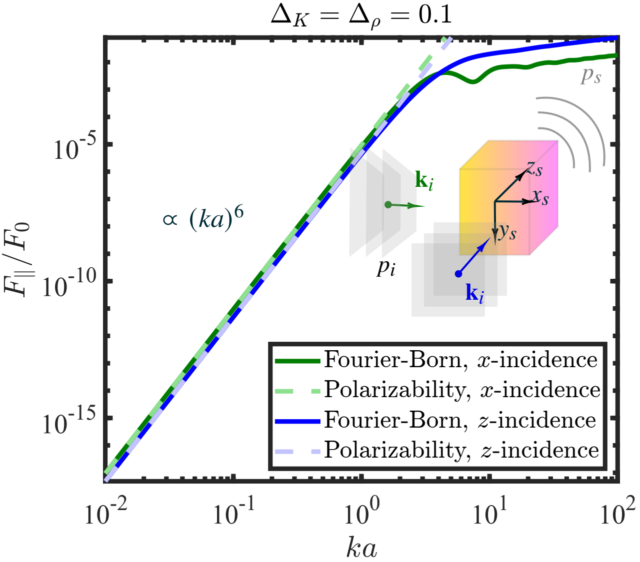
Comparison of the radiation force given by the polarizability formulation [dashed line, Eq. (48)] to the forces on a homogeneous cube [Eq. (52)] and sphere of equal volume [red curves, Eq. (44)] for $\Delta_K = (K_s - K_0)/K_0$ and $\Delta_\rho = (\rho_s - \rho_0)/\rho_0$ ranging four orders of magnitude. For $ka \ll 1$ in all cases, the force on the cube converges to that on the sphere.

Example 3: Spherically symmetric nucleated cell[12]



[12] Y.-Y. Wang et al. *J. Appl. Phys.* 122 (2017), pp. 1–6.

Example 4: Antisymmetric inhomogeneous cube



Conclusion

Summary

- Solved **linear scattering problem** at $O[(ka)^3]$ in Born approximation
- Calculated acoustic radiation force due to progressive waves on **homogeneous and inhomogeneous cubes and spheres**
- Compared results to **partial wave expansions** and **Fourier-Born scattering**

Future work

- Calculate radiation torque
- Develop ray theory for calculation of high-frequency asymptote

Acknowledgments

- T. S. Jerome for the exact solutions on the homogeneous and layered spheres
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Prof. M. R. Haberman

Prof. M. F. Hamilton

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Notation I

Symbol	Description	Dimensions
a	characteristic size of scatterer	m
c_0	speed of sound	m s^{-1}
$\mathbf{F}$	force	kg m s^{-2}
i	complex unit	1
$\mathbf{k}$	wavenumber	m^{-1}
p	acoustic pressure	kg m^{-1} s^{-2}
$\mathbf{R}$	separation vector, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$	m
$\mathbf{r}$	position vector	m
$\mathbf{v}$	particle velocity	m s^{-1}
ρ_0	ambient mass density	kg m^{-3}
$\underline{\mathbf{T}}$	acoustic radiation stress tensor	kg m^{-1} s^{-2}
$\mathbf{S}$	instantaneous Poynting vector	kg s^{-3}
ω	angular frequency, $\omega = 2\pi f$	s^{-1}

Acoustic energy densities

- Acoustic radiation force is a consequence of the energy carried by sound.
- The exact potential and kinetic energy densities of sound are

$$E_T = \tfrac{1}{2}\rho v^2\,, \tag{18a}$$

$$E_U = -\frac{1}{V_0}\int_{V_0}^{V} p\,dV\,, \tag{18b}$$

- Linearization and changes of variables[13] lead to

$$E_T = \tfrac{1}{2}\rho_0 v^2\,, \tag{19a}$$

$$E_U = \frac{p^2}{2\rho_0 c_0^2}\,. \tag{19b}$$

- The Lagrangian density is

$$L = E_T - E_U\,. \tag{20}$$

[13] Noting that $dV \simeq -V_0 dp/\rho_0 c_0^2$ leads from Eq. (18b) to Eq. (19b).

Acoustic radiation stress tensor

- Neglecting viscosity, the conservation of momentum and mass require that

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = -\boldsymbol{\nabla}P\,, \qquad \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0\,. \tag{21}$$

- Combining Eqs. (21), rearranging, and invoking vector and tensor calculus identities yields[14]

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}\otimes\mathbf{v}) = -\boldsymbol{\nabla}P\,. \tag{22}$$

- Assuming that ρ, $\mathbf{v}$, and P in Eq. (22) are time-harmonic, taking its time average, and retaining quadratic terms yields

$$\boldsymbol{\nabla}\cdot\langle\underline{\mathbf{T}}\rangle = \mathbf{0}\,, \quad \langle\underline{\mathbf{T}}\rangle = -\underline{\mathbf{I}}\langle P - P_0\rangle - \rho_0\,\langle\mathbf{v}\otimes\mathbf{v}\rangle \tag{23}$$

- The rank-2 quantity $\langle\underline{\mathbf{T}}\rangle$ is the acoustic radiation stress tensor.

[14] "$\otimes$" is the outer product: $(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{c} = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$.

