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Langevin radiation force as a consequence of linear wave fields

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ABSTRACT:
The question of whether acoustic radiation force is a consequence of linear or nonlinear wave fields has been debated for over a century. It is understood that the Rayleigh radiation pressure depends on the quadratic term in the equation of state, whereas the Langevin radiation pressure does not [R. T. Beyer, J. Acoust. Soc. Am. **63**, 1025–1030 (1978)]. Nevertheless, traditional derivations of the Langevin radiation force begin with the nonlinear momentum equation. Here, the Langevin radiation force is derived from the linearized continuity, state, and momentum equations, proving that "for the calculation of the average force correct up to terms of the second order in the velocity, it is sufficient to find the solution of the linear scattering problem" [L. P. Gor'kov, Sov. Phys. Dokl. **6**, 773–775 (1962)]. The linear origins of the Langevin radiation force are clarified by drawing parallels with radiation forces exerted by electromagnetic waves, surface gravity waves, and waves on a string.
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I. INTRODUCTION

Radiation forces exerted by acoustic waves are generated by stresses that are quadratic in the wave variable,[1–3] leading to the question of whether a nonlinear wave equation must be used to calculate the field producing the force.[4–7] This question has been discussed for over a century. Regarding his calculation of the radiation pressure in an adiabatic gas exerted by a plane progressive wave on a perfectly reflecting wall, Poynting wrote, "Lord Rayleigh kindly pulled me out of the pit into which I fell, pointing out that when we take into account second-order quantities the ordinary sound equation does not hold."[8] Rayleigh's result[9] retains the quadratic term in the state equation in order to conserve mass in a constrained volume.[10] Poynting's neglect of this nonlinearity is not surprising given his background in electromagnetism,[11] in which no such constraint arises, and for which the analogous radiation pressure on a reflecting surface equals twice the energy density of the incident wave.[12,13]

Current interest in radiation force is motivated by acoustic tweezers and acoustofluidic devices,[14,15] for which constraints on the total mass or volume of fluid are rarely imposed, and for which there is contact between the sound field and the background medium to "ensure zero perturbation at infinity."[10,16] In such cases, the nonlinearity considered by Rayleigh is irrelevant, and "the Rayleigh radiation force has no major practical importance."[17] The omission of these constraints is associated with Langevin's treatment of acoustic radiation force, which, like Poynting's approach,[8] was "derived from his knowledge of a similar phenomenon, the radiation pressure exerted by a beam of light."[18] The Langevin radiation force is the "usually measured" quantity that "can be observed down to the lowest sound intensities under certain conditions."[6] A review of the early literature as it relates to the Rayleigh and Langevin radiation pressures is provided by Post.[19] An accessible discussion of the difference between the Rayleigh and Langevin forces is provided by Wang and Lee,[10] and intuitive distinctions between the phenomena are provided by Gol'dberg[16] and Ostrovsky.[20]

Traditional derivations of acoustic radiation force begin with the nonlinear momentum and continuity equations,[1,10,16,21–23] which may create the impression that the Langevin radiation force is a consequence of nonlinear wave fields.[5,24,25] The goal of the present work is to elucidate the connection between the Langevin radiation force and linear acoustics. The equations of linear acoustics for a homogeneous fluid are combined in Sec. II to derive the acoustic momentum conservation theorem,[1,16,22,26]

$$\nabla \cdot \mathbf{T} = \frac{\partial \mathbf{g}}{\partial t}, \tag{1}$$

where $\mathbf{T}$ is the radiation stress tensor and $\mathbf{g}$ is the momentum density vector. The Langevin radiation force on a scatterer is calculated by integrating the time average of $\mathbf{T}$ over any surface enclosing the scatterer.[27–36] Equation (1) and its application to radiation force is well known and widely employed; the contribution of the present work is the manner in which Eq. (1) is obtained from linear acoustic equations. Analogous approaches are applied to derive momentum conservation theorems for electromagnetic waves, surface gravity waves, and

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waves on a string in Secs. III, IV, and V, respectively, showing that similar radiation forces can be obtained from solutions of linear wave equations.

II. ACOUSTIC MOMENTUM CONSERVATION THEOREM

An early form of the acoustic momentum conservation theorem given by Eq. (1) was obtained by Eckart using an approach similar to the one described in this section.[37] However, Eckart's result given by Eq. (11) of Ref. 37 contains a sign error, as noted by Westervelt,[27] and the retention of a term proportional to the curl of the fluid velocity $\mathbf{v}$ prevents his result from being related to Brillouin's radiation stress tensor.[1] For an irrotational fluid ($\nabla \times \mathbf{v} = \mathbf{0}$), Eq. (11) of Ref. 37 should be written as

$$\frac{\partial \mathbf{g}}{\partial t} + \rho_0(\mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \nabla \cdot \mathbf{v}) + \nabla(\mathcal{E} - \rho_0 v^2) = \mathbf{0}, \tag{2}$$

where ρ_0 is the ambient density, $\mathcal{E}$ is the total energy density, and $v^2 = \mathbf{v} \cdot \mathbf{v}$. The following derivation of Eq. (1) from linear fields is based partially on the procedure outlined by Eckart. For a direct reduction of Eq. (2) to Eq. (1), see Appendix A. Eckart's statement that "the acoustic momentum is related to acoustic energy in the same manner as the corresponding electromagnetic quantities"[37] is shown quantitatively in Sec. III.

Begin by considering the equations that govern linear acoustic fields in a homogeneous inviscid fluid:[38]

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = 0, \tag{3}$$

$$\rho' - p/c_0^2 = 0, \tag{4}$$

$$\nabla p + \rho_0 \frac{\partial \mathbf{v}}{\partial t} = \mathbf{0}. \tag{5}$$

Equations (3), (4), and (5) are the linearized continuity, state, and momentum equations, respectively, where ρ' is the density perturbation, p is the acoustic pressure, and c_0 is the speed of sound. The conservation theorem derived below is formed from solutions of the linear wave equation

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = 0, \tag{6}$$

obtained from Eqs. (3)–(5).

Multiplying Eq. (3) by $\mathbf{v}$ yields

$$\frac{\partial \rho'}{\partial t}\mathbf{v} + \rho_0(\nabla \cdot \mathbf{v})\mathbf{v} = \mathbf{0}. \tag{7}$$

Equation (7) is recast using the product rules[39]

$$\frac{\partial \rho'}{\partial t}\mathbf{v} = \frac{\partial(\rho' \mathbf{v})}{\partial t} - \rho' \frac{\partial \mathbf{v}}{\partial t}, \tag{8}$$

$$(\nabla \cdot \mathbf{v})\mathbf{v} = \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - (\nabla \mathbf{v}) \cdot \mathbf{v}, \tag{9}$$

resulting in

$$\frac{1}{c_0^2}\frac{\partial \mathbf{S}}{\partial t} - \frac{p}{c_0^2}\frac{\partial \mathbf{v}}{\partial t} + \rho_0[\nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - (\nabla \mathbf{v}) \cdot \mathbf{v}] = \mathbf{0}, \tag{10}$$

where Eq. (4) has been used to eliminate ρ' and $p\mathbf{v}$ has been identified as the instantaneous intensity $\mathbf{S}$,[25] also referred to as the Kirchhoff-Poynting vector.[40] Solving Eq. (5) for $\partial \mathbf{v}/\partial t$ and multiplying by p shows that

$$p\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0} p \nabla p = -\frac{1}{2\rho_0}\nabla(p^2),$$

