Paraxial and ray approximations of acoustic vortex beams

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What is an acoustic vortex beam?

- Helical wavefronts characterized by orbital number ℓ
- Used for particle trapping, underwater communication
- **Paraxial** and **ray** approximations used to calculate the acoustic pressure field $p(r,\theta,z)e^{-i\omega t}$

Paraxial approximation of Helmholtz eqn $i2k(\partial q/\partial z) + \nabla_\perp^2 q = 0$

- Assume that $q = pe^{ikz}$, where $k =$ wavenumber
- Fresnel diffraction integral yields a compact solution, Eq. (1), for focused Gaussian vortex source
- Eq. (1) reduces to a Gaussian beam for $\ell = 0$
- Unfocused vortex beam recovered for $d/a \to \infty$, where d is the focal length and a is the source radius
- Eq. (1) has power independent of ℓ given by Eq. (2)
- Eq. (3) shows beam radius increases linearly with ℓ, both in focal plane and in far field of unfocused beam

Eq. 1: Solution to paraxial approximation of Helmholtz equation

$$q(r,\theta,z) = \sqrt{8\pi}\frac{p_0 z}{kr^2}\chi^{3/2}e^{-\chi}\left[I_{(\ell-1)/2}(\chi) - I_{(\ell+1)/2}(\chi)\right]e^{i[\ell\theta-(\ell+1)\pi/2+kr^2/2z]}$$

$$\chi(r,z) = \frac{(kar/z)^2/8}{1 - i(ka^2/2z)(1-z/d)}$$

Eq. 2: Power

$$\mathcal{P}_0 = \frac{\pi a^2 p_0^2}{4\rho_0 c_0}$$

density (ρ_0), sound speed (c_0)

Eq. 3: Ring radius

$$r_\ell = \begin{cases} \eta_\ell d/ka, & z = d \\ \eta_\ell z/ka, & d = \infty \end{cases}$$

$$\eta_\ell = 0.94\ell + 0.75$$

Eq. 4: Caustic (thick lines)

$$\frac{\Delta_c(z)}{a} = \sqrt{\frac{\ell z}{Gd}\left|1 - \frac{z}{d}\right|}$$

$G = ka^2/2d =$ focusing gain

Ray approximation ($k \to \infty$):

- Eq. (1) predicts maximum pressure contour resembling “cupped hands”
- Appeal to ray theory (eliminate diffraction) to explain this phenomenon
- Ray theory predicts that annular channels of rays given by Eq. (5) emerge from source and define caustic surfaces given by Eq. (4)
- Caustic explains “cupped hands,” along which pressure is maximum
- Within the caustic given by Eq. (6), there is no sound

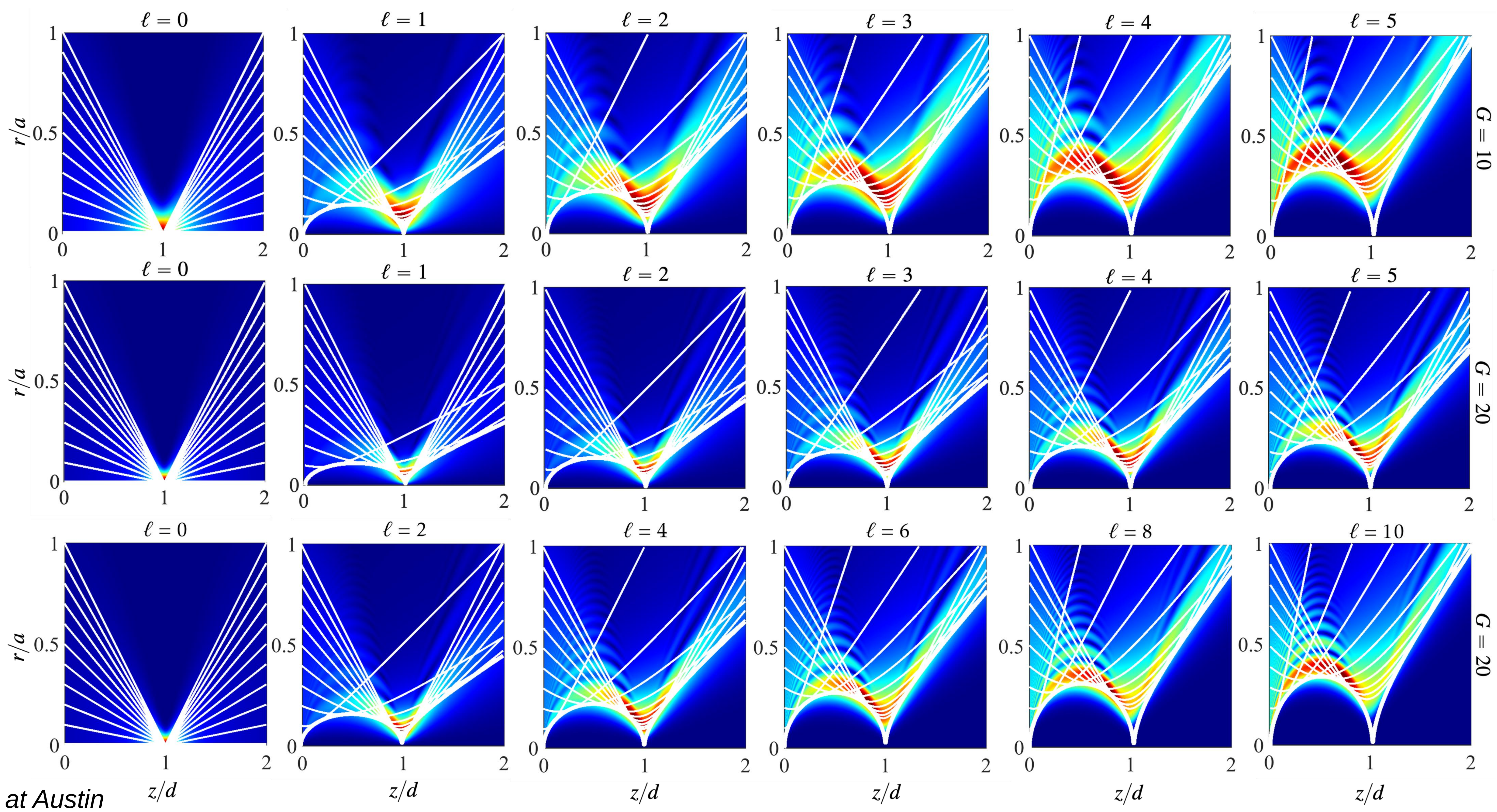
Eq. 5: Ray channels (thin lines)

$$\frac{\Delta(r_0,z)}{a} = \frac{r_0}{a}\sqrt{\left(1-\frac{z}{d}\right)^2 + \left(\frac{z}{d}\right)^2\left(\frac{\ell}{2G}\right)^2\left(\frac{a}{r_0}\right)^4}$$

Eq. 6: Prefocal spheroid

$$\frac{r^2}{\ell a^2/4G} + \frac{(z-d/2)^2}{d^2/4} = 1$$

Volume of spheroid: $V = \ell\lambda d^2/6$



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“Time to go nonlinear!”