Mean excess pressure

- $\langle P - P_0 \rangle$ appearing in Eq. (23) is the "mean excess pressure."
- Evaluating this quantity involves lots of manipulations.
- Begin by Taylor expanding in the enthalpy w:

$$P - P_0 = \rho w + \frac{1}{2}\left(\frac{\partial \rho}{\partial w}\right)_{s,0} w^2 + \ldots \tag{24}$$

- Manipulations and vector calculus identities reduce Eq. (24) to

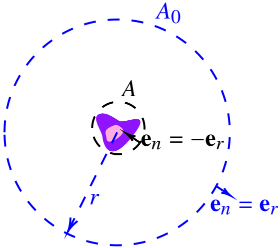
$$\langle P - P_0 \rangle = \frac{\langle p^2 \rangle}{2\rho_0 c_0^2} - \tfrac{1}{2}\rho_0 \langle v^2 \rangle + \rho_0 C\,. \tag{25}$$

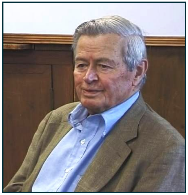
- C in Eq. (25) is 0 in free space, defining *Langevin radiation pressure*.[15]

[15] T. G. Wang and C. P. Lee, "Radiation Pressure and Acoustic Levitation," M. F. Hamilton and D. T. Blackstock, editors. Cham, Switzerland: Springer, 2024.

Westervelt's far-field formulation (more detail)

- $\nabla \cdot \langle \underline{\mathbf{T}} \rangle = \mathbf{0}$ has units of stress per unit length, i.e., force per unit volume.[16]





P. J. Westervelt

- Integrating over the volume within A_0 but outside A gives

$$\int_{V_0} \nabla \cdot \langle \underline{\mathbf{T}}(\mathbf{r}_0) \rangle \, dV_0 - \int_{V_s} \nabla \cdot \langle \underline{\mathbf{T}}(\mathbf{r}_s) \rangle \, dV_s = \mathbf{0} \, . \tag{26}$$

Invoking Eq. (3) and the divergence theorem yields[17]

$$\mathbf{F} = \oint_{A_0} \left[\underline{\mathbf{I}} \langle L \rangle - \rho_0 \langle \mathbf{v} \otimes \mathbf{v} \rangle \right] \cdot \mathbf{e}_n \, dA_0 \, . \tag{27}$$

[16] https://acoustics.whoi.edu/oasl_colleagues.html

[17] P. J. Westervelt. *J. Acoust. Soc. Am.* 29 (1957), pp. 26–29.

Westervelt's far-field formulation (more detail)

- Noting that

$$p = p_i + p_s\,, \quad \mathbf{v} = \mathbf{v}_i + \mathbf{v}_s\,,$$

reduces Eq. (27) to

$$\mathbf{F} = -\frac{1}{c_0}\oint_{A_0} \langle p_s \mathbf{v}_i + p_i v_s \mathbf{e}_r + p_s v_s \mathbf{e}_r \rangle \; dA_0\,. \tag{28}$$

- Meanwhile, the total intensity is

$$\mathbf{I} = (p_i + p_s)(\mathbf{v}_i + \mathbf{v}_s) = p_i \mathbf{v}_i + p_s \mathbf{v}_s + p_s \mathbf{v}_i + p_i \mathbf{v}_s\,. \tag{29}$$

- Removing the scattering object, the intensity is

$$\mathbf{I} = p_i \mathbf{v}_i\,. \tag{30}$$

- Equations (29) and (30) are equal by energy conservation, i.e., adding the scattering object in free space does not add energy to the system:

$$p_i \mathbf{v}_s + p_s \mathbf{v}_i + p_s \mathbf{v}_s = 0 \implies \oint_{A_0} \langle p_i v_s + p_s v_i \cos\psi_0 + p_s v_s \rangle dA_0 = 0. \tag{31}$$

Westervelt's far-field formulation (more detail)

- Dotting Eq. (28) into $\mathbf{e}_i = \mathbf{k}_i/k$ and invoking Eq. (31) yields

$$F_{\parallel} = \frac{1}{c_0} \oint \langle p_s v_s \rangle (1 - \cos\psi_0) dA_0 \,. \tag{32}$$

- Similar reasoning leads to the force at right angles to the incident wave:

$$F_{\perp} = -c_0^{-1} \oint_{A_0} \langle p_s v_s \rangle \mathbf{e}_r \cdot \mathbf{e}_m \, dA_0 \,, \quad \mathbf{e}_m \cdot \mathbf{e}_i = 0 \,. \tag{33}$$

- Denote $p \equiv \Re(\tilde{p})$ and $\tilde{p} = \tilde{p}_\omega e^{-i\omega t}$:

$$\tilde{p}_{s,\omega} = p_0 \frac{e^{ikr}}{r} \Phi_s(\mathbf{k}_s) \,, \quad \tilde{v}_{s,\omega} = \frac{p_0}{\rho_0 c_0} \frac{e^{ikr}}{r} \Phi_s(\mathbf{k}_s) \,, \tag{34}$$

where Φ_s is the scattered directivity. Equations (32) and (33) become

$$F_{\parallel} = \frac{p_0^2}{2\rho_0 c_0^2} \oint |\Phi_s(\mathbf{k}_s)|^2 (1 - \cos\psi_0) d\Omega_0 \,, \tag{35a}$$

$$F_{\perp} = -\frac{p_0^2}{2\rho_0 c_0^2} \oint |\Phi_s(\mathbf{k}_s)|^2 \, \mathbf{e}_m \cdot \mathbf{e}_r \, d\Omega_0 \,, \quad d\Omega_0 = r_0^{-2} \, dA_0 \,. \tag{35b}$$

Why is $\underline{\boldsymbol{\alpha}}_{\mathrm{d}} = \alpha_{\mathrm{d}}\underline{\mathbf{I}}$?