while the term $(\nabla \mathbf{v}) \cdot \mathbf{v}$ in Eq. (10) can be replaced with $\nabla(v^2)/2$.[41] Equation (10) therefore becomes

$$\nabla\left(\tfrac{1}{2}\rho_0 v^2 - \tfrac{1}{2}\beta_0 p^2\right) - \rho_0 \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) = \frac{1}{c_0^2}\frac{\partial \mathbf{S}}{\partial t}, \tag{11}$$

where $\beta_0 = 1/\rho_0 c_0^2$ denotes the compressibility. Since[25]

$$\mathcal{K} = \tfrac{1}{2}\rho_0 v^2 = \text{kinetic energy density}, \tag{12}$$

$$\mathcal{U} = \tfrac{1}{2}\beta_0 p^2 = \text{potential energy density}, \tag{13}$$

$$\mathcal{L} = \mathcal{K} - \mathcal{U} = \text{Lagrangian density}, \tag{14}$$

the gradient in Eq. (11) can be written as[42]

$$\nabla \mathcal{L} = \mathbf{I} \cdot \nabla \mathcal{L} = \nabla \cdot (\mathbf{I}\mathcal{L}) - \mathcal{L}\nabla \cdot \mathbf{I}, \tag{15}$$

where $\mathbf{I}$ is the identity tensor. Equation (15) reduces to $\nabla \mathcal{L} = \nabla \cdot (\mathbf{I}\mathcal{L})$ since $\nabla \cdot \mathbf{I} = \mathbf{0}$, so Eq. (11) becomes

$$\nabla \cdot (\mathbf{I}\mathcal{L} - \rho_0 \mathbf{v} \otimes \mathbf{v}) = \frac{1}{c_0^2}\frac{\partial \mathbf{S}}{\partial t}. \tag{16}$$

Identifying[1,25]

$$\mathbf{g} = \mathbf{S}/c_0^2 = \text{momentum density}, \tag{17}$$

$$\mathbf{T} = \mathbf{I}\mathcal{L} - \rho_0 \mathbf{v} \otimes \mathbf{v} = \text{acoustic radiation stress} \tag{18}$$

allows Eq. (16) to be expressed as Eq. (1), which is the conservation theorem of acoustic momentum in a homogeneous fluid. It is noted that Eq. (18) is defined with a negative sign relative to the standard stress tensor of continuum mechanics.[22,43]

Equation (1) was presented previously by Morse and Ingard,[26] although no derivation was provided. Morse and Ingard's statement that $c_0\mathbf{g}$ "is the radiation pressure exerted by the wave"[26] is true only in the Langevin case for a plane wave incident on a perfectly absorbing wall.[10] Radiation stresses due to waves in solids were studied previously by Cantrell,[44] and a similar derivation of the momentum conservation theorem in an elastic medium was performed by Gurevich and Thellung.[45]

Equation (1) is reminiscent of the acoustic energy conservation theorem,[25]

$$\nabla \cdot \mathbf{S} + \frac{\partial \mathcal{E}}{\partial t} = 0, \tag{19}$$

where $\mathcal{E} = \mathcal{K} + \mathcal{U}$. Like Eq. (1), Eq. (19) can be derived by combining the linearized Eqs. (3)–(5), as shown in Ref. 25. Although Eq. (19) can also be obtained directly from the exact energy equation given by Eq. (1.11.5a) of Ref. 25, Eq. (1) does not follow explicitly from retaining quadratic terms in the exact momentum equation. In particular, while the second term of Eq. (18) and the right-hand side of Eq. (1) follow directly from the exact momentum equation, the first term of Eq. (18) must be determined by integrating Euler's equation over space indefinitely and expanding the pressure in terms of enthalpy. The distinction between the Rayleigh and Langevin radiation pressures is related to the second-order constant C' that results from the spatial integration, as shown on pp. 177–179 of Ref. 10. This constant, which would modify Eq. (18) to $\mathsf{T} = \mathsf{I}(\mathcal{L} - C') - \rho_0 \mathbf{v} \otimes \mathbf{v}$, does not arise by beginning with the linearized Eqs. (3)–(5). The fact that $C = \rho_0 \langle C' \rangle = 0$ in the present derivation of the Langevin radiation pressure[46] is consistent with Wang and Lee's criterion that "the radiation pressure is of Rayleigh type for $C \neq 0$ and of Langevin type for $C = 0$."[10]

The integral form of Eq. (1) is

$$\oint_A \mathsf{T} \cdot d\mathbf{A} = \frac{d}{dt} \int_V \mathbf{g}\, dV, \tag{20}$$

where $d\mathbf{A}$ is the outward-oriented differential area and V is the volume enclosing the surface A. Equation (20) states that "the momentum per unit time flowing in through the surface" equals the rate of change of "momentum stored within the fields."[13] Equation (18) shows that the former quantity includes contributions from both "the surface stress…exerted by the surrounding medium…and the intrinsic [linear] momentum passing through the surface."[47] If the acoustic field is time-harmonic and no object is enclosed by the surface A, the time average of Eq. (20) yields[48]

$$\oint_A \langle \mathsf{T} \rangle \cdot d\mathbf{A} = \mathbf{0}, \tag{21}$$

which equals Westervelt's Eq. (5)[27] and the integral form of Wang and Lee's Eq. (6.5).[10] Equation (21) states that there is no change in momentum in the volume enclosing the surface over an acoustic cycle if the enclosed fluid is lossless and homogeneous.[49] However, if a scatterer lies within the surface, then $\oint_A \langle \mathsf{T} \rangle \cdot d\mathbf{A}$ equals the rate at which momentum is transferred from the acoustic field to the scatterer,[10] which is identified as the Langevin radiation force for T given by Eq. (18):

$$\mathbf{F} = \oint_A \langle \mathsf{T} \rangle \cdot d\mathbf{A}. \tag{22}$$

Equation (22) recovers Westervelt's Eq. (1) of Ref. 49 and Gor'kov's equation preceding Eq. (4) of Ref. 28. Further, if $\nabla \cdot \mathsf{T} = \mathbf{0}$ in the fluid surrounding the scatterer, the surface A in Eq. (22) need not correspond to the surface of the scatterer, and the surface of integration can be taken as any closed surface in the fluid surrounding the scatterer.[32,40,49]

The fact that Eqs. (3)–(5) can be combined to yield both Eqs. (6) and (22) shows that the Langevin radiation force is a consequence of linear wave fields, clarifying the statements of Eckart,[37] Gor'kov,[28] and Beyer[6] quoted above. The derivation also clarifies Yosioka and Kawasima's statement that "the radiation pressure on any moving boundary is thus calculated in terms of the first-order velocity potential."[50] Landau and Lifshitz similarly note that the mean excess pressure "can be expressed in terms of quantities calculated from the linear sound equations, so that it is not necessary to solve directly the nonlinear equations of motion,"[22] as reiterated by Gol'dberg.[16] Despite these statements, the connection between Eqs. (3)–(6) and the quadratic quantities involved in the Langevin radiation force has not previously been made explicit.

The derivation above counters the categorical statements that "any theory dealing with acoustic radiation pressure…must retain at least all second-order terms to be valid even at small amplitudes"[5] and that "nonlinear effects cause a small but nonzero magnitude to be associated with a physical entity, the existence of which the linear model precludes"[25] as they apply to the Langevin radiation force. Linear acoustics predicts a nonzero radiation stress on the same order as the energy and intensity of an acoustic wave, as was originally implied by Eckart.[37]

More discussion of the relationship between Eqs. (1) and (19) as it relates to radiation force is provided by Zhang.[51] The implications of the present work also apply to the conservation of angular momentum and acoustic radiation torque.[40,47,52]

III. ANALOGOUS THEOREM FOR ELECTROMAGNETIC WAVES

Parallels between acoustic and electromagnetic radiation forces have been drawn since Euler noted in 1746 that "there is a propagation of streams of light through the ether …in such a way that this light propagation in the ether resembles that of sound in air."[6] Although the Langevin radiation force is frequently compared with electromagnetic radiation force,[1,4,5,17,53] the traditional derivations of the acoustic and electromagnetic momentum conservation theorems proceed differently. The electromagnetic momentum conservation theorem traditionally begins with the Lorentz force on charges and currents.[13,54–60] The Lorentz force density is expressed in terms of the electromagnetic fields by writing the charge and current densities in terms of the Gauss and Ampère-Maxwell laws. In contrast, the derivation of the acoustic momentum conservation theorem in Sec. II is carried out in free space, where any imbalance in Eq. (21) is identified as the momentum transferred from the acoustic fields to the mechanical momentum of a scatterer.[10]

JASA
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The following derivation parallels that of Sec. II by considering Maxwell's equations in the absence of charges and currents,[61]

$$\nabla \cdot \mathbf{E} = 0, \tag{23}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{24}$$

$$\nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} = \mathbf{0}, \tag{25}$$

$$\nabla \times \mathbf{H} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{0}, \tag{26}$$

where ϵ_0 and μ_0 are the free-space permittivity and permeability. The linearity of Eqs. (25) and (26) in the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ parallels the linearity of Eqs. (3)–(5) in ρ', $\mathbf{v}$, and p. Combining Eqs. (23)–(26) yields wave equations in $\mathbf{E}$ and $\mathbf{H}$ with a wave speed $c_0 = (\epsilon_0\mu_0)^{-1/2}$, similar to how Eqs. (3)–(5) yield Eq. (6). It is also noted that Eqs. (3)–(5) and (23)–(26) are homogeneous, whereas Eqs. (23) and (26) are inhomogeneous in traditional derivations of the electromagnetic momentum conservation theorem to account for the presence of charges and currents.

The cross product of $\epsilon_0\mathbf{E}$ and Eq. (25) yields

$$\epsilon_0\mathbf{E} \times (\nabla \times \mathbf{E}) + \frac{1}{c_0^2}\frac{\partial \mathbf{S}}{\partial t} - \frac{1}{c_0^2}\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{H} = \mathbf{0}, \tag{27}$$

where $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the Poynting vector, and where it has been noted that

$$\mathbf{E} \times \frac{\partial \mathbf{H}}{\partial t} = \frac{\partial \mathbf{S}}{\partial t} - \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{H}.$$

The cross product of $\mu_0\mathbf{H}$ and Eq. (26) similarly yields

$$\mu_0\mathbf{H} \times (\nabla \times \mathbf{H}) + \frac{1}{c_0^2}\frac{\partial \mathbf{S}}{\partial t} + \frac{1}{c_0^2}\frac{\partial \mathbf{H}}{\partial t} \times \mathbf{E} = \mathbf{0}. \tag{28}$$

Adding Eqs. (27) and (28) and noting from Eqs. (25) and (26) that $\partial\mathbf{H}/\partial t = -\nabla \times \mathbf{E}/\mu_0$ and $\partial\mathbf{E}/\partial t = \nabla \times \mathbf{H}/\epsilon_0$ yields

$$\epsilon_0\mathbf{E} \times (\nabla \times \mathbf{E}) + \mu_0\mathbf{H} \times (\nabla \times \mathbf{H}) + \frac{1}{c_0^2}\frac{\partial \mathbf{S}}{\partial t} = \mathbf{0}. \tag{29}$$

The curls in Eq. (29) can be eliminated by invoking Eqs. (23) and (24), resulting in[62]

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \tfrac{1}{2}\nabla(E^2) - \nabla \cdot (\mathbf{E} \otimes \mathbf{E}), \tag{30}$$

$$\mathbf{H} \times (\nabla \times \mathbf{H}) = \tfrac{1}{2}\nabla(H^2) - \nabla \cdot (\mathbf{H} \otimes \mathbf{H}), \tag{31}$$

respectively. Equation (29) therefore becomes[42]

$$\nabla \cdot \left[\epsilon_0\mathbf{E} \otimes \mathbf{E} + \mu_0\mathbf{H} \otimes \mathbf{H} - \mathsf{I}\left(\tfrac{1}{2}\epsilon_0 E^2 + \tfrac{1}{2}\mu_0 H^2\right)\right] = \frac{1}{c_0^2}\frac{\partial \mathbf{S}}{\partial t}. \tag{32}$$

Identifying[13,54]

$$\mathbf{g} = \mathbf{S}/c_0^2 = \text{momentum density}, \tag{33}$$

$$\mathsf{T} = \epsilon_0\mathbf{E} \otimes \mathbf{E} + \mu_0\mathbf{H} \otimes \mathbf{H} - \mathsf{I}\left(\tfrac{1}{2}\epsilon_0 E^2 + \tfrac{1}{2}\mu_0 H^2\right) = \text{Maxwell stress} \tag{34}$$

allows Eq. (32) to be expressed as

$$\nabla \cdot \mathsf{T} = \frac{\partial \mathbf{g}}{\partial t}. \tag{35}$$

Equation (35) recovers the differential form of Eq. (6.121) of Ref. 54, Eq. (8.21) of Ref. 13, Eq. (102) of Ref. 58, and Eq. (13.38) of Ref. 60 in the absence of charges and currents. Electromagnetic radiation force may then be calculated in a manner similar to Eq. (22) above.[63]

IV. ANALOGOUS THEOREM FOR SHALLOW SURFACE GRAVITY WAVES

Wave momentum and radiation stress in surface gravity waves have been studied for arbitrary depths.[64–67] The shallow-depth limit of surface gravity waves provides a two-dimensional analog of the theorems derived in Secs. II and III. The first-order equations involved in the derivation below are discussed by Fetter and Walecka[43] and Elmore and Heald,[68] and the one-dimensional case is presented by Lamb.[64] Unlike the depth-integrated and time-averaged expressions for radiation stress developed in Refs. 65–67, the radiation stress derived below satisfies a local momentum conservation theorem of the same form as Eqs. (1) and (35) for acoustics and electrodynamics, respectively.

Begin by considering a homogeneous inviscid liquid in a uniform gravitational field. Let the height of the liquid be

$$h(\mathbf{r}, t) = h_0 + \zeta(\mathbf{r}, t), \tag{36}$$

where h_0 is the depth as measured from the bottom $z = 0$ to the equilibrium surface $z = h_0$, and ζ is the vertical displacement from equilibrium, as shown in Fig. 1. The position vector $\mathbf{r}$ lies in the x–y plane and is represented by $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y$, where $\mathbf{e}_x$ and $\mathbf{e}_y$ are the Cartesian unit vectors.