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- Strong scattering leads to $\underline{\boldsymbol{\alpha}}_{\mathrm{d}} \neq \alpha_{\mathrm{d}}\underline{\mathbf{I}}$.[18]
- In the Born approximation, $\underline{\boldsymbol{\alpha}}_{\mathrm{d}} = \alpha_{\mathrm{d}}\underline{\mathbf{I}}$:
  - The integral solution of Eq. (10) in the far field is

$$\tilde{p}_{s,\omega}(\mathbf{r}) = -\frac{e^{ikr}}{4\pi r}\Bigg[k^2\int_{V_s} f_1(\mathbf{r}_s)[\tilde{p}_{i,\omega}(\mathbf{r}_s)+\tilde{p}_{s,\omega}(\mathbf{r}_s)]e^{-i\mathbf{k}_s\cdot\mathbf{r}_s}dV_s + i\int_{V_s}\frac{3f_2(\mathbf{r}_s)}{2+f_2(\mathbf{r}_s)}\mathbf{k}_s\cdot\boldsymbol{\nabla}_s[\tilde{p}_{i,\omega}(\mathbf{r}_s)+\tilde{p}_{s,\omega}(\mathbf{r}_s)]e^{-i\mathbf{k}_s\cdot\mathbf{r}_s}dV_s\Bigg]. \tag{36}$$

  - For an incident plane wave and $|p_s| \ll |p_i|$, $\mathbf{k}_s\cdot\boldsymbol{\nabla}_s\tilde{p}_{i,\omega}(\mathbf{r}_s) \propto \mathbf{e}_r\cdot\mathbf{e}_i$.
  - Thus the scattered dipole is oriented in the same direction as $\mathbf{k}_i$:

$$(\underline{\boldsymbol{\alpha}}_{\mathrm{d}}\cdot\mathbf{e}_i)\cdot\mathbf{e}_r = \mathbf{e}_i\cdot\mathbf{e}_r\int_{V_s}\frac{3f_2(\mathbf{r}_s)}{2+f_2(\mathbf{r}_s)}dV_s \implies \underline{\boldsymbol{\alpha}}_{\mathrm{d}} = \alpha_{\mathrm{d}}\underline{\mathbf{I}}.$$

- For plane wave incidence, only when the gradient of the scattered wave is oriented in a direction other than $\mathbf{k}_i$ is $\underline{\boldsymbol{\alpha}}_{\mathrm{d}} \neq \alpha_{\mathrm{d}}\underline{\mathbf{I}}$.

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[18] T. B. A. Senior. *J. Acoust. Soc. Am.* 53 (1973), pp. 742–747; A. D. Pierce. 2nd edition. Woodbury, New York: Acoustical Society of America, 1989.

Distinction from Rayleigh scattering

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$$p_s(\mathbf{r}) = p_0 \frac{e^{ikr}}{r} \Phi_s\,, \quad \Phi_s(\mathbf{k}_s) = \frac{k^2}{4\pi}\left[\alpha_\mathrm{m} + i\boldsymbol{\alpha}_\mathrm{c} \cdot (\mathbf{e}_i - \mathbf{e}_r) + \alpha_\mathrm{d} \cos\psi\right]$$

The Rayleigh limit is sometimes referred to as a quasistatic approximation. This terminology stems from the fact that because of the small size of the body, the phase of any long-wavelength signal will be essentially constant over the extent of the body. This means that any complex exponential representing a phase shift within the scattering body or on its surface may be approximated as having unit value.[19]

A. D. Pierce (left) and J. H. Ginsberg (right)
J. H. Ginsberg. *Acoustics Today* 11 (2015), pp. 10–16

[19] J. H. Ginsberg. Vol. 2. Springer, 2018.

Alternative expressions of polarizabilities

- In terms of Morse and Ingard's contrast factors[20]

$$\gamma_\beta(\mathbf{r}) = \frac{\beta(\mathbf{r})}{\beta_0} - 1 \,, \quad \gamma_\rho(\mathbf{r}) = 1 - \frac{\rho_0}{\rho(\mathbf{r})} \,, \tag{37}$$

Eqs. (15) become easier to interpret:

$$\alpha_\mathrm{m} = \int_{V_s} \gamma_\beta(\mathbf{r}_s)\, dV_s \,, \tag{38a}$$

$$\alpha_\mathrm{d} = \int_{V_s} \gamma_\rho(\mathbf{r}_s) dV_s \,, \tag{38b}$$

$$\boldsymbol{\alpha}_\mathrm{c} = k \left[\int_{V_s} \gamma_\rho(\mathbf{r}_s)\mathbf{r}_s\, dV_s + \cos\psi \int_{V_s} \gamma_\beta(\mathbf{r}_s)\mathbf{r}_s\, dV_s \right]. \tag{38c}$$

- $\alpha_\mathrm{m} = 0$ if $\gamma_\beta(-\mathbf{r}) = -\gamma_\beta(\mathbf{r})$ and $\alpha_\mathrm{d} = 0$ if $\gamma_\rho(-\mathbf{r}) = -\gamma_\rho(\mathbf{r})$.
- $\boldsymbol{\alpha}_\mathrm{c} \neq \mathbf{0}$ unless both γ_β and γ_ρ are *even* about the origin $\mathbf{r} = \mathbf{0}$.
- $\boldsymbol{\alpha}_\mathrm{c} \to \mathbf{0}$ for $\omega = kc_0 \to 0$, i.e., frequency-dependent effect

[20] P. M. Morse and K. U. Ingard. McGraw-Hill, 1968, Eq. (8.1.11).

Rayleigh's comments on non-spherical scatterers

The results which we have obtained are based upon (14), and are as true as the theories from which that equation was derived. In the electromagnetic theory we have treated the molecules as spherical continuous bodies differing from the rest of the medium merely in the value of their dielectric constant. If we abandon the restriction as to sphericity, the results will be modified in a manner that cannot be precisely defined until the shape is specified. On the whole, however, it does not appear probable that this consideration would greatly affect the calculation as to transparency, since the particles must be supposed to be oriented in all directions indifferently. But the theoretical conclusion that the light diffracted in a direction perpendicular to the primary rays should be *completely* polarized may well be seriously disturbed. If the view, suggested in the present paper, that a large part of the light from the sky is diffracted from the molecules themselves, be correct, the observed incomplete polarization at 90° from the Sun may be partly due to the molecules behaving rather as elongated bodies with indifferent orientation than as spheres of homogeneous material.

Lord Rayleigh. *Philos. Mag.* 47 (1899), pp. 375–384

Comparison to prior polarizability formulations

- A procedure for finding α's in 2D was formulated by Su and Norris,[21]

$$\alpha^{pp'} = \frac{-i}{\omega^2 H_0^{(1)}(kr)}\big({}^{+}_{x}p^{-}_{sx} + {}^{+}_{x}p^{+}_{sx} + {}^{+}_{y}p^{-}_{sy} + {}^{+}_{y}p^{+}_{sy} + {}^{-}_{x}p^{-}_{sx} + {}^{-}_{x}p^{+}_{sx} + {}^{-}_{y}p^{-}_{sy} + {}^{-}_{y}p^{+}_{sy}\big),$$

$$\alpha^{pv'}_{x} = \frac{-i}{\sqrt{2}\omega^2 H_0^{(1)}(kr)}\big({}^{+}_{x}p^{-}_{sx} + {}^{+}_{x}p^{+}_{sx} + {}^{+}_{x}p^{-}_{sy} + {}^{+}_{x}p^{+}_{sy} - {}^{-}_{x}p^{-}_{sx} - {}^{-}_{x}p^{+}_{sx} - {}^{-}_{x}p^{-}_{sy} - {}^{-}_{x}p^{+}_{sy}\big),$$

$$\alpha^{pv'}_{y} = \frac{-i}{\sqrt{2}\omega^2 H_0^{(1)}(kr)}\big({}^{+}_{y}p^{-}_{sy} + {}^{+}_{y}p^{+}_{sy} + {}^{+}_{y}p^{-}_{sx} + {}^{+}_{y}p^{+}_{sx} - {}^{-}_{y}p^{-}_{sy} - {}^{-}_{y}p^{+}_{sy} - {}^{-}_{y}p^{-}_{sx} - {}^{-}_{y}p^{+}_{sx}\big),$$

$$\alpha^{vp'}_{x} = \frac{1}{\sqrt{2}\omega^2 H_1^{(1)}(kr)}\big({}^{+}_{x}p^{-}_{sx} - {}^{+}_{x}p^{+}_{sx} + {}^{-}_{x}p^{-}_{sx} - {}^{-}_{x}p^{+}_{sx} + {}^{+}_{y}p^{-}_{sx} - {}^{+}_{y}p^{+}_{sx} + {}^{-}_{y}p^{-}_{sx} - {}^{-}_{y}p^{+}_{sx}\big),$$

$$\alpha^{vp'}_{y} = \frac{1}{\sqrt{2}\omega^2 H_1^{(1)}(kr)}\big({}^{+}_{y}p^{-}_{sy} - {}^{+}_{y}p^{+}_{sy} + {}^{-}_{y}p^{-}_{sy} - {}^{-}_{y}p^{+}_{sy} + {}^{+}_{x}p^{-}_{sy} - {}^{+}_{x}p^{+}_{sy} + {}^{-}_{x}p^{-}_{sy} - {}^{-}_{x}p^{+}_{sy}\big),$$

$$\alpha^{vv'}_{xx} = \frac{1}{\omega^2 H_1^{(1)}(kr)}\big({}^{+}_{x}p^{-}_{sx} - {}^{+}_{x}p^{+}_{sx} - {}^{-}_{x}p^{-}_{sx} + {}^{-}_{x}p^{+}_{sx}\big),$$

$$\alpha^{vv'}_{xy} = \frac{1}{\omega^2 H_1^{(1)}(kr)}\big({}^{+}_{y}p^{-}_{sx} - {}^{+}_{y}p^{+}_{sx} - {}^{-}_{y}p^{-}_{sx} + {}^{-}_{y}p^{+}_{sx}\big),$$

$$\alpha^{vv'}_{yx} = \frac{1}{\omega^2 H_1^{(1)}(kr)}\big({}^{+}_{x}p^{-}_{sy} - {}^{+}_{x}p^{+}_{sy} - {}^{-}_{x}p^{-}_{sy} + {}^{-}_{x}p^{+}_{sy}\big),$$

$$\alpha^{vv'}_{yy} = \frac{1}{\omega^2 H_1^{(1)}(kr)}\big({}^{+}_{y}p^{-}_{sy} - {}^{+}_{y}p^{+}_{sy} - {}^{-}_{y}p^{-}_{sy} + {}^{-}_{y}p^{+}_{sy}\big).$$

  which is coordinate-dependent, FEM-based, and hard to interpret.

- Meanwhile, Sepehrirahnama's polarizabilities[22] are

$$\alpha_{pp} = -\alpha_{\mathrm{m}}/c_0^2\,, \qquad \boldsymbol{\alpha}_{pv} = -i\rho_0\boldsymbol{\alpha}_{\mathrm{c}}/c_0 \tag{39a}$$

$$\underline{\boldsymbol{\alpha}}_{vv} = i\rho_0\underline{\boldsymbol{\alpha}}_{\mathrm{d}}/kc_0\,, \quad \boldsymbol{\alpha}_{vp} = \boldsymbol{\alpha}_{\mathrm{c}}/kc_0^2 \tag{39b}$$

- Eliminating $\boldsymbol{\alpha}_{\mathrm{c}}$ between Eqs. (39a) and (39b) recovers $-\boldsymbol{\alpha}_{pv}/\rho_0 c_0 = ik\boldsymbol{\alpha}_{vp}$, which according to Quan et al. is required to satisfy reciprocity.[23]

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[21] X. Su and A. N. Norris. *Phys. Rev. B* 98 (2018), pp. 1–8.