The following analysis rests on the same footing as the linear wave phenomena considered in Secs. II and III by supposing that the wavelength λ is much larger than the depth h_0. The velocity $\mathbf{v}$ of the liquid may then be approximated as having no dependence on depth:

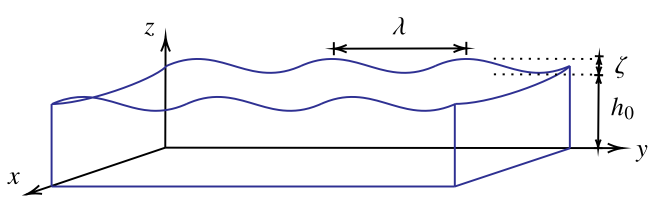

FIG. 1. Surface gravity waves of height $z = h$ as measured from the bottom, $z = 0$.

$$\mathbf{v} = \mathbf{v}(x, y, t) = v_x \mathbf{e}_x + v_y \mathbf{e}_y. \tag{37}$$

The validity of Eq. (37) is assessed in Ref. 43 (pp. 362–363) by considering modal solutions of the resulting wave equation in a closed rectangular tank. It is also assumed that the liquid is incompressible, and therefore the pressure in the liquid at height z is

$$P(\mathbf{r}, z, t) = P_0 + \rho_0 \gamma [\zeta(\mathbf{r}, t) - z], \tag{38}$$

where P_0 is the atmospheric pressure and γ is the acceleration due to gravity. Equations (36)–(38) allow the liquid motion to be described by the linearized continuity and momentum equations given by[43]

$$h_0 \nabla \cdot \mathbf{v} + \frac{\partial \zeta}{\partial t} = 0, \tag{39}$$

$$\gamma \nabla \zeta + \frac{\partial \mathbf{v}}{\partial t} = \mathbf{0}, \tag{40}$$

respectively. Combination of Eqs. (39) and (40) yields the linear wave equation $\nabla^2 \zeta - c_0^{-2} \partial^2 \zeta / \partial t^2 = 0$, where $c_0 = \sqrt{\gamma h_0}$ is the wave speed. The corresponding kinetic energy density, potential energy density, and intensity equal

$$\mathcal{K} = \tfrac{1}{2} \rho_0 v^2, \tag{41}$$

$$\mathcal{U} = \tfrac{1}{2} \frac{\rho_0 \gamma}{h_0} \zeta^2, \tag{42}$$

$$\mathbf{S} = \rho_0 \gamma \zeta \mathbf{v}. \tag{43}$$

The momentum conservation theorem is obtained by multiplying Eqs. (39) and (40) by $\mathbf{v}$ and ζ, respectively. Adding the resulting equations and noting that $\mathbf{v} \partial \zeta / \partial t + \zeta \partial \mathbf{v} / \partial t = \partial (\zeta \mathbf{v}) / \partial t$ yields

$$h_0 (\nabla \cdot \mathbf{v}) \mathbf{v} + \gamma \zeta \nabla \zeta + \frac{\partial (\zeta \mathbf{v})}{\partial t} = \mathbf{0}. \tag{44}$$

Equation (9) for the two-dimensional velocity field given by Eq. (37) is combined with Eq. (44), the result of which is multiplied by ρ_0 / h_0, yielding

$$\rho_0 [\nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - (\nabla \mathbf{v}) \cdot \mathbf{v}] + \frac{\rho_0 \gamma}{h_0} \zeta \nabla \zeta + \frac{\rho_0}{h_0} \frac{\partial (\zeta \mathbf{v})}{\partial t} = \mathbf{0}. \tag{45}$$

Taking the curl of Eq. (40) shows that $\nabla \times \mathbf{v} = \mathbf{0}$;[69] thus $(\nabla \mathbf{v}) \cdot \mathbf{v} = \nabla (v^2)/2$,[41] allowing the second term of Eq. (45) to be expressed as[42]

$$-\rho_0 (\nabla \mathbf{v}) \cdot \mathbf{v} = -\tfrac{1}{2} \rho_0 \nabla \cdot (\mathbf{I} v^2) = -\nabla \cdot (\mathbf{I} \mathcal{K}), \tag{46}$$

where the second equality follows from Eq. (41). In view of Eq. (42), the third term of Eq. (45) can be written as[42]

$$\frac{\rho_0 \gamma}{h_0} \zeta \nabla \zeta = \tfrac{1}{2} \frac{\rho_0 \gamma}{h_0} \nabla \zeta^2 = \nabla \mathcal{U} = \nabla \cdot (\mathbf{I} \mathcal{U}). \tag{47}$$

Finally, invoking Eq. (43) and the fact that $c_0^2 = \gamma h_0$ allows the last term of Eq. (45) to be written as $\partial \mathbf{g} / \partial t$, where

$$\mathbf{g} = \mathbf{S} / c_0^2 \tag{48}$$

is the momentum density. In view of Eqs. (46)–(48), Eq. (45) becomes

$$\rho_0 \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nabla \cdot (\mathbf{I} \mathcal{L}) + \frac{\partial \mathbf{g}}{\partial t} = \mathbf{0}, \tag{49}$$

where $\mathcal{L} = \mathcal{K} - \mathcal{U}$. Rearranging terms in Eq. (49) yields the momentum conservation theorem,

$$\nabla \cdot \mathbf{T} = \frac{\partial \mathbf{g}}{\partial t}, \tag{50}$$

where

$$\mathbf{T} = \mathbf{I} \mathcal{L} - \rho_0 \mathbf{v} \otimes \mathbf{v} \tag{51}$$

is the radiation stress, which can be represented in a chosen coordinate system by a 2×2 matrix. Although Eq. (50) applies to two-dimensional wave fields, it has the same symbolic form as Eqs. (1) and (35) for acoustic and electromagnetic waves, respectively. The radiation force exerted on a scatterer within any surface A enclosed by the path l is

$$\mathbf{F} = \int_A \langle \nabla \cdot \mathbf{T} \rangle \, dA = \oint_l \langle \mathbf{T} \rangle \cdot \mathbf{e}_n \, dl. \tag{52}$$

The second equality in Eq. (52) holds by Green's theorem, where $\mathbf{e}_n$ denotes the outward normal to the boundary of A and dl is a differential line element along the boundary. In contrast with Eq. (22), which involves integrating the radiation stress over a closed surface, Eq. (52) involves integrating the stress over a closed path. This distinction is simply due to the reduction of the wave field from three dimensions in Secs. II and III to two dimensions in the present section.

V. ANALOGOUS THEOREM FOR WAVES ON A STRING

Waves on a string provide a one-dimensional analog of the theorems derived in Secs. II–IV. The primary advantage of the following discussion is pedagogical: tensor- and vector-valued quantities are reduced to scalars, facilitating straightforward calculations of radiation force. The instructional value of waves on a string is evident in the early literature: Rayleigh[70] and Brillouin[4] considered waves on a string before addressing the more involved acoustical problem, as reviewed by Beyer[6] and Rowland.[71] The momentum conservation theorem derived below for waves on a string was obtained previously by Elmore and Heald,[68] although their discussion of radiation force is restricted to a perfectly absorbing scatterer. Examples of partially reflecting scatterers are presented in Appendix B. Fetter and Walecka[43] show that the energy conservation theorem for waves on a string analogous to Eq. (19) for acoustic waves can be derived using the least-action principle.

Let $\xi(x,t)$ denote the transverse displacement of a string of density ρ_0 under tension τ. For $|\partial\xi/\partial x| \ll 1$, the equation of motion is the linear wave equation,[26,38]

$$\frac{\partial^2\xi}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2\xi}{\partial t^2} = 0, \tag{53}$$

where $c_0 = \sqrt{\tau/\rho_0}$ is the transverse wave speed. Longitudinal waves are not considered in either the present work or in standard references on strings;[13,26,38,68,72–74] a discussion of how longitudinal waves affect momentum conservation is provided by Rowland and Pask.[75]

To obtain the momentum conservation theorem, Eq. (53) is multiplied by $\partial\xi/\partial x$, yielding

$$\frac{1}{c_0^2}\frac{\partial}{\partial t}\left(\tau\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial t}\right) = \tau\frac{\partial^2\xi}{\partial x^2}\frac{\partial\xi}{\partial x} + \rho_0\frac{\partial^2\xi}{\partial t\partial x}\frac{\partial\xi}{\partial t}, \tag{54}$$

where the product rule has been used to write

$$\frac{\partial\xi}{\partial x}\frac{\partial^2\xi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial t}\right) - \frac{\partial^2\xi}{\partial t\partial x}\frac{\partial\xi}{\partial t}.$$

Also, noting from the product rule that

$$\frac{\partial^2\xi}{\partial x^2}\frac{\partial\xi}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial x}\right)^2 \quad \text{and} \quad \frac{\partial^2\xi}{\partial t\partial x}\frac{\partial\xi}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial t}\right)^2$$

allows Eq. (54) to be written as

$$\frac{\partial T}{\partial x} = \frac{\partial g}{\partial t}, \tag{55}$$

where[26,38]

$$S = -\tau\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial t} = \text{instantaneous intensity}, \tag{56}$$

$$g = S/c_0^2 = \text{momentum density}, \tag{57}$$

$$\mathcal{K} = \tfrac{1}{2}\rho_0(\partial\xi/\partial t)^2 = \text{kinetic energy density}, \tag{58}$$

$$\mathcal{U} = \tfrac{1}{2}\tau(\partial\xi/\partial x)^2 = \text{potential energy density}, \tag{59}$$

$$\mathcal{L} = \mathcal{K} - \mathcal{U} = \text{Lagrangian density}, \tag{60}$$

$$T = \mathcal{L} - \rho_0(\partial\xi/\partial t)^2 = \text{radiation stress}. \tag{61}$$

Equation (55) recovers Elmore and Heald's Eq. (1.11.15) of Ref. 68 and Morse and Ingard's in-line equation developed qualitatively in Ref. 26 (p. 106). Equation (56) can be understood by noting that the instantaneous force exerted by an element of the string on an element to its right is $-\tau\partial\xi/\partial x$; since power equals force times velocity $\partial\xi/\partial t$, the quantity in parentheses on the left-hand side of Eq. (54) is identified as minus the energy flux.

Integrating Eq. (55) over x from point x_1 to x_2, assuming that ξ is time-harmonic and that no object lies between the points, and taking the time average yields[48]

$$\langle T(x_2) - T(x_1)\rangle = 0. \tag{62}$$

Equation (62), which is analogous to Eq. (21) of the present work and Eq. (7) of Ref. 49, states that waves passing through the control length $x_2 - x_1$ experience no change in momentum. If a mass lies between points x_1 and x_2, however, waves will generally reflect from and transmit through the mass. In such cases, the difference in $\langle T\rangle$ at points x_2 and x_1 equals the rate at which momentum is transferred from the waves to the mass, which by Newton's second law equals the radiation force

$$F = \langle T(x_2) - T(x_1)\rangle. \tag{63}$$

Equation (63) states that the radiation force equals the difference between the stresses at either end of the mass. The radiation stress, in turn, contains squares of partial derivatives of ξ, which is a solution of Eq. (53). Calculation of Eq. (63) therefore amounts to solving the linear scattering problem, as in the case for calculating the radiation forces in Secs. II–IV. Whereas radiation force is calculated by evaluating a surface integral of a tensor in Secs. II and III and a line integral in Sec. IV, radiation force exerted by waves on a string is calculated by simply taking the difference between scalars.

The pedagogical value of waves on a string is demonstrated in Appendix B, in which Eq. (63) is used to calculate the radiation force on a point mass on a string. Beyond classroom instruction, one-dimensional analyses of radiation pressure serve as a starting point toward understanding more complicated wave phenomena. For example, Sharfarets considered waves on a string to clarify three-dimensional scattering based on partial scattering amplitudes,[76] and Zhang generalized the one-dimensional result for the Langevin radiation pressure [similar to Eq. (B8) for waves on a string] to calculate three-dimensional acoustic radiation forces.[51]

VI. CONCLUSION

The acoustic momentum conservation theorem was derived from the equations of linear acoustics, paralleling Pierce's derivation of the acoustic energy corollary[25] and proving the frequently stated fact that the Langevin radiation force may be calculated using solutions of the linear acoustic wave equation. The derivation of the acoustic momentum conservation theorem paralleled the momentum conservation theorems of electromagnetism, shallow surface gravity waves, and waves on a string, which serve as three-, two-, and one-dimensional analogs of linear acoustic fields, respectively.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: SIMPLIFICATION OF ECKART'S RESULT

Equation (1) can be obtained directly from the irrotational form of Eq. (11) of Ref. 37, where the sign error has been corrected in footnote ‡ of Ref. 27 and in Eq. (2) of the present work. Begin by noting that the last term of Eq. (2) equals $\mathcal{E} - \rho_0 v^2 = -\mathcal{L}$. Thus[42]

$$\frac{\partial \mathbf{g}}{\partial t} = -\rho_0(\mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{v}\nabla\cdot\mathbf{v}) + \nabla\cdot(\mathbf{I}\mathcal{L}). \tag{A1}$$

The first term on the right-hand side of Eq. (A1) can be simplified by noting that[41]

$$\mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{v}\nabla\cdot\mathbf{v} = (\nabla\mathbf{v})^{\mathrm{T}}\cdot\mathbf{v} + (\nabla\cdot\mathbf{v})\mathbf{v}, \tag{A2}$$

which follows from the definition of the transpose operation and the fact that $\nabla\cdot\mathbf{v}$ is a scalar. Since $(\nabla\mathbf{v})^{\mathrm{T}} = \nabla\mathbf{v}$,[77] Eq. (A2) equals $(\nabla\mathbf{v})\cdot\mathbf{v} + (\nabla\cdot\mathbf{v})\mathbf{v} = \nabla\cdot(\mathbf{v}\otimes\mathbf{v})$,[27,39] and Eq. (A1) therefore reduces to

$$\frac{\partial \mathbf{g}}{\partial t} = \nabla\cdot(\mathbf{I}\mathcal{L} - \rho_0\mathbf{v}\otimes\mathbf{v}), \tag{A3}$$

which equals Eq. (1) in view of Eq. (18). Similar manipulations were made by Westervelt to the time-averaged version of Eq. (A1) to obtain Eq. (5) of Ref. 27, which equals Eq. (21) of the present work.

APPENDIX B: RADIATION FORCE ON A POINT SCATTERER ON A STRING

To illustrate the pedagogical value of Sec. V, Prob. 9.6(a) of Ref. 13 is extended to consider the radiation force exerted on a point scatterer of mass m located at the origin $x = 0$ of an infinitely long string. The density of the string is ρ_1 for $x < 0$ and ρ_2 for $x > 0$, as shown in Fig. 2.

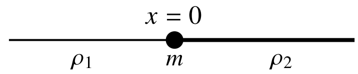

FIG. 2. Point scatterer of mass m at $x = 0$ on a string with density ρ_1 for $x < 0$ and ρ_2 for $x > 0$. The string is driven harmonically in time from $x = -\infty$ with amplitude ξ_0.