[22] S. Sepehrirahnama, S. Oberst, Y. K. Chiang, and D. A. Powell. *Phys. Rev. E* 104 (2021), pp. 1–11, Eqs. (7).

[23] L. Quan, Y. Ra'di, D. L. Sounas, and A. Alù. *Phys. Rev. Lett.* 120 (2018), pp. 1–7, Eq. (5).

Outline

Example 1: Homogeneous sphere

- Consider a sphere of radius $r = a$ with material properties f_1 and f_2.
- The polarizabilities are calculated by Eqs. (15):[24]

$$\alpha_{\mathrm{m}} = -\frac{4\pi a^3}{3} f_1 \,, \tag{40a}$$

$$\alpha_{\mathrm{d}} = \frac{4\pi a^3}{3} \frac{3f_2/2}{1 + f_2/2} \,, \tag{40b}$$

$$\boldsymbol{\alpha}_{\mathrm{c}} = \mathbf{0} \,. \tag{40c}$$

- The angle-distribution function is formed from Eqs. (16) and (40):

$$\Phi_s(\theta) = \frac{k^2}{4\pi} \left[\alpha_{\mathrm{m}} + \alpha_{\mathrm{d}} \cos\theta\right] , \tag{41}$$

---

[24] Equation (40c) is obtained by noting that $\boldsymbol{\alpha}_{\mathrm{c}} = k \left[\frac{3f_2}{2+f_2} \cos(\psi) - f_1\right] \int_{V_s} \mathbf{r}_s dV_s \propto$

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{a} (\mathbf{e}_x r_s \sin\theta_s \cos\phi_s + \mathbf{e}_y r_s \sin\theta_s \sin\phi_s + \mathbf{e}_z r_s \cos\theta_s) r_s^2 \sin\theta_s \, dr_s \, d\theta_s \, d\phi_s$$

$$= \mathbf{e}_x \frac{\pi}{8} r_s^4 \Big|_s^a \sin\phi_s \Big|_s^{2\pi} + -\mathbf{e}_y \frac{\pi}{8} r_s^4 \Big|_s^a \cos\phi_s \Big|_s^{2\pi} - \mathbf{e}_z \frac{\pi}{8} r_s^4 \Big|_0^a \cos 2\theta_s \Big|_0^{\pi} = 0\mathbf{e}_x + 0\mathbf{e}_y + 0\mathbf{e}_z \,.$$

Example 1: Homogeneous sphere

- Inserting Eqs. (41) into Eq. (5a) yields the radiation force

$$\boxed{F_z = \frac{4\pi\langle I\rangle}{9c_0} a^2 (ka)^4 \left[f_1^2 + \frac{f_1 f_2}{1 + f_2/2} + \frac{3f_2^2}{4(1+f_2^2/2)^2} \right].} \tag{42}$$

  where $p_0^2/2\rho_0 c_0 = \rho_0 c_0 v_0^2/2 = \langle I\rangle$.

- Gor'kov's result[25] is recovered by retaining the lowest order terms in Eq. (42):

$$F_z = \frac{4\pi\langle I\rangle}{9c_0} a^2 (ka)^4 \left(f_1^2 + f_1 f_2 + \tfrac{3}{4} f_2^2 \right). \tag{43}$$

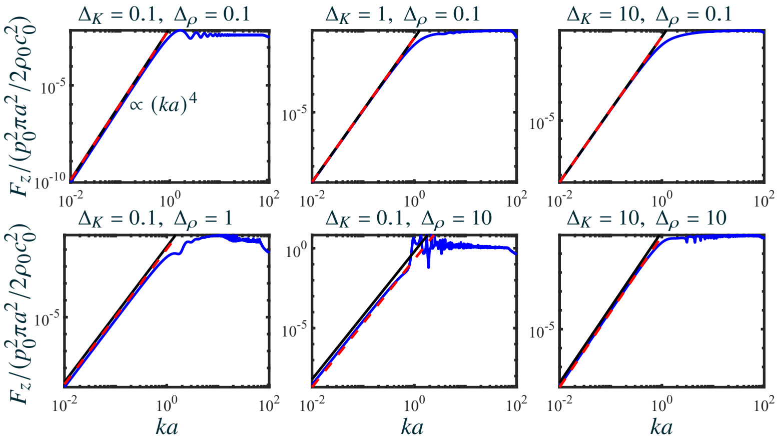
- Equations (42) and (43) are compared to[26] the exact solution in terms of spherical wave expansions:

$$F_z = \frac{i\pi}{\rho_0 c_0^2 k^2} \sum_{n=0}^{\infty} \frac{n+1}{(2n+1)(2n+3)} (A_n^* + A_{n+1} + 2A_n^* A_{n+1}) a_n^* a_{n+1} + \text{c.c.}. \tag{44}$$

[25] L. P. Gor'kov. *Sov. Phys. Dokl.* 6 (1962), pp. 773–775.

[26] Y. A. Ilinskii, E. A. Zabolotskaya, B. C. Treweek, and M. F. Hamilton. *J. Acoust. Soc. Am.* 144 (2018), pp. 568–576.

Example 1: Homogeneous sphere



Comparison of the polarizability formulation (42), Gor'kov's result (43), and the exact solution based on spherical wave expansions (44), for values of $\Delta_K = (K_s - K_0)/K_0$ and $\Delta_\rho = (\rho_s - \rho_0)/\rho_0$ ranging two orders of magnitude.

Outline

Example 2: Homogeneous cube

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- A cube of side length $b = a(4\pi/3)^{1/3}$ is now considered:

$$f_1(x_0, y_0, z_0) = f_1 \operatorname{rect}(x_0/b) \operatorname{rect}(y_0/b) \operatorname{rect}(z_0/b)\,, \tag{45a}$$

$$f_2(x_0, y_0, z_0) = f_2 \operatorname{rect}(x_0/b) \operatorname{rect}(y_0/b) \operatorname{rect}(z_0/b)\,, \tag{45b}$$

where

$$\operatorname{rect} x \equiv \begin{cases} 1, & |x| \le 1/2 \\ 0, & |x| > 1/2\,. \end{cases} \tag{46}$$

- Inserting Eqs. (45) into Eqs. (15) yields the polarizabilities

$$\alpha_\mathrm{m} = -b^3 f_1\,, \tag{47a}$$

$$\alpha_\mathrm{d} = b^3 \frac{3f_2/2}{1 + f_2/2}\,, \tag{47b}$$

$$\boldsymbol{\alpha}_\mathrm{c} = \mathbf{0}\,. \tag{47c}$$

- Inserting Eqs. (47) into Eqs. (17) and (5a) yields the same result as before:

$$F_z = \frac{4\pi\langle I\rangle}{9c_0} a^2 (ka)^4 \left[f_1^2 + \frac{f_1 f_2}{1 + f_2/2} + \frac{3f_2^2}{4(1 + f_2^2/2)^2} \right]. \tag{48}$$

Example 2: Homogeneous cube

- Equation (48) is compared to the force calculated using Fourier transforms in the Born approximation but not in the subwavelength limit:[27]

$$\Phi_s(\mathbf{k}_s) = -\frac{k^2}{4\pi}\left[\mathcal{F}_{3\mathrm{D}}\left\{f_1(\mathbf{r}_s)e^{i\mathbf{k}_i\cdot\mathbf{r}_s}\right\} - \mathcal{F}_{3\mathrm{D}}\left\{\tfrac{3f_2(\mathbf{r}_s)}{2+f_2(\mathbf{r}_s)}e^{i\mathbf{k}_i\cdot\mathbf{r}_s}\right\}\cos\psi\right]. \quad (49)$$

- The directivity factor is

$$\Phi_s(\mathbf{k}_s) = \frac{4\pi}{3}\frac{k^2a^3}{4\pi}f(\mathbf{k}_s)\left(-f_1 + \tfrac{3f_2/2}{1+f_2/2}\cos\theta\right), \quad (50)$$

where $\theta = \arctan(\sqrt{x^2+y^2}/z)$ is the spherical polar coordinate, and where

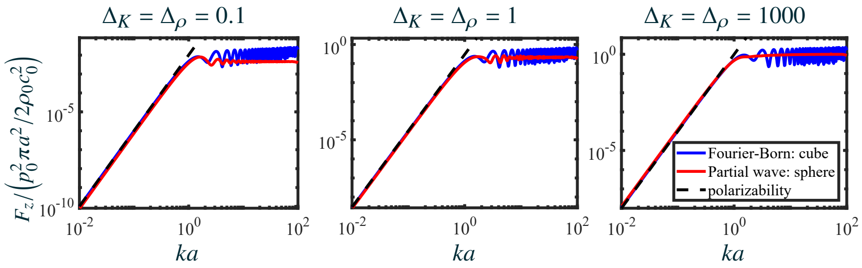
$$f(\mathbf{k}_s) = \frac{\sin[(k_{s,x}-k_{i,x})b/2]}{(k_{s,x}-k_{i,x})b/2}\,\frac{\sin[(k_{s,y}-k_{i,y})b/2]}{(k_{s,y}-k_{i,y})b/2}\,\frac{\sin[(k_{s,z}-k_{i,z})b/2]}{(k_{s,z}-k_{i,z})b/2}. \quad (51)$$

- Equation (5a) provides the corresponding radiation force

$$F_z = \frac{p_0^2}{2\rho_0c_0^2}\int_0^{2\pi}\int_0^{\pi}|\Phi_s(\theta_0,\phi_0)|^2(1-\cos\theta_0)\sin\theta_0 d\theta_0 d\phi_0\,. \quad (52)$$

[27] P. M. Morse and K. U. Ingard. McGraw-Hill, 1968, Eq. (8.1.20).

Example 2: Homogeneous cube



Comparison of the radiation force given by the polarizability formulation [dashed line, Eq. (48)] to the forces on a homogeneous cube [Eq. (52)] and sphere of equal volume [red curves, Eq. (44)] for $\Delta_K = (K_s - K_0)/K_0$ and $\Delta_\rho = (\rho_s - \rho_0)/\rho_0$ ranging four orders of magnitude. For $ka \ll 1$ in all cases, the force on the cube converges to that on the sphere.

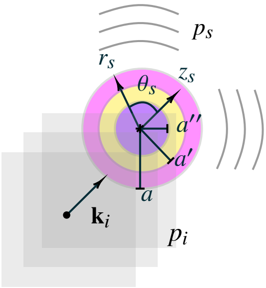
Insights on forces due to progressive waves

- The radiation force on the cube and sphere for $ka \lesssim 1$ converges to the same value, indicating that **the shape does not matter for radiation force on subwavelength objects**.
  - Physical explanation: The features of the cube (like its edges and corners) cannot be resolved for $ka \ll 1$.
- The agreement between the exact solution and Born approximation holds for all material contrasts, indicating that **the Born approximation can always be made at low frequencies**, no matter how greatly the scatterer's material properties differ from those of the background medium.
  - Physical explanation: The Born approximation assumes that the amplitude of the scattered wave is much less than that of the incident wave, which is guaranteed by the smallness of the scatterer in the $ka \ll 1$ limit.