The wave motion is considered to be time-harmonic,

$$\xi(x,t) = \tfrac{1}{2}\tilde{\xi}(x)e^{-i\omega t} + \text{c.c.}, \tag{B1}$$

where "c.c." denotes the complex conjugate of the preceding term, and ω is the angular frequency. Combining Eqs. (B1) and (53) shows that the string obeys

$$\begin{cases} d^2\tilde{\xi}/dx^2 + k_1^2\tilde{\xi} = 0\,, & x < 0\,, \\ d^2\tilde{\xi}/dx^2 + k_2^2\tilde{\xi} = 0\,, & x > 0, \end{cases} \tag{B2}$$

where $k_1 = \omega\sqrt{\rho_1/\tau}$ and $k_2 = \omega\sqrt{\rho_2/\tau}$ are the wavenumbers to the left and right of the mass, respectively. The solution of Eqs. (B2) may be expressed as

$$\tilde{\xi}(x) = \begin{cases} \xi_0 e^{ik_1 x} + \mathrm{R}\xi_0 e^{-ik_1 x}\,, & x < 0\,, \\ \mathrm{T}\xi_0 e^{ik_2 x}\,, & x > 0, \end{cases} \tag{B3}$$

where R and T are the reflection and transmission coefficients.[78] Since both sides of the string are attached to the mass at $x = 0$, Eq. (B3) satisfies

$$1 + \mathrm{R} = \mathrm{T}. \tag{B4}$$

The mass obeys Newton's second law at $x = 0$, requiring that

$$-m\omega^2\mathrm{T} = i\tau[k_2\mathrm{T} - k_1(1-\mathrm{R})]. \tag{B5}$$

Solving Eqs. (B4) and (B5) for R and T yields

$$\mathrm{R} = \frac{1 - k_2/k_1 + imk_1/\rho_1}{1 + k_2/k_1 - imk_1/\rho_1}, \tag{B6}$$

$$\mathrm{T} = \frac{2}{1 + k_2/k_1 - imk_1/\rho_1}. \tag{B7}$$

Evaluating Eq. (63) in terms of Eqs. (58)–(61) and Eq. (B3) yields the radiation force

$$F = \langle\mathcal{E}_{\mathrm{i}}\rangle\Big(1 + |\mathrm{R}|^2 - |\mathrm{T}|^2 k_2^2/k_1^2\Big), \tag{B8}$$

$$\to 2\langle\mathcal{E}_{\mathrm{i}}\rangle\,, \quad m \to \infty\ (\text{or}\ k_2 \gg k_1), \tag{B9}$$

$$\to 2\langle\mathcal{E}_{\mathrm{i}}\rangle\,, \quad m \to 0\ \text{and}\ k_1 \gg k_2, \tag{B10}$$

$$\to 0\,, \quad m \to 0\ \text{and}\ k_1 = k_2, \tag{B11}$$

where $\langle\mathcal{E}_{\mathrm{i}}\rangle = \xi_0^2\tau k_1^2/2$ is the time-averaged energy density of the incident wave. The fact that Eqs. (B9) and (B10) are equal suggests that "the phase between the incident and reflected waves does not play a role on the radiation pressure."[51] The acoustical analog of Eqs. (B9) and (B10) is the well-known Langevin radiation pressure on a perfect reflector[6,8,10] that was mentioned in Sec. I. The analogs of Eqs. (B9) and (B10) in electrodynamics are discussed on p. 382 of Ref. 13. The trivial limit taken in Eq. (B11) is consistent with the fact that "the unperturbed incident wave loses no momentum in passing through the control [length],"[49] as expressed by Eq. (62).

The acoustical analogs of Eqs. (B8)–(B11) are widely understood; for a more general version of the result that applies to Bessel beams, see Wang and Lee's discussion on pp. 185–188 of Ref. 10. Nevertheless, the simplicity afforded by waves on a string, particularly in the derivation of Eqs. (55) and (63), makes Sec. V a useful pedagogical tool for understanding radiation forces exerted by waves of other varieties.