Outline

Example 3: Spherically symmetric nucleated cell

- Let the inner, middle, and outer radii be a'', a', and a, respectively.
- Let the material properties be given by[28]
  - f_1'', f_2'' for $r \leq a''$
  - f_1', f_2' for $a'' < r \leq a'$
  - f_1, f_2 for $a' < r \leq a$
- Volume fractions $\chi' \equiv (a'/a)^3 < 1$ and $\chi'' \equiv (a''/a')^3 < 1$ are introduced.



Three-layered sphere with inner radius a'', middle radius a', and outer radius a.

[28] Y.-Y. Wang et al. *J. Appl. Phys.* 122 (2017), pp. 1–6, Table I.

Example 3: Spherically symmetric nucleated cell

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- Insertion of the material properties in Eq. (15) yields

$$\alpha_\mathrm{m} = -\tfrac{4\pi a^3}{3}\left[f_1''\chi'' + f_1'(\chi' - \chi'') + f_1(1-\chi')\right], \tag{53a}$$

$$\alpha_\mathrm{d} = \tfrac{4\pi a^3}{3}\left[\tfrac{3f_2''/2}{1+f_2''/2}\chi'' + \tfrac{3f_2'/2}{1+f_2'/2}(\chi' - \chi'') + \tfrac{3f_2/2}{1+f_2/2}(1-\chi')\right], \tag{53b}$$

$$\boldsymbol{\alpha}_\mathrm{c} = \mathbf{0}\,. \tag{53c}$$

- The force is given by Eqs. (53) in combination with

$$F_z = \frac{p_0^2}{2\rho_0 c_0^2}\frac{k^4}{4\pi}\left(\alpha_\mathrm{m}^2 - \tfrac{2}{3}\alpha_\mathrm{m}\alpha_\mathrm{d} + \tfrac{1}{3}\alpha_\mathrm{d}^2\right). \tag{54}$$

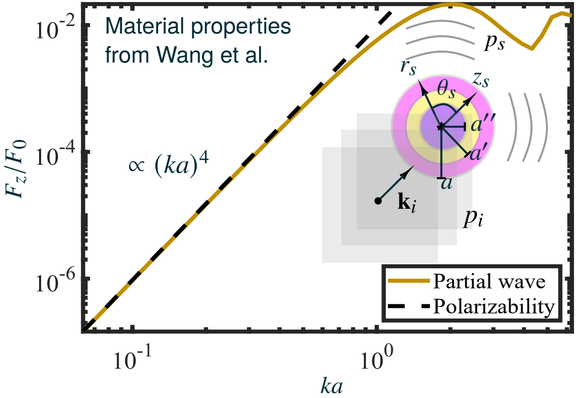
- Equation (54) is validated by comparison to[29]

$$F_z = \frac{i\pi}{\rho_0 c_0^2 k^2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{(n+m+1)(n+m)!}{(2n+1)(2n+3)(n-m)!} \times (A_n^* + A_{n+1} + 2A_n^*A_{n+1})a_n^{m*}a_{n+1}^m + \text{c.c.}. \tag{55}$$

[29] C. A. Gokani, T. S. Jerome, M. R. Haberman, and M. F. Hamilton. *Proc. Mtgs. Acoust.* 48 (2022), pp. 1–10.

Example 3: Spherically symmetric nucleated cell

The forces are normalized to $F_0 = p_0^2 \pi a^2 / 2\rho_0 c_0^2$:



Outline

Example 4: Antisymmetric inhomogeneous cube

TEXAS
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- An inhomogeneous cube is now considered:

$$f_1(x_0, y_0, z_0) = -(2f_1 x_0/a)\,\mathrm{rect}(x_0/a)\,\mathrm{rect}(y_0/a)\,\mathrm{rect}(z_0/a)\,, \tag{56a}$$

$$f_2(x_0, y_0, z_0) = \frac{2(2f_2 x_0/a)}{3 - 2f_1 x_0/a}\,\mathrm{rect}(x_0/a)\,\mathrm{rect}(y_0/a)\,\mathrm{rect}(z_0/a)\,, \tag{56b}$$

- Inserting Eqs. (56) into Eqs. (15) for the polarizabilities yields

$$\alpha_{\mathrm{m}} = 0\,, \tag{57a}$$

$$\alpha_{\mathrm{d}} = 0\,, \tag{57b}$$

$$\boldsymbol{\alpha}_{\mathrm{c}} = \frac{ka^4}{6}\,(f_1 + f_2\cos\theta)\,\mathbf{e}_x\,. \tag{57c}$$

- Φ_s is given by Eq. (49), where the 3D Fourier transform is

$$f(\mathbf{k}_s) = \frac{\sin[(k_{s,y} - k_{i,y})a/2]}{(k_{s,y} - k_{i,y})a/2}\,\frac{\sin[(k_{s,z} - k_{i,z})a/2]}{(k_{s,z} - k_{i,z})a/2} \times \frac{i}{(k_{s,x} - k_{i,x})a/2}\left\{\cos[(k_{s,x} - k_{i,x})a/2] - \frac{\sin[(k_{s,x} - k_{i,x})a/2]}{(k_{s,x} - k_{i,x})a/2}\right\}. \tag{58}$$

Example 4: Antisymmetric inhomogeneous cube

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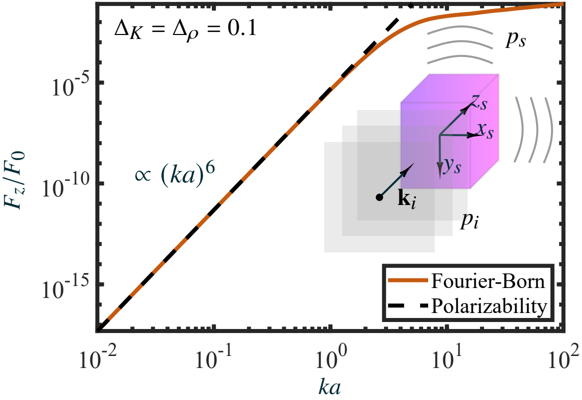
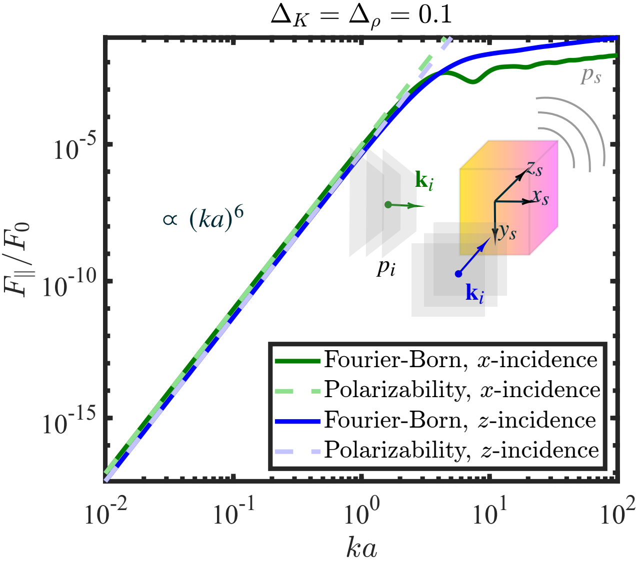
- Denoting $F_0 = p_0^2 a^2 / 2\rho_0 c_0^2$, the force in the direction of the incident wave is

$$\frac{F_z}{F_0} = \frac{(ka)^6}{576\pi^2} \int_0^{2\pi} \int_0^{\pi} \left[(f_1 + f_2 \cos\theta_0) \sin\theta_0 \cos\phi_0\right]^2 (1 - \cos\theta_0) \sin\theta_0 \, d\theta_0 \, d\phi_0$$

$$= \boxed{\frac{(ka)^6}{144\pi} \left[\frac{1}{3} f_1^2 + \frac{1}{15}(f_2^2 - 2 f_1 f_2)\right]} . \qquad (59)$$



Example 4: Antisymmetric inhomogeneous cube



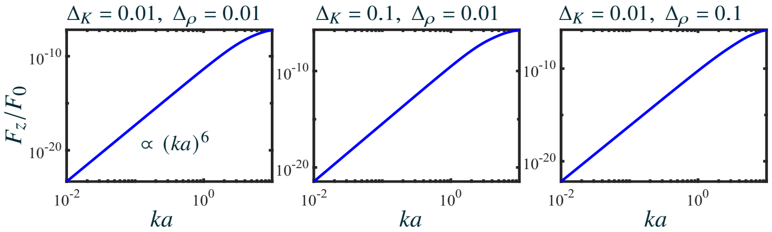
Example 4: Antisymmetric inhomogeneous cube

- To calculate the force in the x direction, Eq. (33) is used:[30]

$$\frac{F_x}{F_0} = -\frac{(ka)^6}{576\pi^2}\int_0^{2\pi}\int_0^{\pi}\left[(f_1+f_2\cos\theta_0)\sin\theta_0\cos\phi_0\right]^2\sin^2\theta_0\cos\phi_0\,d\theta_0\,d\phi_0$$
$$= \boxed{0}\,. \qquad (60)$$

- In terms of Fourier transforms, the force in the x direction is calculated by numerically evaluating Eq. (5b):

$$\frac{F_x}{F_0} = -\frac{1}{a^2}\int_0^{2\pi}\int_0^{\pi}|\Phi_s(\mathbf{k}_s)|^2\ \sin^2\theta_0\cos\phi_0\,d\theta_0\,d\phi_0\,. \qquad (61)$$



[30] $\mathbf{e}_x \cdot (\mathbf{e}_z - \mathbf{e}_r) = -\sin\theta\cos\phi$

Example 4: Antisymmetric inhomogeneous cube

- Similarly, the force in the y direction is found by noting that $\mathbf{e}_y \cdot \mathbf{e}_r = \sin\theta \sin\phi$:

$$\frac{F_y}{F_0} = -\frac{K^6}{576\pi^2} \int_0^{2\pi} \int_0^{\pi} [(f_1 + f_2 \cos\theta_0) \sin\theta_0 \cos\phi_0]^2 \sin^2\theta_0 \sin\phi_0 \, d\theta_0 \, d\phi_0$$
$$= \boxed{0}, \qquad (62)$$

while in terms of Fourier transforms, the force in the y direction is

$$\frac{F_y}{F_0} = -\frac{1}{a^2} \int_0^{2\pi} \int_0^{\pi} |\Phi_s(\mathbf{k}_s)|^2 \sin^2\theta_0 \sin\phi_0 \, d\theta_0 \, d\phi_0 \,. \qquad (63)$$

- Equation (63) numerically evaluates to zero.
- Transverse forces due to plane progressive waves are of quadrupolar and higher order.[31]

[31] S. Sepehrirahnama, S. Oberst, Y. K. Chiang, and D. A. Powell. *Phys. Rev. E* 104 (2021), pp. 1–11.