[1]L. Brillouin, "On radiation stresses," Ann. Phys. **4**, 528–586 (1925), Eqs. (6)–(8).
[2]F. E. Borgnis, "Acoustic radiation pressure of plane-compressional waves at oblique incidence," J. Acoust. Soc. Am. **24**, 468–469 (1952).
[3]E. M. J. Herrey, "Experimental studies on acoustic radiation pressure," J. Acoust. Soc. Am. **27**, 891–896 (1955).
[4]L. Brillouin, "Les pressions et tensions de radiation," Rev. Acoust. **5**, 99–111 (1936), Secs. I and II.
[5]F. E. Borgnis, "Acoustic radiation pressure of plane compressional waves," Rev. Mod. Phys. **25**, 653–664 (1953).
[6]R. T. Beyer, "Radiation pressure: The history of a mislabeled tensor," J. Acoust. Soc. Am. **63**, 1025–1030 (1978).
[7]T. Tini, "Radiation pressure and nonlinear wave-body interaction in acoustics," Ph.D. thesis, University of Rome, Rome, 2024.
[8]J. H. Poynting, "Radiation pressure," Philos. Mag. **9**, 393–406 (1905).
[9]L. Rayleigh, "On the momentum and pressure of gaseous vibrations, and on the connection with the virial theorem," Philos. Mag. **10**, 364–374 (1905).
[10]T. G. Wang and C. P. Lee, "Radiation pressure and acoustic levitation," in *Nonlinear Acoustics*, edited by M. F. Hamilton and D. T. Blackstock, 3rd ed. (Springer, Cham, Switzerland, 2024), Chap. 6, pp. 179–182, 187–188, 191, Eqs. (6.8) and (6.45).
[11]P. L. Marston, "James Clerk Maxwell: Life and science," J. Quant. Spectrosc. Radiat. Transfer. **178**, 50–65 (2016).
[12]J. Hartmann and T. Mortensen, "A comparison of the Rayleigh disk and the acoustic radiometer methods for the measurement of sound-wave energy," Philos. Mag. **39**, 377–394 (1948).
[13]D. J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Pearson, Upper Saddle River, NJ, 1999), pp. 182–183, 345–356, 380–382, and Prob. 9.6.
[14]P. L. Marston and D. B. Thiessen, "Manipulation of fluid objects with acoustic radiation pressure," Ann. N. Y. Acad. Sci. **1027**, 414–434 (2004).
[15]J. Rufo, F. Cai, J. Friend, M. Wiklund, and T. J. Huang, "Acoustofluidics for biomedical applications," Nat. Rev. Methods Primers **2**, 30 (2022).
[16]Z. A. Gol'dberg, "Acoustic radiation pressure," in *High-Intensity Ultrasonic Fields*, edited by L. D. Rozenberg (Plenum Press, New York, 1971), pp. 73–133, Eqs. (5), (6), and (101).
[17]G. R. Torr, "The acoustic radiation force," Am. J. Phys. **52**, 402–408 (1984).
[18]B. Bensaude-Vincent and F. Duck, *Paul Langevin: Physicist and Social Activist* (Springer, Cham, Switzerland, 2025), p. 81.
[19]E. J. Post, "Radiation pressure and dispersion," J. Acoust. Soc. Am. **25**, 55–60 (1953).
[20]L. A. Ostrovsky, "Notes on the acoustic radiation force," Radiophys. Quantum El. **66**, 421–430 (2023).
[21]H. Olsen, W. Romberg, and H. Wergeland, "Radiation force on bodies in a sound field," J. Acoust. Soc. Am. **30**, 69–76 (1958).
[22]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon Press, Oxford, 1987), Eqs. (7.1), (7.2), and p. 257.
[23]O. A. Sapozhnikov, "High-intensity ultrasonic waves in fluids: Nonlinear propagation and effects," in *Power Ultrasonics* (Woodhead, Cambridge, MA, 2023), pp. 7–22, Sec. 2.4.2.
[24]J. H. Ginsberg, *Acoustics: A Textbook for Engineers and Physicists*, Vol. 2 (Springer, Cham, Switzerland, 2018), p. 655.
[25]A. D. Pierce, *Acoustics: An Introduction to Its Physical Principles and Applications*, 3rd ed. (Springer, Cham, Switzerland, 2019) Eq. (1.11.2), Secs. 1.11.1, 1.11.2, and 9.1.4, and p. 649. Pierce's Eq. (1.11.4) states that the rate at which acoustic energy decreases in a volume equals the power radiated on its surface.
[26]P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (Princeton University Press, Princeton, NJ, 1968), pp. 100–106 and Eq. (6.2.17). Morse and Ingard's $\mathfrak{W}$ = Eq. (18) and $\mathbf{M} = -$ Eq. (17). Note on p. 250 that $-c\mathbf{M} = p\mathbf{u}/c$ (not $c\mathbf{M}$) is the instantaneous radiation pressure.
[27]P. J. Westervelt, "The theory of steady forces caused by sound waves," J. Acoust. Soc. Am. **23**, 312–315 (1951), Eqs. (5), (8), and footnote ‡.
[28]L. P. Gor'kov, "On the forces acting on a small particle in an acoustical field in an ideal fluid," Sov. Phys. Dokl. **6**, 773–775 (1962).
[29]M. J. Marr Lyon, D. B. Thiessen, and P. L. Marston, "Passive stabilization of capillary bridges in air with acoustic radiation pressure," Phys. Rev. Lett. **86**, 2293–2296 (2001).
[30]W. Wei, D. B. Thiessen, and P. L. Marston, "Acoustic radiation force on a compressible cylinder in a standing wave," J. Acoust. Soc. Am. **116**, 201–208 (2004), Eqs. (5), (8), and (9).
[31]P. L. Marston, "Axial radiation force of a Bessel beam on a sphere and direction reversal of the force," J. Acoust. Soc. Am. **120**, 3518–3524 (2006).
[32]L. Zhang and P. L. Marston, "Axial radiation force exerted by general non-diffracting beams," J. Acoust. Soc. Am. **131**, EL329–EL335 (2012), Sec. 3.
[33]B. E. Simon and M. F. Hamilton, "Analytical solution for acoustic radiation force on a sphere near a planar boundary," J. Acoust. Soc. Am. **153**, 627–642 (2023), Eqs. (5) and (6).
[34]P. L. Marston, "Position dependence of the standing-wave radiation pressure quadrupole projection on a sphere applied to drop shape," J. Acoust. Soc. Am. **156**, 1586–1593 (2024).
[35]D. Zhang and C. Ma, "Acoustic metamaterials for remote manipulation of large objects in water," Mater. Horiz. **12**, 4639–4647 (2025), Eq. (8).
[36]C. A. Gokani, T. S. Jerome, M. R. Haberman, and M. F. Hamilton, "Acoustic radiation force exerted by progressive waves on subwavelength inhomogeneous scatterers," J. Acoust. Soc. Am. **159**, 3030–3047 (2026), Eq. (A1).
[37]C. Eckart, "Vortices and streams caused by sound waves," Phys. Rev. **73**, 68–76 (1948). Eckart's "$\mathbf{u}_0$," "ρ_0," "ρ_1," and "W" equal $\mathbf{v}$, ρ_0, ρ', and $\mathcal{E}$, respectively, of the present work. Eckart's Eq. (7.0) should equal "$\mathbf{u} = \mathbf{u}_0 + N\mathbf{u}_1 + N^2\mathbf{u}_2 + \cdots$," and the third term in Eckart's Eq. (11) should be added to, not subtracted from, the equation.
[38]D. T. Blackstock, *Fundamentals of Physical Acoustics* (Wiley, New York, 2000), Eqs. (1C-13), (1C-20), (1C-21), (1C-55), and (1D-4), p. 48.
[39]For differentiable vector fields $\mathbf{u}$ and $\mathbf{w}$, $\nabla\cdot(\mathbf{u}\otimes\mathbf{w}) = (\nabla\mathbf{u})\cdot\mathbf{w} + (\nabla\cdot\mathbf{w})\mathbf{u}$, where $(\mathbf{u}\otimes\mathbf{w})\cdot\mathbf{x} = (\mathbf{w}\cdot\mathbf{x})\mathbf{u}$ defines the outer product.
[40]L. Zhang and P. L. Marston, "Angular momentum flux of nonparaxial acoustic vortex beams and torques on axisymmetric objects," Phys. Rev. E **84**, 065601 (2011).
[41]$\nabla(\mathbf{u}\cdot\mathbf{w}) = (\nabla\mathbf{u})^{\mathrm{T}}\cdot\mathbf{w} + (\nabla\mathbf{w})^{\mathrm{T}}\cdot\mathbf{u}$, where "T" denotes the transpose operation. For $\mathbf{u} = \mathbf{w} = \mathbf{v}$, $\nabla(v^2) = 2(\nabla\mathbf{v})^{\mathrm{T}}\cdot\mathbf{v}$. Taking the curl of Eq. (5) shows that $\nabla\times\mathbf{v} = 0$; therefore, $(\nabla\mathbf{v})^{\mathrm{T}} = \nabla\mathbf{v}$, as shown in Ref. 77. Thus, $(\nabla\mathbf{v})\cdot\mathbf{v} = \nabla(v^2)/2$.
[42]For a differentiable scalar field f and rank-2 tensor field A, $\nabla\cdot(\mathsf{A}f) = \mathsf{A}\cdot\nabla f + f\nabla\cdot\mathsf{A}$.
[43]A. L. Fetter and J. D. Walecka, *Theoretical Mechanics of Particles and Continua* (Dover, Mineola, New York, 2003), Secs. 45 and 53, Eqs. (48.32), (48.34), (53.28)–(53.32), (53.34), (53.35), and (53.40).
[44]J. H. Cantrell, Jr. "Acoustic-radiation stress in solids: I. Theory," Phys. Rev. B **30**, 3214–3220 (1984), Sec. VI.
[45]V. L. Gurevich and A. Thellung, "Quasimomentum in the theory of elasticity and its conservation," Phys. Rev. B **42**, 7345–7349 (1990).
[46]If ω is the angular frequency, the time average of $f(t)$ is $\langle f\rangle = (\omega/2\pi)\int_0^{2\pi/\omega} f(t)\,dt$.
[47]L. Zhang and P. L. Marston, "Acoustic radiation torque and the conservation of angular momentum (L)," J. Acoust. Soc. Am. **129**, 1679–1680 (2011).

[48]Suppose $f(t) = \cos\omega t$ and $g(t) = \cos(\omega t + \phi)$, so $f(t)g(t) = [\cos(2\omega t + \phi) + \cos\phi]/2$. Thus, $d[f(t)g(t)]/dt = -\omega\sin(2\omega t + \phi)$, the time average of which is zero.

[49]P. J. Westervelt, "Acoustic radiation pressure," J. Acoust. Soc. Am **29**, 26–29 (1957). Note that $d\mathbf{A}$ in Eq. (22) points outward while the differential area in Westervelt's Eq. (1) points inward.

[50]K. Yosioka and Y. Kawasima, "Acustica radiation pressure on a compressible sphere," Acustica **5**, 167–173 (1955).

[51]L. Zhang, "From acoustic radiation pressure to three-dimensional acoustic radiation forces," J. Acoust. Soc. Am. **144**, 443–447 (2018).

[52]G. Maidanik, "Torques due to acoustical radiation pressure," J. Acoust. Soc. Am. **30**, 620–623 (1958), Eq. (3).

[53]R. T. Beyer, "Radiation pressure in a sound wave," Am. J. Phys. **18**, 25–29 (1950).

[54]J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, Hoboken, NJ, 1999), Eqs. (6.120)–(6.123).

[55]J. B. Marion, *Classical Electromagnetic Radiation* (Academic Press, San Diego, 1965), pp. 437–442, Eqs. (13.142a) and (13.146).

[56]J. R. Oppenheimer, *Lectures on Electrodynamics* (Gordon and Breach, New York, 1970), pp. 9–16, Eqs. (5.6), (5.8), and (5.9).

[57]E. J. Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw-Hill, New York, 1981), pp. 156–164.

[58]H. C. Ohanian, *Classical Electrodynamics* (Allyn and Bacon, Newton, MA, 1988), pp. 244–251, Eq. (102).

[59]J. Schwinger, L. L. DeRaad, K. A. Milton, and W.-Y. Tsai, *Classical Electrodynamics* (Perseus, Reading, MA, 1998), Sec. 3.2.

[60]W. Greiner, *Classical Electrodynamics* (Springer, New York, 1998), pp. 256–262, Eq. (13.38).

[61]Although $\mathbf{B}$ is used in standard electrodynamics references (Refs. 13, 54–58, and 60), using $\mathbf{H} = \mathbf{B}/\mu_0$ establishes a closer parallel to the fluid velocity vector $\mathbf{v}$ in Sec. II. For example, acoustic and electromagnetic intensity equal $p\mathbf{v}$ and $\mathbf{E}\times\mathbf{H}$, respectively; if $\mathbf{B}$ were used, the latter expression would be $\mathbf{E}\times\mathbf{B}/\mu_0$.

[62]Equation (30) can be obtained in index notation by expressing the ith component of the left-hand side as $\varepsilon_{ijk}\varepsilon_{klm}E_j\partial_l E_m$, where ε_{ijk} are the components of the Levi-Civita tensor and ∂_l denotes the partial derivative with respect to the lth component. Noting that $\varepsilon_{ijk}\varepsilon_{klm} = -\varepsilon_{kji}\varepsilon_{klm} = \delta_{jm}\delta_{il} - \delta_{jl}\delta_{im}$,[79] where δ_{jm} is the Kronecker delta, yields $\frac{1}{2}\partial_i(E_m E_m) - \partial_l(E_i E_l) + E_i\partial_l E_l$, which is the ith component of $\frac{1}{2}\boldsymbol{\nabla}(E^2) - \boldsymbol{\nabla}\cdot(\mathbf{E}\otimes\mathbf{E}) + \mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E})$. Invoking Eq. (23) yields Eq. (30). Equation (31) is obtained similarly.

[63]X. Wu, Y. Liu, S. Jin, and M. Yu, "Particle dynamics in optical tweezer systems," Rev. Chem. Eng. **41**, 197–223 (2025), Eq. (6).

[64]H. Lamb, *Hydrodynamics*, 6th ed. (Dover, New York, 1945), Secs. 10, 169–174, and 250. As noted on p. 357 of Ref. 43, Lamb refers to shallow surface gravity waves as "tidal waves." A brief exploration of how tidal forces generate such waves is provided on pp. 195–202 of Ref. 68.

[65]M. S. Longuet-Higgins and R. W. Stewart, "Radiation stress and mass transport in gravity waves, with application to 'surf beats,'" J. Fluid Mech. **13**, 481–504 (1962).

[66]G. B. Whitham, "Mass, momentum and energy flux in water waves," J. Fluid Mech. **12**, 135–147 (1962).

[67]M. S. Longuet-Higgins and R. W. Stewart, "Radiation stresses in water waves; a physical discussion, with applications," Deep-Sea Res. **11**, 529–562 (1964).

[68]W. C. Elmore and M. A. Heald, *Physics of Waves* (Dover, New York, 1985), Sec. 1.11, Eq. (1.11.15), and Chap. 6.

[69]The curl is defined in three dimensions. The curls of the two-dimensional vector fields $\boldsymbol{\nabla}\zeta$ and $\mathbf{v}$ are evaluated by setting the z components to zero, e.g., $v = v_x\mathbf{e}_x + v_y\mathbf{e}_y + 0\mathbf{e}_z$.

[70]L. Rayleigh, "On the pressure of vibrations," Philos. Mag. **3**, 338–346 (1902).

[71]D. R. Rowland, "Novel insights into the radiation pressure of transverse waves on strings using numerical modelling," Eur. J. Phys. **47**, 015005 (2026), Sec. 1.

[72]R. P. Feynman, *Feynman Lectures on Physics* (Basic, New York, 2013) Vol. I, Chap. 49.

[73]L. E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders, *Fundamentals of Acoustics*, 3rd ed. (Wiley, New York, 1982), Chap. 2.

[74]S. L. Garrett, *Understanding Acoustics: An Experimentalist's View of Sound and Vibration*, 2nd ed. (Springer, Cham, Switzerland, 2020), Chap. 3.

[75]D. R. Rowland and C. Pask, "The missing wave momentum mystery," Am. J. Phys. **67**, 378–388 (1999). Radiation force resolves "Paradox 1": the time-averaged force applied by the wave at the interface connecting the two strings in Fig. 3 of Ref. 75 equals the restoring force exerted by the interface to maintain its position at $x = 0$.

[76]B. P. Sharfarets, "Radiation pressure of a string on a ring: A solution in terms of the scattering theory," Acoust. Phys. **58**, 533–536 (2012).

[77]To show that $\boldsymbol{\nabla}\times\mathbf{v} = \mathbf{0}$ implies that $(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} = \boldsymbol{\nabla}\mathbf{v}$, consider the ith component of $\boldsymbol{\nabla}\times\mathbf{v} = \mathbf{0}$, which equals $\varepsilon_{ijk}\partial_j v_k = 0_i = -\varepsilon_{ijk}\partial_j v_k$. Permuting the Levi-Civita symbol yields $\varepsilon_{ijk}\partial_j v_k = \varepsilon_{ikj}\partial_j v_k$, and renaming indices $j \leftrightarrow k$ yields $\varepsilon_{ijk}\partial_j v_k = \varepsilon_{ijk}\partial_k v_j$. Thus, $\partial_j v_k = \partial_k v_j$, which is the indicial form of $(\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}} = \boldsymbol{\nabla}\mathbf{v}$.

[78]The Roman letter "T" is used for the transmission coefficient and should not be confused with the italic "T" denoting the scalar radiation stress or "τ" denoting the string tension.

[79]M. L. Boas, *Mathematical Methods in the Physical Sciences*, 3rd ed. (John Wiley & Sons, Hoboken, New Jersey, 2006), Chap. 10, Sec. 5, Eqs. (5.8) and (5.13).